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Non-Solvable Groups whose all Vanishing Class Sizes are Odd-Square-Free

Roghayeh Hafezieh — Gülsemin Çonoğlu

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Abstract

Given a finite group G, a vanishing element is an element $x \in G$ for which there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$. The conjugacy class of a vanishing element is called a vanishing class of G. Considering G as a finite non-solvable group with $\mathrm{Sol}(G)$ as its solvable radical, in this paper, we prove that if all vanishing class sizes of G are odd-square-free, then either $G/\mathrm{Sol}(G)$ is an almost simple group, or it has exactly two chief factors with the properties mentioned in Theorem 1.

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1 Introduction

Given a finite group G, a celebrated theorem of Burnside which relates the influence of conjugacy class sizes of G on the group structure with irreducible characters, implies that any nonlinear $\chi \in \mathrm{Irr}(G)$ always vanishes on some element of G. Such an element is called a vanishing element of G and its associated conjugacy class is called a vanishing class of G. There have been many studies on the group structure of G, considering some restrictions on the sizes of the vanishing classes. As an example, in [6] under the condition where no vanishing class size of G is divisible by a given prime p, Dolfi et al.

proved that G has a normal p-complement and abelian Sylow p-subgroups. Brough in [3] proved that a group whose vanishing class sizes are all square-free numbers is supersolvable. In [2], Bianchi et al. described those groups whose vanishing classes all have size p for some prime p. In [10], Robati has showed that if each vanishing class size of a non-solvable group G has at most two prime divisors then $G/\mathrm{Sol}(G)$ is isomorphic to $\mathrm{PSL}(2,4)$ or $\mathrm{PSL}(2,8)$. By the inspiration of the outcomes attained in [9], recently in [11], Robati and the first author of this paper discussed those non-solvable groups whose all conjugacy class sizes are odd-square-free.

Motivated by the results in [10] and [11], in this paper, we consider those finite non-solvable groups whose all vanishing class sizes are odd-square-free, (briefly denoted by OSF). Our aim is to prove the following theorem which is the main theorem of this paper.

Theorem 1 *Let* G *be a non-solvable group all of whose vanishing class sizes are OSF,* $\mathrm{Sol}(G)$ *be the solvable radical of* G*, and* $\overline{G} := G/\mathrm{Sol}(G)$*. Then one of the following cases holds for* $\overline{G}$*:*

(i) $\overline{G}$ *is an almost simple group with socle* $\overline{N}$ *which is the unique non-abelian chief factor of* $\overline{G}$ *and* $\overline{N}$ *is isomorphic with one of the groups listed in* Theorem 6*. Moreover, if* $\overline{N} < \overline{G}$*, then* $\overline{G}$ *is one of the groups mentioned in* Proposition 7*;*

(ii) $\overline{G}$ *has exactly two non-abelian chief factors, say* $\overline{N}$ *and* $\overline{M}$*, with the following properties:*

  (a) $(\overline{N},\overline{M})$ *is either* $\big(\mathrm{PSL}(2,p),{}^2B_2(q)\big)$ *or* $\big(\mathrm{PSL}(2,2^n),{}^2B_2(q)\big)$ *with the following properties:*

    (1) $\rho\big(\mathrm{cs}(\overline{N})\big) \cap \rho\big(\mathrm{cs}(\overline{M})\big) = \{2\}$.

    (2) $n \geqslant 2$ *and* $2^{2n}-1$ *is prime-square-free.*

    (3) $p \geqslant 5$ *is an odd prime where* p^2-1 *is an odd-square-free number.*

    (4) $q = 2^{2m+1} \geqslant 8$ *and* $(q-1)(q^2+1)$ *is prime-square-free.*

  (b) *The direct product* $\mathrm{Aut}(\overline{N})\times\mathrm{Aut}(\overline{M})$ *is either isomorphic with* $\mathrm{PGL}(2,p)\times{}^2B_2(q)\langle\alpha\rangle$*, or* $\mathrm{PSL}(2,2^n)\langle\varphi\rangle\times{}^2B_2(q)\langle\alpha\rangle$*, where* α *is the generator of the outer automorphism of* ${}^2B_2(q)$ *of order* $2m+1$ *and* φ *is a field automorphism of* $\mathrm{PSL}(2,2^n)$ *of order* n*.*

  (c) $\overline{N}\times\overline{M} \leqslant \overline{G} \leqslant \mathrm{Aut}(\overline{N})\times\mathrm{Aut}(\overline{M})$

Notation Given a finite group G, we denote the conjugacy class of x in G by x^G; the set of conjugacy class sizes by $\mathrm{cs}(G)$; the set of primes dividing the integers in $\mathrm{cs}(G)$ by $\rho\big(\mathrm{cs}(G)\big)$; and the set of all vanishing elements by $\mathrm{Van}(G)$. Other notation throughout the paper is standard.

