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The NEPS of Gain Graphs over Arbitrary Groups and its Spectra *

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Abstract

We provide two definitions for a gain function on a *non-complete extended* p*-sum* (*NEPS*) of gain graphs over arbitrary groups. In the first one, the gain function takes values in the direct product of the gain groups of the factors. In the second one, the gain group is the codomain of a homomorphism defined on the direct product of the gain groups of the factors. In both cases, we show that these gain functions are well-defined with respect to the switching isomorphism and the balance. Moreover, by using matrices with entries in the group algebra and their Fourier transforms at representations of the gain groups, we manage to describe the adjacency matrices and the spectra. We also give an application for the search of pairs of switching nonisomorphic cospectral gain graphs, extending in this way some techniques from spectral graph theory to the context of gain graphs over arbitrary groups which are not necessarily Abelian. Finally, we classify all possible gain functions for a NEPS of gain graphs over simple groups and over the group $\mathrm{U}(\mathbb{H})$ of quaternion units.

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1 Introduction

The idea of constructing a graph by building upon smaller graphs is intuitive and extensively explored in the literature. It achieves theoretical significance across various branches of mathematics, including algebra, combinatorics, probability, and harmonic analysis. Moreover, it finds practical applications in mathematical chemistry, where graphs naturally emerge by replacing atoms with vertices and bonds with edges. One of the most general constructions, which recovers many classical products, is certainly the *non-complete extended* p*-sum* (*NEPS*) of graphs. It was introduced by Cvetković in [12] and, for this reason, it is sometimes also referred to as the *Cvetković product*. Starting with n graphs $\Gamma_1, \dots, \Gamma_n$ and a set $\mathcal{B}$ of binary words of length n, the composition $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$ is a graph whose vertex set is the Cartesian product of the vertex sets and whose adjacency rules depend on $\mathcal{B}$ and on the adjacencies of the factor graphs (see Section 2.1). Many investigations were conducted about this construction, for example, the analysis of the diameter [6], the examination of the connected components [30, 32], or it has been used as a source of remarkable examples of families of graphs [24, 25]. But the most interesting perspective, which is one of the motivations behind this construction, is certainly the spectral and linear algebra viewpoint. Indeed, the adjacency matrix $A_{\mathcal{B}}$ of $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$ can be precisely described in terms of $\mathcal{B}$ and the adjacency matrices A_i of the factor graphs [12, 14]:

$$A_{\mathcal{B}} = \sum_{\beta=(\beta_1,\dots,\beta_n)\in\mathcal{B}} A_1^{\beta_1} \otimes A_2^{\beta_2} \otimes \dots \otimes A_n^{\beta_n}. \tag{1.1}$$

Just to mention one of the many subsequent investigations, the *almost cospectrality* of the connected components of a NEPS has been studied in [13, 31]. The interested reader is referred to the survey [15] and references therein for more results about NEPS.

Graph theory is widely applicable because it provides a model for systems of interactions. When these interactions may be positive or negative, a more suitable representation is a *signed graph* [23, 35]. A signed graph consists of a pair (Γ, σ), where Γ represents the *underlying graph* and σ is the *signature*, assigning a sign to each adjacency. There exists a further generalization, called *gain graph*, which allows to accommodate even more diverse types of interactions, for example

in the geometry of the hyperplanes [36, 39] or in the theory of spin glasses [38]. A gain graph is represented by a triple (Γ, G, ψ) where Γ is the underlying graph, G is the *gain group* and ψ is the *gain function*. The gain function assigns a gain value, denoted by $\psi(u, v) \in G$, to each ordered pair (u, v) of adjacent vertices. It is only required that $\psi(v, u) = \psi(u, v)^{-1}$. Signed graphs can be considered as a special instance of gain graphs with $G = \mathbb{T}_2 = \{\pm 1\}$. Another special case is $G = \mathbb{T}$, the multiplicative group of complex numbers with modulo 1 (*complex unit gain graphs* [26], or $\mathbb{T}$*-gain graphs*), or $G = \mathrm{U}(\mathbb{H})$, the group of quaternions having modulo 1 (*quaternion unit gain graphs* [4]). But in general, the gain group G can be abstract and arbitrary; the ones mentioned are not the only interesting cases, for example gain graphs are also related to *voltage graphs*, which constitute a largely investigated topic in topological graph theory [20], where gains are permutations. The reader is referred to [37, 38] for a rich and periodically updated glossary and bibliography on signed and gain graphs. Signed and gain graphs are studied up to *switching isomorphism* [34, 35, 36], as classical graphs are studied up to isomorphism. A gain (signed) graph is *balanced* [23] when the product of the gains (signs) along any closed walk equals the unit element of the group. It is known that a gain graph is balanced if and only if it is switching isomorphic to its underlying graph, endowed with the trivial gain function.

In [18] (resp. [5]) a NEPS construction for signed (resp. $\mathbb{T}$-gain) graphs was introduced. The underlying graphs is the NEPS graph of the underlying graph of each factor, so what was really introduced in those works is a gain function on the construction of Cvetković. The idea in both cases is the same: associating with an oriented edge related to a word $\beta \in \mathcal{B}$ a gain obtained from the product of gains of the edges in the factor graphs corresponding to the non-zero entries of β. The very first assurance needed to have a well-posed definition in this context is that by replacing a factor graph with another switching isomorphic factor, the resulting compound gain graph is switching isomorphic to the original one [5, Proposition 3.6]. This property, among other things, also ensures that the product of balanced graphs is balanced (see [18, Theorem 3.3] and [5, Proposition 3.7]). Just like for classical graphs, a spectral theory has been developed for these constructions. Indeed, the adjacency matrix and its spectrum have been described in [18, Theorem 3.1] and [5, Proposition 3.3], by providing an analogous formula to that which holds for classical graphs (cf. Eq. (1.1)).

The aim of this article is to provide a comprehensive definition for a NEPS of gain graphs over arbitrary groups and to explore the corresponding spectrum. There are two main obstructions to achieve this goal, which make the task significantly different from a mere generalization of [5, 18].

1. The fact that the gain groups are arbitrary means that we do not assume them to be Abelian. Consequently, even in the case of gain graphs over the same group G, the natural generalization associating with an oriented edge related to a word $\beta \in \mathcal{B}$ a gain obtained from the product of gains of the edges in the factor graphs, does not result in a valid gain function. In fact, different orientations of the same edge may not correspond to inverse elements of the group G. Even for the classical *direct product* of graphs, which is a particular case of NEPS, this construction indeed does not work.

2. Discussing the spectrum and adjacency matrix of gain graphs over an arbitrary group G is not as straightforward and unambiguous as when G is $\mathbb{T}_2$ or $\mathbb{T}$. Indeed, in those particular cases, the natural matrices are Hermitian complex, and all the tools of complex linear algebra and spectral theory work efficiently.

For the first issue, our initially proposed solution in Section 3 is to construct a gain function on the group that is the direct product of the gain groups of the factors. In this way, we somehow enforce the commutativity that was lacking to obtain a well-defined gain function. This approach also has the advantage of allowing the multiplication of gain graphs on different groups. We reach one of our first goals in Theorem 3.5 by demonstrating the well-definedness with respect to the switching isomorphism. The less favorable aspect, however, is that even in the case of factor gain graphs all on the same group G, the final result is not a gain function on G itself, but on its direct power. Therefore, this definition is not compatible with the existing definitions for signed graphs and complex unit gain graphs. For this reason, in Section 4, we provide a more general version of gain function that takes values in a group H which is the codomain of a homomorphism ϕ from the aforementioned direct product. Regardless of the specific ϕ chosen, we have demonstrated the same stability properties of this construction with respect to the switching isomorphism (see Theorem 4.4). This way, with a suitable ϕ (that works every time the group is Abelian, see Section 4.2) we recover

the existent definitions of NEPS of signed graphs and complex unit gain graphs. Another case in which this approach is advantageous is that of symmetric groups, where the direct product of permutations can be seen as a permutation itself. Given the relevance also in relation to voltage graphs, Section 4.3 is devoted to this specific case. Finally, in Section 4.4, we explore how much this new approach can produce new gain functions, compared to the approach based on the direct product. In other words, we attempt to answer the question:"Is expanding to the direct product really necessary, or is it an unnecessary complication?" For non-Abelian simple groups, for example, we prove that the only sensible gain function is the one described in Section 3 (Corollary 4.21). For the product of two quaternion unit gain graphs the situation is similar (Theorem 4.23): non-trivial gain functions may only take values in $U(\mathbb{H}) \times U(\mathbb{H})$, or $U(\mathbb{H}) \times SO(3)$, or $SO(3) \times SO(3)$, or $SO(4)$, where $SO(n)$ is the group of rotations of n-dimensional Euclidean space.