2 Some simple groups

Given a finite non-abelian simple group S, in this section, we aim to prove that if all vanishing class sizes of S are OSF, then S has an irreducible character of p-defect zero for each prime p. In order to progress we start our discussion by working on alternating groups of degree at least 7. It should be mentioned that some results of this section is a part of the second writer's master thesis [5], which we prefer to present them.

Lemma 2 *Let $S = \mathrm{Alt}(n)$ and $n \geqslant 7$. Then S has a conjugacy class α^S whose size is divisible by the square of an odd prime and a character $\theta \in \mathrm{Irr}(S)$ which extends to $\mathrm{Aut}(S)$ and vanishes on α^S.*

Proof — Let $S = \mathrm{Alt}(n)$. Define $\theta : S \to \mathbb{C}$ with $\theta(x) = |\mathrm{Fix}(x)| - 1$ for each $x \in S$, where $|\mathrm{Fix}(x)|$ denotes the number of fixed points of x. It is a well-known fact that θ is an irreducible character of S which extends to $\mathrm{Sym}(n) = \mathrm{Aut}(S)$. First consider the case where $n \geqslant 10$ is an even number. Now, let

$$\alpha = (1, 2, \ldots, n-5)(n-4, n-3)(n-2, n-1)(n)$$

It is obvious that $\alpha \in S$ and $\theta(\alpha^S) = 0$. The fact that α commutes with an odd permutation results in $\alpha^S = \alpha^{\mathrm{Sym}(n)}$, so

$$|\alpha^{\mathrm{Sym}(n)}| = \frac{n \cdot (n-1) \cdot \ldots \cdot 7 \cdot 6}{n-5} \cdot 5 \cdot 3 \qquad \text{(I)}$$

Considering all possible cases for n modulo 3, we observe that 3 divides some $a \in \{n, n-1, n-2\}$. Now the equality (I) results in the fact that 3^2 divides $|\alpha^S|$ in each case. While $n = 8$, considering $\gamma := (1, 2, \ldots, n-1)(n)$, we can observe that $\theta(\gamma) = 0$ and γ^S is a vanishing class of size divisible by 3^2.

On the other hand, considering the case where $n \geqslant 7$ is an odd number, let

$$\beta = (1, 2, \ldots, n-3)(n-2, n-1)(n).$$

Since $\beta \in S$ with $|\mathrm{Fix}(\beta)| = 1$, we have $\theta(\beta) = 0$. As β commutes with an odd permutation, we deduce that $\beta^S = \beta^{\mathrm{Sym}(n)}$. Now, similar to the previous part, considering all possible cases for n modulo 3, we can observe that $|\beta^S|$ is divisible by 3^2. □

Corollary 3 *Let $S = \mathrm{Alt}(n)$ be the non-abelian alternating group of degree n. Then each vanishing class size of S is OSF if and only if $S = \mathrm{Alt}(5)$.*

Proof — This is proved by using ATLAS [4] and Lemma 2. □

Lemma 4 *Let S be one of the following sporadic simple groups:*

$$\{M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, BM\}$$

Then S has a conjugacy class α^S whose size is divisible by the square of an odd prime and a character $\theta \in \mathrm{Irr}(S)$ which extends to $\mathrm{Aut}(S)$ and vanishes on α^S.

Proof — By application of the results in the ATLAS [4], we can construct Table 1 and we can observe that if S is one of the above mentioned sporadic simple groups, then S has a conjugacy class α^S whose size is divisible by 3^2 or by 5^2. Moreover, in each case, there exists $\theta \in \mathrm{Irr}(S)$ which extends to $\mathrm{Aut}(S)$ and vanishes on α^S.