For the second issue, we use the spectral theory introduced in [7, 8] for gain graphs over arbitrary groups. The adjacency matrix of a gain graph (Γ, G, ψ) is a matrix whose entries are in the group algebra $\mathbb{C}G$. Equivalently, it is a function with finite support from the group G to the space of complex matrices. Spectral graph theory involves studying graphs using linear algebra, treating them as operators on a vector space; group representation theory applies the same idea to groups. Combining these two approaches appears quite natural when one aims at developing a spectral theory for gain graphs. Once a fixed (unitary) representation π of G is established, a (Hermitian) complex adjacency matrix can be associated with a G-gain graph, by means of the *Fourier transform at* π. Consequently, a notion of π-spectrum and π-cospectrality can be defined. Notably, spectral computations for signed and complex unit gain graphs can be obtained as specific cases with a suitable choice of the representation π. There exists a stronger notion of cospectrality, called G-cospectrality, which, in the case of a finite group, coincides with cospectrality with respect to each representation. Despite this theory has already been established, there are efforts needed to precisely describe the analogues of Kronecker products for these special types of matrices, considering the two possible formulations of gain functions on a NEPS. Subsequently, we will investigate the spectra associated with different representations. These efforts yield their best results in Theorem 3.7 and Theorem 4.8, that prove the analogue of Eq. (1.1) for our two constructions, and Theorem 3.10 and Theorem 4.9, that allow us to

write the eigenvalues of the NEPS graphs, in this broader meaning of spectrum, in terms of the eigenvalues of the factors.

An immediate application of our research is the ability to generate infinite pairs of G-cospectral gain graphs over arbitrary groups, starting from pairs of classically cospectral graphs (see Remark 3.13): just like for classical graphs, the search for non-switching isomorphic cospectral pairs of gain graphs is a highly discussed and extensively studied topic in recent literature [1, 2, 3, 4, 9, 19, 21, 28]. Moreover, this technique is also valuable for signed graphs: given any pair of cospectral unsigned graphs, their composition with the same signed graph produces pairs of graphs that are not only cospectral but also have cospectral underlying graphs (which is precisely $\mathbb{T}_2$-cospectrality), see Corollary 3.14. As G-cospectrality is a very strong condition, we believe that this can be useful also for generalizing techniques which are standard in spectral graph theory, but that have not been extended to gain graphs yet.

2 Preliminaries

2.1 NEPS of graphs

In this section we recall the classical definition of NEPS for graphs. For each $i = 1, \dots, n$, let $\Gamma_i = (V_i, E_i)$ be a graph with vertex set V_i and edge set E_i. We use the notation $\sim$ to denote the adjacency of two vertices. We define the set of oriented edges $\overrightarrow{E_i}$ as the set of the ordered pairs of adjacent vertices, so that if $\{u, v\} \in E_i$, then (u, v) and (v, u) are element of $\overrightarrow{E_i}$.

Let $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0, \dots, 0)\}$ be a family of nonzero binary words of length n. The NEPS (*non-complete extended p-sum*) of the graphs $\Gamma_1, \dots, \Gamma_n$ with respect to the basis $\mathcal{B}$ is the graph whose vertex set is the Cartesian product $V_1 \times \dots \times V_n$, where two vertices $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are adjacent if and only if there exists an n-tuple $\beta = (\beta_1, \dots, \beta_n) \in \mathcal{B}$ such that $x_i = y_i$ whenever $\beta_i = 0$, and $x_i \sim y_i$ in Γ_i whenever $\beta_i = 1$. If this is the case, we use the notation

$$e = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

to denote an oriented edge in the NEPS graph, and we write:

$$\beta_i(e) = \begin{cases} 1 & \text{if } x_i \sim y_i \text{ in } \Gamma_i \\ 0 & \text{if } x_i = y_i \text{ in } \Gamma_i, \end{cases} \qquad \text{for each } i = 1, \dots, n. \tag{2.1}$$

Observe that there exist edges $e \neq f$ in the NEPS graph such that $\beta_i(e) = \beta_i(f)$ for every $i = 1, \dots, n$.
Such a new graph will be denoted by $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$ in the rest of the paper.

The NEPS construction produces a number of graph operations in which the vertex set of the new graph is the Cartesian product of the vertex sets of the factor graphs: it contains and extends some already existing constructions. More precisely:

- in a p-sum of graphs, the number of 1's of any n-tuple of the basis $\mathcal{B}$ is p;
- in a complete p-sum of graphs, the basis $\mathcal{B}$ contains exactly all the n-tuples of $\{0,1\}^n$ whose number of 1's is p;
- in an extended p-sum, the basis $\mathcal{B}$ contains all n-tuples in which the number of 1's belongs to a given subset $J \subseteq \{1, \dots, n\}$.

It is also worth mentioning that some graph operations which are classical in the literature (see, for instance, [11, Chapter 8] or [22]) can be obtained as particular cases of NEPS. For instance, for $n = 2$:

1. if $\mathcal{B} = \{(1,0),(0,1)\}$, then the $\mathrm{NEPS}(\Gamma_1, \Gamma_2; \mathcal{B})$ is the classical *Cartesian product* of graphs $\Gamma_1 \square \Gamma_2$;
2. if $\mathcal{B} = \{(1,1)\}$, then the $\mathrm{NEPS}(\Gamma_1, \Gamma_2; \mathcal{B})$ is the classical *direct (or tensor) product* of graphs $\Gamma_1 \times \Gamma_2$;
3. if $\mathcal{B} = \{(1,0),(0,1),(1,1)\}$, then the $\mathrm{NEPS}(\Gamma_1, \Gamma_2; \mathcal{B})$ is the classical *strong product* of graphs $\Gamma_1 \boxtimes \Gamma_2$.

On the other hand, some known operations on graphs do not belong to the family of NEPS. This is the example of the so-called *lexicographic product* $\Gamma_1[\Gamma_2]$ of two graphs Γ_1 and Γ_2, whose vertex set is given by $V_1 \times V_2$, where two vertices (x_1, x_2) and (y_1, y_2) are adjacent if either $x_1 \sim y_1$ in Γ_1 or $x_1 = y_1$ and $x_2 \sim y_2$ in Γ_2.

To describe the adjacency matrix of NEPS, we recall that the *Kronecker product* of two matrices $A = (a_{ij})_{i=1,\dots,m;j=1,\dots,n} \in M_{m \times n}(\mathbb{C})$

and $B = (b_{hk})_{h=1,\dots,p;k=1,\dots,q} \in M_{p\times q}(\mathbb{C})$ is the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

The Kronecker product is, in general, not commutative. However, it satisfies the following properties (see, for instance, [40]):

1. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
2. $A \otimes (B + C) = A \otimes B + A \otimes C; \quad (A + B) \otimes C = A \otimes C + B \otimes C$;
3. $(A \otimes B)^T = A^T \otimes B^T$;
4. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
5. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Now, for each $i=1,\dots,n$, let A_i be the adjacency matrix of $\Gamma_i=(V_i, E_i)$, that is, the square $\{0,1\}$-matrix indexed by $V_i = \{x_1, x_2, \dots, x_{|V_i|}\}$ whose entry (h,k) is equal to 1 if $x_h \sim x_k$ in Γ_i, and to 0 otherwise. We will use both the notations $(A_i)_{h,k}$ and $(A_i)_{x_h,x_k}$ for this entry.