Table 1: Some sporadic simple groups

S	Char.	α^S	some divisors
M_{12}	χ_7	8B	3^2
M_{22}	χ_3	5A	3^2
M_{24}	χ_5	12B	3^2
J_2	χ_{10}	10C	3^2
HS	χ_7	20A	3^2
Ru	χ_2	12A	5^2
Suz	χ_3	10B	3^2
Co_3	χ_9	11A	5^2
Co_1	χ_2	23A	3^2
BM	χ_2	70A	3^2

The statement is proved. □

Now we are ready to prove the following theorem.

Theorem 5 *Let S be a finite non-abelian simple group and p be a prime for which S does not have any irreducible character of p-defect zero. Then S has a conjugacy class α^S whose size is divisible by the square of an odd prime and a character $\theta \in \mathrm{Irr}(S)$ which vanishes α and extends to $\mathrm{Aut}(S)$.*

Proof — Let S be a finite non-abelian simple group and p be a prime such that S does not have any irreducible character of p-defect zero, then [7, Corollary 2] verifies that $p = 2$ or 3 and S is either isomorphic with one of the following sporadic simple groups

$$\{M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, BM\}$$

or isomorphic with $\mathrm{Alt}(n)$ where $n \geqslant 7$. On the other hand, as a result of Lemma 2 and Lemma 4, each of these groups contains a class α^S whose size is divisible by 3^2 or 5^2 and a character $\theta \in \mathrm{Irr}(S)$ which vanishes α and extends to $\mathrm{Aut}(S)$. □

3 Main theorem

In this section we aim to prove the main theorem of this paper. We start by mentioning the following theorem from [11] which plays an important role in our discussion.

Theorem 6 (see [11], Theorem 2.5) *Let S be a finite non-abelian simple group. Then conjugacy class sizes of S are all OSF if and only if one of the following statements holds:*

(1) $S \simeq \mathrm{Alt}(5)$;

(2) $S \simeq J_1$;

(3) $S \simeq \mathrm{PSL}(2, q)$, *where* $q = p \geqslant 5$ *is an odd prime and* $q^2 - 1$ *is odd-square-free;*

(4) $S \simeq \mathrm{PSL}(2, 2^n)$, *where* $n \geqslant 2$ *and* $2^{2n} - 1$ *is prime-square-free;*

(5) $S \simeq {}^2B_2(q)$, *where* $q = 2^{2m+1} \geqslant 8$ *and* $(q-1)(q^2+1)$ *is prime-square-free.*

Proposition 7 *Let* H *be an almost simple group whose all vanishing class sizes are OSF with socle* $S \neq H$*, where* S *is one of the groups mentioned in* Theorem 6*. Then one of the following cases holds:*

(i) $H = S_5$*;*

(ii) $H = \mathrm{PGL}(2, p)$ *where* $p \geqslant 5$ *is a prime;*

(iii) $H = \mathrm{PSL}(2, 2^n)\langle \varphi^k \rangle$ *for some* $1 \leqslant k < n$ *with* $k|n$*, where* φ *is a field automorphism of order* n*;*

(iv) $[H : S]$ *divides* $2m + 1$*, where* $S \simeq {}^2\mathrm{B}_2(q)$ *with* $q = 2^{2m+1} \geqslant 8$*.*

Proof — Let S be one of the groups mentioned in Theorem 6 and H be an almost simple group whose all vanishing class sizes are OSF with socle $S \neq H$. The case $S = J_1$ does not occur as $\mathrm{Aut}(J_1) = J_1$ implies that $H = J_1$ which is not possible. Looking at the character table of S_5, one can observe that $\mathrm{Van}(S_5)$ is the set of non-trivial conjugacy classes of S_5. As

$$\mathrm{cs}(S_5) \setminus \{1\} = \{10, 15, 20, 24, 30\},$$

we have that S_5 is an almost simple group whose all vanishing class sizes are OSF. Indeed, for $S = A_5$ the property that $S \neq H$ results in $H = \mathrm{Aut}(S) = S_5$. Hence we have part (i).

Assume $p \geqslant 5$ is an odd prime and $p^2 - 1$ is an OSF number. It should be noted that the hypothesis on $p^2 - 1$ guarantees that $|\mathrm{PGL}(2, p)| = p(p^2 - 1)$ is OSF, results in the fact that each vanishing class size of $\mathrm{PGL}(2, p)$ is OSF. Additionally, in this case, the outer automorphism of $\mathrm{PSL}(2, p)$ has order 2 which indicates that

$$H = \mathrm{Aut}\big(\mathrm{PSL}(2, p)\big) = \mathrm{PSL}(2, p)\langle \overline{\delta} \rangle = \mathrm{PGL}(2, p)$$

where $\overline{\delta}$ is a diagonal automorphism of order 2. Hence we have part (ii).

Consider the case where $n \geqslant 2$ and $2^{2n} - 1$ is prime-square-free. Since $\mathrm{Aut}\big(\mathrm{PSL}\big(2, 2^n\big)\big) = \mathrm{PSL}(2, 2^n)\langle \varphi \rangle$, where φ is a field automorphism of order n, we conclude that

$$H = \mathrm{PSL}(2, 2^n)\langle \varphi^k \rangle$$

for some $1 \leqslant k < n$ with $k|n$ and we have part (iii).

Finally, if $S = {}^2B_2(q)$, where $q = 2^{2m+1} \geqslant 8$, then the outer automorphism of S is generated by an element of order $2m+1$, say α and

$$\mathrm{Aut}({}^2B_2(q)) = {}^2B_2(q)\langle\alpha\rangle.$$

Therefore H is a group with the property that $[H : S]$ divides $2m+1$ and we have part (iv). □

Now, we are ready to discuss the main idea of this paper by presenting the following proof of Theorem 1.

Proof of Theorem 1 — Let $\mathrm{Sol}(G)$ be the solvable radical of G and $\overline{G} := G/\mathrm{Sol}(G)$. It is easy to understand that all minimal normal subgroups of $\overline{G}$ are non-abelian. Let $\overline{N}$ be a minimal normal subgroup of $\overline{G}$. Since $\overline{N}$ is non-abelian, we have that $\overline{N} \simeq S_1 \times \ldots \times S_k$, where all S_i s are isomorphic with a non-abelian simple group S.

If S does not have any irreducible character of p-defect zero for a prime p, then Theorem 5 implies that S has a vanishing class, say α^S whose size is divisible by the square of an odd prime and a character $\theta \in \mathrm{Irr}(S)$ which vanishes α and extends to $\mathrm{Aut}(S)$. Now [1, Lemma 5] guarantees that $\theta \times \ldots \times \theta \in \mathrm{Irr}(\overline{N})$ extends to $\overline{G}$. Thus $\overline{G}$ (so does G) has a vanishing class whose size is not OSF, which is a contradiction. Therefore, S has an irreducible character of p-defect zero for each prime p which in particular implies that $\overline{N}$ has an irreducible character of p-defect zero for each prime p.

Let $\overline{n}$ be a non-trivial element of $\overline{N}$, p be a prime divisor of the order of $\overline{n}$, and λ be an irreducible character of $\overline{N}$ of p-defect zero. Let $\chi \in \mathrm{Irr}(\overline{G})$ with $[\chi_{\overline{N}}, \lambda] \neq 0$. Since every conjugate of λ in $\overline{G}$ is an irreducible character of $\overline{N}$ of p-defect zero, [8, Theorem 8.17] shows that $\overline{n}$ is a zero of any such conjugate of λ. Consequently, $\chi(\overline{n}) = 0$. On that account, we conclude that each non-trivial element of $\overline{N}$ is a vanishing element of $\overline{G}$. This verifies that $\overline{N}$ and so S are groups whose all conjugacy class sizes are OSF which forces S to be isomorphic to one of the groups mentioned in Theorem 6. This assures the existence of a vanishing class in G whose size is divisible by 3^2 or 5^2, unless $k = 1$ and $\overline{N} \simeq S$.