By definition, the adjacency matrix of $\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})$ is

$$A_{\mathcal{B}} = \sum_{\beta=(\beta_1,\dots,\beta_n)\in\mathcal{B}} A_1^{\beta_1} \otimes A_2^{\beta_2} \otimes \dots \otimes A_n^{\beta_n}, \tag{2.2}$$

where we use the convention that $A_i^1 = A_i$ and $A_i^0 = I_i$, and where I_i is the identity matrix of size $|V_i|$.

Now put $|V_i| = n_i$. It follows from the previous description that the adjacency spectrum of the graph $\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})$ can be easily deduced from the spectrum of the factor graphs. More precisely, let $\{\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,n_i}\}$ be the spectrum of the matrix A_i, and let $\mathbf{x}_{i,j_i}$ be an eigenvector of A_i associated with the eigenvalue λ_{i,j_i}, with $j_i \in \{1,2,\dots,n_i\}$. Then it can be easily checked, by using Eq. (2.2), that the vector $\mathbf{x} = \mathbf{x}_{1,j_1} \otimes \mathbf{x}_{2,j_2} \otimes \dots \otimes \mathbf{x}_{n,j_n}$ is an eigenvector of $A_{\mathcal{B}}$ associated with the eigenvalue $\lambda_{1,j_1}\lambda_{2,j_2}\dots\lambda_{n,j_n}$, so that the adjacency spectrum of the graph $\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})$ consists of all possible values of type

$$\Lambda_{j_1,j_2,\dots,j_n} = \sum_{\beta=(\beta_1,\dots,\beta_n)\in\mathcal{B}} \lambda_{1,j_1}^{\beta_1}\lambda_{2,j_2}^{\beta_2}\dots\lambda_{n,j_n}^{\beta_n},$$

with $j_i \in \{1, 2, \dots, n_i\}$ for each $i = 1, 2, \dots, n$.

The connectedness of the graph $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$ has been investigated in [30], where it is proven that the connectedness depends only on the part of the basis corresponding to the bipartite factors and not on the factors themselves. More precisely, one can construct a $|\mathcal{B}| \times n$ matrix whose rows coincide with the n-tuples of $\mathcal{B}$, and then consider the $|\mathcal{B}| \times k$ submatrix obtained by taking into account only the columns corresponding to the k factors which are bipartite. Then it turns out that $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$ is connected if and only if such submatrix has rank k.

2.2 Gain graphs

Let G be a group with unit 1_G and let $\Gamma = (V_\Gamma, E_\Gamma)$ be a finite simple graph, with vertex set V_Γ and edge set E_Γ. Then Γ is a *gain graph over* G, or a G-gain graph for short, if there exists a map $\psi\colon \overrightarrow{E_\Gamma} \to G$ such that $\psi(u, v) = \psi(v, u)^{-1}$. The graph Γ is called the *underlying graph* and ψ is said a *gain function* on Γ. We denote such a gain graph by (Γ, G, ψ).

Consider the vector space $\mathbb{C}G$ consisting of all finite $\mathbb{C}$-linear combinations of elements of G, endowed with a product and an involution as follows:

$$\left(\sum_{x \in G} f_x x\right)\left(\sum_{y \in G} h_y y\right) := \sum_{x,y \in G} f_x h_y\, xy,$$

for each $f = \sum_{x \in G} f_x x$ and $h = \sum_{y \in G} h_y y \in \mathbb{C}G$, and

$$f^* := \sum_{x \in G} \overline{f_{x^{-1}}}\, x.$$

Endowed with these operations, $\mathbb{C}G$ is an algebra with involution, called the *group algebra* of G.

We denote by $M_{m \times n}(\mathbb{C}G)$ the space of $m \times n$ matrices with coefficients in the (complex) group algebra $\mathbb{C}G$ of G. The product of $F \in M_{m \times n}(\mathbb{C}G)$ and $H \in M_{n \times t}(\mathbb{C}G)$ is the matrix $FH \in M_{m \times t}(\mathbb{C}G)$ such that $(FH)_{i,j} = \sum_{k=1}^{n} F_{i,k} H_{k,j}$, where $F_{i,k} H_{k,j}$ is the product in $\mathbb{C}G$. When $m = n$, we use the notation $M_n(\mathbb{C}G)$. Actually $M_n(\mathbb{C}G)$ is an algebra with the involution * defined by $(F^*)_{i,j} = (F_{j,i})^*$, where the * on the right is the involution in $\mathbb{C}G$. The unit in $M_n(\mathbb{C}G)$ is the identity matrix $I_{M_n(\mathbb{C}G)}$, which is the diagonal matrix whose di-

agonal entries are all equal to 1_G. An element of $F \in M_{m\times n}(\mathbb{C}G)$ can also be interpreted as a map $F\colon G \to M_{m\times n}(\mathbb{C})$. Specifically, for $x \in G$, the entry $F(x)_{i,j}$ is the coefficient of x in $F_{i,j} \in \mathbb{C}G$. Vice versa, if we have a (finitely supported) map $F\colon G \to M_{m\times n}(\mathbb{C})$, we can write $F = \sum_{x\in G} F(x)\,x$ and, by identifying the complex number 1 with the element $1_G \in \mathbb{C}G$, we obtain an element of $M_{m\times n}(\mathbb{C}G)$. In the case $m = n$, this description corresponds to an isomorphism of algebras with involutions (see [7, Proposition 3.2]). Throughout the article, we will use both approaches interchangeably, depending on which one will be more convenient and clear.

The adjacency matrix of the gain graph (Γ, G, ψ) is the matrix $A_{(\Gamma,\psi)}$ of $M_n(\mathbb{C}G)$, where $n = |V_\Gamma|$, with entries

$$(A_{(\Gamma,\psi)})_{i,j} = \begin{cases} \psi(v_i, v_j) & \text{if } v_i \sim v_j; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $A_{(\Gamma,\psi)}$ is Hermitian in the group algebra sense, that is, $A^*_{(\Gamma,\psi)} = A_{(\Gamma,\psi)}$. If we want to interpret $A_{(\Gamma,\psi)} \in M_n(\mathbb{C}G)$ as a map from G to $M_n(\mathbb{C})$, then the image $A_{(\Gamma,\psi)}(g)$ of an element $g \in G$ is the complex adjacency matrix of the directed graph (without gains) obtained from (Γ, G, ψ) by considering only the oriented edges with gain g.

In order to give the definition of a balanced gain graph, one has to define the gain of a walk in (Γ, G, ψ). Let ω be a walk of length h, that is, an ordered sequence of $h+1$ vertices $v_0, v_1, \dots, v_h$ of Γ such that $v_i \sim v_{i+1}$. Then the gain $\psi(\omega) \in G$ of ω is defined as

$$\psi(\omega) := \psi(v_0, v_1) \dots \psi(v_{h-1}, v_h).$$

A closed walk of length h is a walk of length h such that $v_0 = v_h$. A gain graph (Γ, G, ψ) is *balanced* if $\psi(\omega) = 1_G$ for every closed walk ω. Easy examples of balanced gain graphs are obtained by considering an arbitrary underlying graph endowed with the *trivial gain function*, or an underlying graph which is a tree, endowed with any gain function.

Two gain functions ψ_1 and ψ_2 on the same underlying graph Γ are said *switching equivalent* if there exists $f\colon V_\Gamma \to G$ such that

$$\psi_2(v_i, v_j) = f(v_i)^{-1}\psi_1(v_i, v_j)f(v_j) \tag{2.3}$$

for any pair of adjacent vertices v_i and v_j of Γ. If Eq. (2.3) holds, we write $\psi_2 = \psi_1^f$. With a small abuse, we will equivalently say that the gain graphs (Γ, G, ψ_1) and (Γ, G, ψ_2) are switching equivalent.

It is known that a gain graph (Γ, G, ψ) is balanced if and only if it is switching equivalent to the gain graph Γ endowed with the trivial gain function (see [36, Lemma 5.3]). Moreover, in analogy with the complex case (see [26, Lemma 3.2]), the following result holds.

Theorem 2.1 (see [7], Theorem 4.1) *Let ψ_1 and ψ_2 be two gain functions on the same underlying graph Γ, with respectively adjacency matrices $A_{(\Gamma,\psi_1)}$ and $A_{(\Gamma,\psi_2)} \in M_n(\mathbb{C}G)$. Then ψ_1 and ψ_2 are switching equivalent if and only if there exists a diagonal matrix $F \in M_n(\mathbb{C}G)$, with $F_{i,i} \in G$ for each $i = 1, \ldots, n$, such that $F^* A_{(\Gamma,\psi_1)} F = A_{(\Gamma,\psi_2)}$.*

The notion of switching equivalence can be further generalized as follows. Two gain graphs (Γ_1, G, ψ_1) and (Γ_2, G, ψ_2) are said *switching isomorphic* if there exists a graph isomorphism $\theta\colon V_{\Gamma_1} \to V_{\Gamma_2}$ such that ψ_1 is switching equivalent to the gain function $\psi_2 \circ \theta$ on Γ_1 defined as $(\psi_2 \circ \theta)(u, v) = \psi_2(\theta(u), \theta(v))$. When the underlying graphs are implicitly specified, we will equivalently say that the gain functions ψ_1 and ψ_2 are switching isomorphic. The matrix counterpart of switching isomorphism requires the existence of a diagonal matrix $F \in M_n(\mathbb{C}G)$, with $F_{i,i} \in G$ for each $i = 1, \ldots, n$, and of a matrix $P \in M_n(\mathbb{C}G)$ whose nonzero entries are equal to 1_G and where the positions of the nonzero entries correspond to those of a suitable permutation matrix, satisfying the equation

$$(PF)^* A_{(\Gamma_1,\psi_1)}(PF) = A_{(\Gamma_2,\psi_2)}.$$

2.3 Group representations

Denote by $M_k(\mathbb{C})$ the algebra of all complex square matrices of size k, by $GL_k(\mathbb{C})$ the group of all invertible matrices in $M_k(\mathbb{C})$, and by $U_k(\mathbb{C}) = \{M \in GL_k(\mathbb{C}) : M^{-1} = M^*\}$ the subgroup of $GL_k(\mathbb{C})$ consisting of unitary matrices, where M^* is the Hermitian transpose of M.

A representation π of *degree* k (we write $\deg \pi = k$) of a group G is a group homomorphism $\pi\colon G \to GL_k(\mathbb{C})$. The representation π is *unitary* if $\pi(g) \in U_k(\mathbb{C})$, for every $g \in G$. Equivalently, π can be regarded as a group homomorphism from G to the general linear group $GL(V)$ of a k-dimensional complex vector space V or, in the unitary case, to the unitary group $U(V)$.

Two representations π and π' are said to be equivalent, and we write $\pi \sim \pi'$, if there exists $S \in GL_k(\mathbb{C})$ such that $\pi'(g) = S^{-1}\pi(g)S$ for every $g \in G$. It is known that, for a finite group G, each equivalence class contains a unitary representative: this is the reason why working with unitary representations is not restrictive in the finite case. The *kernel* of a representation π is $\ker\pi = \{g \in G : \pi(g) = I_k\}$, and π is *faithful* if $\ker\pi = \{1_G\}$.

Given two representations $\pi\colon G \to GL(U)$ and $\tau\colon G \to GL(V)$ the *direct sum* representation $\pi \oplus \tau$ of G is defined by considering the action of G coordinate-wise on the vector space $U \oplus V$, so that

$$(\pi \oplus \tau)(g)(u, v) = (\pi(g)(u), \tau(g)(v)),$$

for all $g \in G, u \in U, v \in V$. Notice that $\deg(\pi \oplus \tau) = \deg\pi + \deg\tau$. We will use the notation $\pi^{\oplus i}$ for the i-th iterated direct sum of π with itself.

A representation π of G is said to be *irreducible* if it admits no non-trivial invariant subspace for the action of G. It is known that π is irreducible if and only if it is not equivalent to any direct sum of representations. Also, if G has m conjugacy classes, then there exist m irreducible, pairwise inequivalent, representations $\pi_0, \dots, \pi_{m-1}$, which constitute a *complete system of irreducible representations of* G such that, for each representation π of G, there are $k_0, \dots, k_{m-1} \in \mathbb{N} \cup \{0\}$ such that $\pi \sim \bigoplus_{i=0}^{m-1} \pi_i^{\oplus k_i}$.

Let us recall some remarkable representations of a group G. The first one is the *trivial representation* $\pi_0\colon G \to \mathbb{C}$, such that $\pi_0(g) = 1$ for every $g \in G$. It is unitary and irreducible and it always occurs in a complete system of irreducible representations of G. For a group $G \leqslant \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, one can also consider the *identical representation* $\pi_{id} : G \to \mathbb{C}$, defined by $\pi_{id}(z) = z$.

A group G naturally acts by left multiplication on itself. This action can be identified with the action of G on the vector space

$$\mathbb{C}G = \Big\{ \sum_{x \in G} c_x x : \ c_x \in \mathbb{C} \Big\}.$$

When G is finite, one gets the *left regular representation* λ_G, which is faithful and has degree $|G|$. It is known that, if $\pi_0, \dots, \pi_{m-1}$ is a complete system of irreducible representations, then λ_G decomposes

as:

$$\lambda_G \sim \bigoplus_{i=0}^{m-1} \pi_i^{\oplus \deg \pi_i}.$$

For further information and a general discussion on group representation theory, we refer to [10, 17].

A representation π of degree k of G can be naturally extended by linearity to the group algebra $\mathbb{C}G$, thus giving rise to the algebra homomorphism:

$$\begin{aligned} \pi\colon \mathbb{C}G &\longrightarrow M_k(\mathbb{C}) \\ \sum_{x\in G} f_x x &\longmapsto \sum_{x\in G} f_x \pi(x). \end{aligned} \tag{2.4}$$

Moreover, if π is unitary, then $\pi(f^*) = \pi(f)^*$ for any $f \in \mathbb{C}G$. In addition, π can be extended to $M_n(\mathbb{C}G)$. We still use the notation

$$\pi\colon M_n(\mathbb{C}G) \longrightarrow M_{nk}(\mathbb{C}).$$

For $A \in M_n(\mathbb{C}G)$, the element $\pi(A) \in M_{nk}(\mathbb{C})$ is called the *Fourier transform of* A *at* π. Roughly speaking, $\pi(A)$ is obtained from A by replacing each entry $a_{ij} \in \mathbb{C}G$ with the block $\pi(a_{ij}) \in M_k(\mathbb{C})$, where π is the map defined in Eq. (2.4). An alternative but equivalent way (see [7, Definition 3.2, Proposition 3.3]) to define the Fourier transform of A at π is

$$\pi(A) = \sum_{x\in G} A(x) \otimes \pi(x), \tag{2.5}$$

where we interpreted $A \in M_n(\mathbb{C}G)$ as a map from G to $M_n(\mathbb{C})$ as described in Section 2.2.

Moreover, equivalence relations and subrepresentations are also preserved in their extensions to $M_n(\mathbb{C}G)$, as the next proposition claims.

Proposition 2.2 (see [7], Proposition 3.4) *Let* $A \in M_n(\mathbb{C}G)$ *and let* π, π' *be two representations of* G. *Then:*

- *if* $\pi \sim \pi'$ *then the matrices* $\pi(A)$ *and* $\pi'(A)$ *are similar;*
- *the matrices* $(\pi \oplus \pi')(A)$ *and* $\pi(A) \oplus \pi'(A)$ *are similar;*
- *if* π *is unitary, then* $\pi(A^*) = \pi(A)^*$.