Let $\overline{C} := C_{\overline{G}}(\overline{N})$. It is clear that $\overline{C} \lhd \overline{G}$ and has trivial intersection with $\overline{N}$. Besides, $\overline{G}/\overline{C}$ can be embedded in $\mathrm{Aut}(\overline{N})$. We can observe that if $\overline{M}$ is another minimal normal subgroup of $\overline{G}$, then $\overline{M} \subset \overline{C}$. If $\overline{C} = 1$, then $\overline{N}$ is the unique minimal normal subgroup of $\overline{G}$ and $\overline{G}$ is an almost simple group with socle $\overline{N}$ which brings about part (i). As a matter of fact, if $\overline{N} < \overline{G}$, then $\overline{G}$ is one of the groups mentioned

in Proposition 7.

On the other hand, $\overline{C} \neq 1$ guarantees the existence of a minimal normal subgroup of $\overline{G}$, say $\overline{M}$ which is isomorphic with one of the groups mentioned in Theorem 6, inside $\overline{C}$. Let $\overline{D} = C_{\overline{G}}(\overline{M})$. Similarly we can see that $\overline{D} \cap \overline{M} = 1$, $\overline{D} \lhd \overline{G}$, $\overline{N} \subset \overline{D}$, and $\overline{G}/\overline{D}$ can be embedded in $\mathrm{Aut}(\overline{M})$. Suppose

$$p \in \rho\big(\mathrm{cs}(\overline{N})\big) \cap \rho\big(\mathrm{cs}(\overline{M})\big).$$

Let $\overline{m}$ and $\overline{n}$ be those elements of $\overline{M}$ and $\overline{N}$, respectively whose conjugacy class sizes are divisible by the prime p. As $\overline{n} \in \mathrm{Van}(\overline{G})$, there exists $\chi \in \mathrm{Irr}(\overline{G})$ which vanishes $\overline{n}$. Let $\mu \in \mathrm{Irr}(\overline{N})$ be an irreducible constituent of $\chi_{\overline{N}}$. As $\overline{N} \times \overline{M} \lhd \overline{G}$, the Clifford correspondence implies that $\psi^{\overline{G}} \in \mathrm{Irr}(\overline{G})$, where ψ is an irreducible character of $I_{\overline{G}}(\mu \times 1_{\overline{M}})$ with $\big[\psi_{\overline{N}\times\overline{M}}, \mu \times 1_{\overline{M}}\big] \neq 0$. Since $\psi^{\overline{G}}(\overline{n}\,\overline{m}) = 0$ and p^2 divides $|(\overline{n}\,\overline{m})^{\overline{G}}|$, we conclude that p is not an odd prime. Therefore, $\rho\big(\mathrm{cs}(\overline{N})\big) \cap \rho\big(\mathrm{cs}(\overline{M})\big) = \{2\}$.

In addition, the fact that $\mathrm{Alt}(5)$ and J_1 both have some conjugacy class sizes divisible by 15, gives rise to the property that neither $\overline{N}$ nor $\overline{M}$ is isomorphic with $\mathrm{Alt}(5)$ or J_1. On the flip side, it is easy to observe that if $p \geqslant 5$ is a prime and $n \geqslant 2$, then 3 is a common divisor of $p^2 - 1$ and $2^{2n} - 1$. Hence, among the groups mentioned in Theorem 6, the case $(\overline{N}, \overline{M}) = \big(\mathrm{PSL}(2,p), \mathrm{PSL}(2,2^n)\big)$ does not occur. This completes the proof of part (a).

Part (b) is a direct conclusion of part (a) and the results mentioned in the proof of Proposition 7.

Over and above the previous notes, $\overline{G}/\overline{C} \cap \overline{D}$ is isomorphic with a subgroup of $\mathrm{Aut}(\overline{N}) \times \mathrm{Aut}(\overline{M})$. Eventually, if $\overline{C} \cap \overline{D} = 1$, we are done by part (c). The case $\overline{C} \cap \overline{D} \neq 1$ does not occur, otherwise, it contains a minimal normal subgroup of $\overline{G}$, say $\overline{T}$. Similar to the previous paragraph, we can see that $\overline{N} \times \overline{M} \times \overline{T}$ contains a vanishing element of $\overline{G}$ whose conjugacy class size is divisible by 3^2 or 5^2 which contradicts our hypothesis. The proof is complete now. □

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Roghayeh Hafezieh (*corresponding author*), Gülsemin Çonoğlu
Department of Mathematics
Gebze Technical University
41400, Gebze (Turkey)
e-mail: roghayeh@gtu.edu.tr; gulseminconoglu@gmail.com