In the rest of the paper, we will frequently make use of the notion of tensor product of two representations

$$\pi\colon G \to GL(U) \quad \text{and} \quad \tau\colon H \to GL(V)$$

of two groups G and H, which provides a representation of the direct product $G \times H$ defined by $(\pi \otimes \tau)(g,h)(u \otimes v) = \pi(g)(u) \otimes \tau(h)(v)$, for all $g \in G, h \in H, u \in U, v \in V$. Notice that $\deg(\pi \otimes \tau) = \deg \pi \cdot \deg \tau$. We will use the notation $\pi^{\otimes i}$ for the i-th iterated tensor product of π with itself.

2.4 π-Spectrum, π-cospectrality and G-cospectrality

We recall here some basic definitions and properties about π-spectrum and cospectrality properties for a gain graph (Γ, G, ψ), where π is a representation of G; cf. [8].

Consider a G-gain graph (Γ, G, ψ) whose adjacency matrix is $A_{(\Gamma,\psi)} \in M_n(\mathbb{C}G)$, and let π be a representation of G with $\deg \pi = k$. We will refer to the matrix $\pi(A_{(\Gamma,\psi)}) \in M_{nk}(\mathbb{C})$ as the *represented adjacency matrix* of (Γ, G, ψ) with respect to π. Notice that, if π is unitary, then $\pi(A_{(\Gamma,\psi)})$ is a Hermitian matrix in the classical sense. We say that the (real) spectrum of $\pi(A_{(\Gamma,\psi)})$ is the *π-spectrum of* (Γ, G, ψ), denoted by $\sigma_\pi(A_{(\Gamma,\psi)})$. Two gain graphs are π-cospectral if they have the same π-spectrum.

Although the spectrum of a gain graph with respect to a given representation π was explicitly introduced only in [7], particular cases of π-spectra have already been considered in the literature, as for the gain groups $\mathbb{T}_2$ or $\mathbb{T}$, where the spectrum with respect to the identical representation π_{id} equals the usual spectrum for signed and complex unit gain graphs.

Observe that, if π and π' are two equivalent representations of G, then $\sigma_\pi(A_{(\Gamma,\psi)}) = \sigma_{\pi'}(A_{(\Gamma,\psi)})$ by Proposition 2.2. Note that if G-gain graphs (Γ_1, G, ψ_1) and (Γ_2, G, ψ_2) are π_1-cospectral and π_2-cospectral, with π_1 and π_2 unitary representations, then they are $(\pi_1 \oplus \pi_2)$-cospectral. As a consequence, if (Γ_1, G, ψ_1) and (Γ_2, G, ψ_2) are cospectral with respect to each representation belonging to a complete system of irreducible representations, then they are cospectral with respect to all representations of G.

Let us denote by $\mathcal{C}^h(\Gamma)$ the set of all closed walks of length h in Γ. The following theorem holds.

Theorem 2.3 (see [8], Theorem 4.14) *Let* (Γ_1, G, ψ_1) *and* (Γ_2, G, ψ_2) *be two* G*-gain graphs, with* G *finite. Let* $\pi_0, \dots, \pi_{m-1}$ *be a complete system of irreducible, unitary representations of* G*. The following are equivalent.*

1. (Γ_1, G, ψ_1) *and* (Γ_2, G, ψ_2) *are* π*-cospectral, for every unitary representation* π *of* G*;*
2. (Γ_1, G, ψ_1) *and* (Γ_2, G, ψ_2) *are* π_i*-cospectral, for each* $i = 0, \dots, m-1$*;*
3. *if* $[g]$ *denotes the conjugacy class of* $g \in G$*, then for all* $g \in G$ *and* $h \in \mathbb{N}$ *we have*

$$\left|\{\omega \in \mathcal{C}^h(\Gamma_1) : \psi_1(\omega) \in [g]\}\right| = \left|\{\omega \in \mathcal{C}^h(\Gamma_2) : \psi_2(\omega) \in [g]\}\right|.$$

If the third condition of the previous theorem is satisfied, even when G is infinite, we say that (Γ_1, G, ψ_1) and (Γ_2, G, ψ_2) are G-*cospectral*. In some sense, in the finite case, one can define the G-*spectrum* of a G-gain graph as the list of its π_i-spectra, with π_i varying in a complete system of irreducible representations. It has been proven in [8] that a G-gain graph (Γ, G, ψ) is balanced if and only if it is G-cospectral with its underlying graph, endowed with the trivial gain function. Moreover, if two G-gain graphs (Γ_1, G, ψ_1) and (Γ_2, G, ψ_2) are switching isomorphic, then they are G-cospectral, but the converse implication does not hold in general.

3 NEPS of gain graphs

3.1 Gain functions on a NEPS

Let (Γ_i, G_i, ψ_i) be a G_i-gain graph with gain function ψ_i, for each $i = 1, \dots, n$, and let $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0, \dots, 0)\}$. Denote by $G_1 \times \ldots \times G_n$ the direct product of the groups $G_1, \dots, G_n$, with group operation

$$(g_1, \dots, g_n) \cdot (h_1, \dots, h_n) = (g_1 h_1, \dots, g_n h_n),$$

defined componentwise for all $g_i, h_i \in G_i$.

Our aim is to define a gain function taking values in $G_1 \times \ldots \times G_n$ on the graph $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$ introduced in Section 2.1.

Definition 3.1 Let $\overrightarrow{E}(\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B}))$ denote the set of oriented edges of the graph $\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})$. The *gain function* Ψ on $\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})$ is the function

$$\Psi\colon \overrightarrow{E}(\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})) \to G_1\times\dots\times G_n$$

defined, for each oriented edge $e=\begin{pmatrix} x_1 & x_2 & \cdots & x_n\\ y_1 & y_2 & \cdots & y_n\end{pmatrix}$, as

$$\Psi(e) := (z_1,\dots,z_n),\quad \text{where } z_i=\begin{cases}\psi_i(x_i,y_i) & \text{if } \beta_i(e)=1\\ 1_{G_i} & \text{if } \beta_i(e)=0.\end{cases}$$

Remark 3.2 The function Ψ is in fact a gain function: if

$$e_-=\begin{pmatrix} y_1 & y_2 & \cdots & y_n\\ x_1 & x_2 & \cdots & x_n\end{pmatrix}$$

is the opposite of the oriented edge e, then

$$\Psi(e)\cdot\Psi(e_-)=(z_1,\dots,z_n)$$

where $z_i=\psi_i(x_i,y_i)\psi_i(y_i,x_i)=1_{G_i}$ if $\beta_i(e)=\beta_i(e_-)=1$, and $z_i=1_{G_i}1_{G_i}=1_{G_i}$ if $\beta_i(e)=\beta_i(e_-)=0$.

In the next lemma we prove that if all (Γ_i,G_i,ψ_i) are balanced, then also their NEPS is balanced. This result can be also obtained as a consequence of Theorem 3.5, which is proved later. However we would like to give a direct proof without using Theorem 3.5.

Lemma 3.3 *Let* (Γ_i,G_i,ψ_i), $i=1,\dots,n$, *be balanced* G_i*-gain graphs. Let* $\mathcal{B}\subseteq\{0,1\}^n\setminus\{(0,\dots,0)\}$. *Then* $(\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B}),G_1\times\dots\times G_n,\Psi)$ *is a balanced* $(G_1\times\dots\times G_n)$*-gain graph.*

Proof — Let ω be the walk $v_1,v_2,\dots,v_k$ in $\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})$, with

$$v_i=(x_{i,1},x_{i,2},\dots,x_{i,n}),\qquad i=1,\dots,k.$$

Let us define, for each $j=1,\dots,n$, the projection $p_j(\omega)\in V_j^k$ as

$$p_j(\omega)=(x_{1,j},\dots,x_{k,j}).$$

Notice that, up to eliminating any possible repetition of consecutive vertices, the projection $p_j(\omega)$ is a walk on Γ_j. In particular, if ω is a

closed walk, then also $p_j(\omega)$ is a closed walk in Γ_j and

$$\Psi(\omega) = (\psi_1(p_1(\omega)), \dots, \psi_n(p_n(\omega))) = (1_{G_1}, \dots, 1_{G_n}),$$

since each factor is supposed to be balanced. □

Remark 3.4 The reverse implication of Lemma 3.3 does not hold. It is not difficult to check that, with the choice of the basis $\mathcal{B} = \{(1,1)\}$, the NEPS of the signed graphs in Fig. 6 is a balanced $\mathbb{T}_2^2$-gain graph, even if the first factor is not balanced. For a description of this property in the case of the classical tensor product of signed graphs, see [29, Theorem 2.6].

Theorem 3.5 *Let* (Γ_i, G_i, ψ_i) *and* $(\Gamma_i', G_i, \psi_i')$ *be two switching isomorphic* G_i*-gain graphs, for each* $i = 1, \dots, n$*. Then the two gain graphs*

$$(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B}), G_1 \times \dots \times G_n, \Psi)$$

and

$$(\mathrm{NEPS}(\Gamma_1', \dots, \Gamma_n'; \mathcal{B}), G_1 \times \dots \times G_n, \Psi')$$

are switching isomorphic. In particular, if (Γ_i, G_i, ψ_i) *and* (Γ_i, G_i, ψ_i') *are switching equivalent for each* i*, then also the corresponding NEPS gain graphs are switching equivalent.*

Proof — Since, for each $i = 1, \dots, n$, the graphs

$$(\Gamma_i, G_i, \psi_i) \quad \text{and} \quad (\Gamma_i', G_i, \psi_i')$$

are switching isomorphic, there are a graph isomorphism $\theta_i \colon V_i \to V_i'$ and a function $f_i \colon V_i \to G_i$ such that

$$\psi_i'(\theta_i(x_i), \theta_i(y_i)) = f_i(x_i)^{-1} \psi_i(x_i, y_i) f_i(y_i),$$

for each pair of adjacent vertices $x_i, y_i \in V_i$. Let us define the map

$$\begin{aligned} \Theta \colon V(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})) &\to V(\mathrm{NEPS}(\Gamma_1', \dots, \Gamma_n'; \mathcal{B})) \\ (x_1, \dots, x_n) &\to (\theta_1(x_1), \dots, \theta_n(x_n)). \end{aligned}$$

This map is clearly a graph homomorphism; in particular, if

$$e = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

is an edge in $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$, then the element

$$\begin{pmatrix} \theta_1(x_1) & \theta_2(x_2) & \cdots & \theta_n(x_n) \\ \theta_1(y_1) & \theta_2(y_2) & \cdots & \theta_n(y_n) \end{pmatrix}$$

is an edge in $\mathrm{NEPS}(\Gamma_1', \dots, \Gamma_n'; \mathcal{B})$, and $\beta_i(\Theta(e)) = \beta_i(e)$ for every $i = 1, \dots, n$. Note that Θ is an isomorphism, with $\Theta^{-1} = (\theta_1^{-1}, \dots, \theta_n^{-1})$. Let us define the function

$$\begin{aligned} F\colon V(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})) &\longrightarrow G_1 \times \ldots \times G_n \\ (x_1, \dots, x_n) &\longmapsto (f_1(x_1), \dots, f_n(x_n)). \end{aligned}$$

Then, for each pair of adjacent vertices $\overline{x}, \overline{y}$ in $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$, one has:

$$\begin{aligned} \Psi'(\Theta(\overline{x}), \Theta(\overline{y})) &= \Psi'((\theta_1(\overline{x}_1), \dots, \theta_n(\overline{x}_n)), (\theta_1(\overline{y}_1), \dots, \theta_n(\overline{y}_n))) \\ &= (\psi_1'(\theta_1(\overline{x}_1), \theta_1(\overline{y}_1)), \dots, \psi_n'(\theta_n(\overline{x}_n), \theta_n(\overline{y}_n))) \\ &= (f_1^{-1}(\overline{x}_1)\psi_1(\overline{x}_1, \overline{y}_1)f_1(\overline{y}_1), \dots, f_n^{-1}(\overline{x}_n)\psi_n(\overline{x}_n, \overline{y}_n)f_n(\overline{y}_n)) \\ &= F(\overline{x})^{-1}\Psi(\overline{x}, \overline{y})F(\overline{y}). \end{aligned} \tag{3.1}$$

As a particular case, if

$$(\Gamma_i, G_i, \psi_i) \quad \text{and} \quad (\Gamma_i, G_i, \psi_i')$$

are switching equivalent, the isomorphism Θ is exactly the identity map on $V(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B}))$ and so, by Eq. (3.1), one gets switching equivalence. □

3.2 Spectral properties of a NEPS of gain graphs

Our next step is the computation of the adjacency matrix of the gain graph $(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B}), G_1 \times \ldots \times G_n, \Psi)$ in terms of $\mathcal{B}$ and of the adjacency matrices of the factor gain graphs (Γ_i, G_i, ψ_i). In order to do that, we need the notion of Kronecker product of group algebra valued matrices. We denote such a product by $\otimes_M$ and we refer to it as the *mixed* Kronecker product because, if A is a matrix with entries in $\mathbb{C}G_i$ and B is a matrix with entries in $\mathbb{C}G_j$, then the entries of $A \otimes_M B$ belong to $\mathbb{C}(G_i \times G_j)$.

Definition 3.6 Let G_1 and G_2 be groups. Given $A \in M_{m\times n}(\mathbb{C}G_1)$ and $B \in M_{p\times q}(\mathbb{C}G_2)$, the *mixed Kronecker product* of A and B is the

element $A \otimes_M B \in M_{mp\times nq}(\mathbb{C}(G_1 \times G_2))$ defined as:

$$(A \otimes_M B)(g_1, g_2) = A(g_1) \otimes B(g_2), \qquad \forall (g_1, g_2) \in G_1 \times G_2.$$

By considering the Kronecker product of elements of two distinct group algebras, it is also possible to describe the block structure of $A \otimes_M B$. Let $\sum_{x\in G_1} f_x x \in \mathbb{C}G_1$ and $\sum_{y\in G_2} h_y y \in \mathbb{C}G_2$, then

$$\Big(\sum_{x\in G_1} f_x x\Big) \otimes \Big(\sum_{y\in G_2} h_y y\Big) = \sum_{(x,y)\in G_1\times G_2} f_x h_y (x, y) \in \mathbb{C}(G_1 \times G_2). \quad (3.2)$$

Thus the element $A \otimes_M B$, seen as a group algebra valued matrix, has the following block structure:

$$\begin{pmatrix} a_{11} \otimes_M B & a_{12} \otimes_M B & \cdots & a_{1n} \otimes_M B \\ a_{21} \otimes_M B & a_{22} \otimes_M B & \cdots & a_{2n} \otimes_M B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \otimes_M B & a_{m2} \otimes_M B & \cdots & a_{mn} \otimes_M B \end{pmatrix} \quad (3.3)$$

where $a_{ij} \otimes_M B$ is the $p \times q$ matrix whose entry (r, s) is $(a_{ij} \otimes b_{rs})$.

In the next theorem, we give a description of the adjacency matrix of the NEPS of gain graphs. This is a generalization to gain graphs of a classical result of Cvetković and Lučić [14]. Let $|V_i| = n_i$ and fix an ordering in V_i. Let us choose for the vertices of $\mathrm{NEPS}(\Gamma_1, \ldots \Gamma_n; \mathcal{B})$ the lexicographic order induced by the orderings on V_i.

Theorem 3.7 *Let (Γ_i, G_i, ψ_i) be a G_i-gain graph, for each $i = 1, \ldots, n$. Let $A_i \in M_{n_i}(\mathbb{C}G_i)$ be the adjacency matrix of (Γ_i, G_i, ψ_i). Consider*

$$\mathcal{B} \subseteq \{0, 1\}^n \setminus \{(0, \ldots, 0)\}.$$

Then the adjacency matrix of $(\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B}), G_1 \times \ldots \times G_n, \Psi)$ is

$$A_\Psi = \sum_{\beta=(\beta_1,\ldots,\beta_n)\in\mathcal{B}} A_1^{\beta_1} \otimes_M \ldots \otimes_M A_n^{\beta_n} \quad (3.4)$$

where $A_i^1 = A_i$ and A_i^0 is the diagonal matrix of size n_i whose nonzero entries are 1_{G_i}.

Proof — Let $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$ be two vertices of $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$. Recall that we write directly $(A_\Psi)_{\overline{x},\overline{y}}$ to indicate the entry of the adjacency matrix A_Ψ corresponding to the posi-

tion $(\overline{x}, \overline{y})$. Notice that, by definition of adjacency matrix, $(A_\psi)_{\overline{x},\overline{y}} = 0$ when $\overline{x}$ and $\overline{y}$ are not adjacent. When instead $\overline{x}$ and $\overline{y}$ are adjacent, there exists $\beta = (\beta_1, \dots, \beta_n) \in \mathcal{B}$ such that

$$\begin{cases} x_s \sim y_s \text{ in } \Gamma_s & \text{if } \beta_s = 1 \\ x_s = y_s \text{ in } \Gamma_s & \text{if } \beta_s = 0 \end{cases} \qquad \text{for } s = 1, \dots, n. \tag{3.5}$$

Observe that if such an n-tuple β exists, then it is unique, in fact if

$$e = \begin{pmatrix} x_1 & x_1 & \cdots & x_n \\ y_1 & y_1 & \cdots & y_n \end{pmatrix}$$

is an edge of $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$, then $\beta_i(e) = \beta_i$ for each $i = 1, \dots, n$, according with Eq. (2.1). By Eq. (3.2), we have

$$(A_\psi)_{\overline{x},\overline{y}} = \begin{cases} g_1 \otimes \ldots \otimes g_n & \text{if } \overline{x} \sim \overline{y} \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

where $g_s = \psi_s(x_s, y_s)$ if $\beta_s = 1$ and $g_s = 1_{G_s}$ if $\beta_s = 0$.

Now we study the right-hand side of Eq. (3.4). First, note that each term of that summation is an element of $M_{\prod n_i}(\mathbb{C}(G_1 \times \ldots \times G_n))$ and then there exists a bijection between the pairs of vertices of $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$ and the entries of

$$\sum_{\beta=(\beta_1,\dots,\beta_n)\in\mathcal{B}} A_1^{\beta_1} \otimes_M \dots \otimes_M A_n^{\beta_n}.$$

Precisely, by virtue of the block structure described in Eq. (3.3), we have

$$\Big(\sum_{\beta=(\beta_1,\dots,\beta_n)\in\mathcal{B}} A_1^{\beta_1} \otimes_M \dots \otimes_M A_n^{\beta_n} \Big)_{\overline{x},\overline{y}} =$$

$$= \sum_{\beta=(\beta_1,\dots,\beta_n)\in\mathcal{B}} \left(A_1^{\beta_1}\right)_{x_1,y_1} \otimes_M \dots \otimes_M \left(A_n^{\beta_n}\right)_{x_n,y_n}.$$

Notice that, if for the fixed $\overline{x} = (x_1, \dots, x_n)$ and $\overline{y} = (y_1, \dots, y_n)$

the element $\beta \in \mathcal{B}$ is such that Eq. (3.5) holds, then

$$\left(A_s^{\beta_s}\right)_{x_s,y_s} = \begin{cases} \psi_s(x_s,y_s) & \text{if } \beta_s = 1 \\ 1_{G_s} & \text{if } \beta_s = 0; \end{cases}$$

in other words, the summand

$$\left(A_1^{\beta_s}\right)_{x_1,y_1} \otimes_M \ldots \otimes_M \left(A_n^{\beta_n}\right)_{x_n,y_n} \quad \text{and} \quad g_1 \otimes \ldots \otimes g_n$$

of Eq. (3.6) coincide.

If instead $\beta \in \mathcal{B}$ is such that Eq. (3.5) does not hold, then there is s such that $\beta_s = 1$ but $x_s \nsim y_s$, and/or there is s such that $\beta_s = 0$ but $x_s \neq y_s$: in both cases $\left(A_s^{\beta_s}\right)_{x_s,y_s} = 0$ and then the contribution of $\left(A_1^{\beta_1}\right)_{x_1,y_1} \otimes_M \ldots \otimes_M \left(A_n^{\beta_n}\right)_{x_n,y_n}$ in the sum is null. This demonstrates, all at once, both that the sum

$$\Big(\sum_{\beta=(\beta_1,\ldots,\beta_n)\in\mathcal{B}} A_1^{\beta_1} \otimes_M \ldots \otimes_M A_n^{\beta_n} \Big)_{\overline{x},\overline{y}}$$

is equal to $g_1 \otimes \ldots \otimes g_n$ when $\overline{x} \sim \overline{y}$, and that it is equal to zero when $\overline{x}$ and $\overline{y}$ are not adjacent. The thesis follows from a comparison with Eq. (3.6). □

As already mentioned in the introduction, the construction of Ψ given in Definition 3.1 is not consistent with the one existing in the literature for signed graphs. In fact, the gain group of Ψ is not even $\mathbb{T}_2$.

Example 3.8 Consider the $\mathbb{T}_2$-gain graphs

$$(K_3, \mathbb{T}_2, \psi_3) \quad \text{and} \quad (K_2, \mathbb{T}_2, \psi_2)$$

depicted in Fig. 6. They are indeed signed graphs, and we do not



Figure 6: The gain graphs $(K_3, \mathbb{T}_2, \psi_3)$ and $(K_2, \mathbb{T}_2, \psi_2)$.

need to represent oriented edges in Fig. 6. Fix the basis

$$\mathcal{B} = \{(1,0),(1,1)\}$$

and consider the graph $(\mathrm{NEPS}(K_3, K_2; \mathcal{B}), \mathbb{T}_2^2, \Psi)$, according to our general construction of Definition 3.1. In this way, we get a gain graph over the Klein group $\mathbb{T}_2^2 = \{(1,1),(-1,1),(1,-1),(-1,-1)\}$. The associated underlying graph $\mathrm{NEPS}(K_3, K_2; \mathcal{B})$ is depicted in Fig. 7.

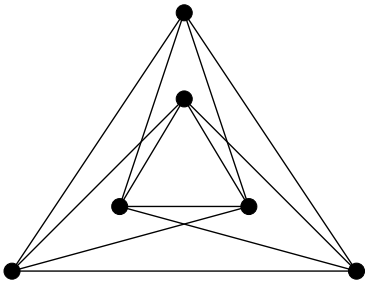


Figure 7: The graph $\mathrm{NEPS}(K_3, K_2; \mathcal{B})$.

If we fix in the NEPS graph the lexicographic order of the vertex set induced by the orderings $\{u_1, u_2, u_3\}$ and $\{v_1, v_2\}$ of the vertex sets of K_3 and K_2, then the adjacency matrix of the gain graph $(\mathrm{NEPS}(K_3, K_2; \mathcal{B}), \mathbb{T}_2^2, \Psi)$ is

$$A_\Psi = A_{(K_3,\psi_3)} \otimes_M I_2 + A_{(K_3,\psi_3)} \otimes_M A_{(K_2,\psi_2)}$$

$$= \left(\begin{array}{cc|cc|cc} 0 & 0 & (1,1) & (1,-1) & (-1,1) & (-1,-1) \\ 0 & 0 & (1,-1) & (1,1) & (-1,-1) & (-1,1) \\ \hline (1,1) & (1,-1) & 0 & 0 & (1,1) & (1,-1) \\ (1,-1) & (1,1) & 0 & 0 & (1,-1) & (1,1) \\ \hline (-1,1) & (-1,-1) & (1,1) & (1,-1) & 0 & 0 \\ (-1,-1) & (-1,1) & (1,-1) & (1,1) & 0 & 0 \end{array}\right).$$

Let $\pi_i \colon G_i \to \mathrm{GL}_{n_i}(\mathbb{C})$ be a representation of G_i. Let us denote by Π the tensor product representation $\pi_1 \otimes \ldots \otimes \pi_n$ of $G_1 \times \ldots \times G_n$ on $\mathrm{GL}_{\prod_{i=1}^n n_i}(\mathbb{C})$ of Section 2.3, and recall that, for each $(g_1, \ldots, g_n)$, it is defined as

$$\Pi(g_1, \ldots, g_n) := \pi_1(g_1) \otimes \ldots \otimes \pi_n(g_n). \tag{3.7}$$

Theorem 3.10 clarifies the relationship between the Π-spectrum of the NEPS gain graph and the π_i-spectra of (Γ_i, G_i, ψ_i). We first need a lemma about Kronecker product.

Lemma 3.9 *For each* $i = 1, \ldots, n$, *let* $A_i \in M_{n_i}(\mathbb{C}G_i)$. *Consider a representation*

$$\Pi = \pi_1 \otimes \ldots \otimes \pi_n$$

of $G_1 \times \ldots \times G_n$. *Then there exists a permutation matrix* P, *which only depends on* n_i *and on* $\deg \pi_i$, *such that*

$$\Pi(A_1 \otimes_M \ldots \otimes_M A_n) = P^{-1} \cdot (\pi_1(A_1) \otimes \ldots \otimes \pi_n(A_n)) \cdot P.$$

In particular, if $\deg \pi_i = 1$ *for each* i, *then* P *is the identity matrix.*

Proof — We will only prove the case $n = 2$, as the proof in the general case is an inductive generalization of this. Let

$$A_1 \in M_k(\mathbb{C}G_1) \quad \text{and} \quad A_2 \in M_m(\mathbb{C}G_2).$$

Combining Eq. (3.7) and Eq. (2.5), we get:

$$\Pi(A_1 \otimes_M A_2) = (\pi_1 \otimes \pi_2)(A_1 \otimes_M A_2)$$

$$= \sum_{(g_1,g_2) \in G_1 \times G_2} (A_1 \otimes_M A_2)(g_1, g_2) \otimes \pi_1(g_1) \otimes \pi_2(g_2)$$

$$= \sum_{g_1 \in G_1} \sum_{g_2 \in G_2} A_1(g_1) \otimes A_2(g_2) \otimes \pi_1(g_1) \otimes \pi_2(g_2);$$

where in the last equality we used the Definition 3.6 for $A_1 \otimes_M A_2$. On the other hand, by using Eq. (2.5), we have:

$$\pi_1(A_1) \otimes \pi_2(A_2)$$

$$= \Big(\sum_{g_1 \in G_1} A_1(g_1) \otimes \pi_1(g_1) \Big) \otimes \Big(\sum_{g_2 \in G_2} A_2(g_2) \otimes \pi_2(g_2) \Big)$$

$$= \sum_{g_1 \in G_1} \sum_{g_2 \in G_2} A_1(g_1) \otimes \pi_1(g_1) \otimes A_2(g_2) \otimes \pi_2(g_2).$$

At first observe that if $\deg \pi_i = 1$ for every i, the Kronecker product

by $\pi_1(g_1)$ in particular is just the multiplication by the scalar $\pi_1(g_1)$ that is clearly commutative: the claim easily follows by choosing $P = I$. In the general case, by [27, Proposition 1], further generalized in [16], there is a permutation matrix Q (of size $n_2 \cdot \deg \pi_1$) which only depends on the sizes of $\pi_1(g_1)$ and $A_2(g_2)$, such that

$$Q^{-1} (\pi_1(g_1) \otimes A_2(g_2)) Q = A_2(g_2) \otimes \pi_1(g_1),$$

therefore:

$$\begin{aligned} &\sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} A_1(g_1) \otimes A_2(g_2) \otimes \pi_1(g_1) \otimes \pi_2(g_2) \\ = &\sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} A_1(g_1) \otimes Q^{-1} (\pi_1(g_1) \otimes A_2(g_2)) Q \otimes \pi_2(g_2) \\ &= (I_{n_1} \otimes Q \otimes I_{\deg \pi_2})^{-1} \chi (I_{n_1} \otimes Q \otimes I_{\deg \pi_2}), \end{aligned}$$

where

$$\chi = \sum_{\substack{g_1 \in G_1 \\ g_2 \in G_2}} A_1(g_1) \otimes \pi_1(g_1) \otimes A_2(g_2) \otimes \pi_2(g_2).$$

The thesis follows by setting $P = I_{n_1} \otimes Q \otimes I_{\deg \pi_2}$. □

Theorem 3.10 *For each $i = 1, \dots, n$, let (Γ_i, G_i, ψ_i) be a G_i-gain graph, with $|V_i| = n_i$, and let π_i be a unitary representation of G_i. Let*

$$\Pi = \bigotimes_{i=1}^{n} \pi_i.$$

Then the Π-spectrum of $(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B}), G_1 \times \dots \times G_n, \Psi)$ consists of the elements

$$\Lambda_{j_1, \dots, j_n} = \sum_{\beta \in \mathcal{B}} \lambda_{1,j_1}^{\beta_1} \dots \lambda_{n,j_n}^{\beta_n}$$

where $\{\lambda_{i,1}, \dots, \lambda_{i, n_i \deg \pi_i}\}$ is the π_i-spectrum of (Γ_i, G_i, ψ_i).

Proof — Let A_Ψ be the adjacency matrix of

$$(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B}), G_1 \times \dots \times G_n, \Psi)$$

as in Theorem 3.7. Then from Lemma 3.9 we obtain

$$\begin{aligned}\Pi(A_\Psi) &= \Pi\Big(\sum_{\beta\in\mathcal{B}} A_1^{\beta_1}\otimes_M\ldots\otimes_M A_n^{\beta_n}\Big)\\ &= \sum_{\beta\in\mathcal{B}}\Pi\Big(A_1^{\beta_1}\otimes_M\ldots\otimes_M A_n^{\beta_n}\Big)\\ &= \sum_{\beta\in\mathcal{B}} P^{-1}\Big(\pi_1(A_1^{\beta_1})\otimes\ldots\otimes\pi_n(A_n^{\beta_n})\Big)P\\ &= P^{-1}\Big(\textstyle\sum_{\beta\in\mathcal{B}}\pi_1(A_1)^{\beta_1}\otimes\ldots\otimes\pi_n(A_n)^{\beta_n}\Big)P.\end{aligned}$$

This implies that it is enough to focus on the spectrum of

$$\sum_{\beta\in\mathcal{B}}\pi_1(A_1)^{\beta_1}\otimes\ldots\otimes\pi_n(A_n)^{\beta_n}$$

in order to study the spectrum of $\Pi(A_\Psi)$. Now notice that, if $\mathbf{x}_{i,j_i}$ is a λ^1_{i,j_i}-eigenvector of $\pi_i(A_i)^1=\pi_i(A_i)$, then it is also a λ^0_{i,j_i}-eigenvector of $\pi_i(A_i)^0=I_{n_i\deg\pi_i}$.

Now for each $i=1,\ldots,n$, let λ_{i,j_i} be an element of the spectrum of $\pi_i(A_i)$ and let $\mathbf{x}_{i,j_i}$ be an associated eigenvector. Define

$$\overline{\mathbf{x}}:=\mathbf{x}_{1,j_1}\otimes\ldots\otimes\mathbf{x}_{n,j_n}.$$

Then we obtain

$$\begin{aligned}&\Big(\sum_{\beta\in\mathcal{B}}(\pi_1(A_1)^{\beta_1}\otimes\ldots\otimes\pi_n(A_n)^{\beta_n})\Big)\cdot(\mathbf{x}_{1,j_1}\otimes\ldots\otimes\mathbf{x}_{n,j_n})\\ &\quad=\sum_{\beta\in\mathcal{B}}(\pi_1(A_1)^{\beta_1}\cdot\mathbf{x}_{1,j_1})\otimes\ldots\otimes(\pi_n(A_n)^{\beta_n}\cdot\mathbf{x}_{n,j_n})\\ &\quad\quad=\sum_{\beta\in\mathcal{B}}(\lambda_{1,j_1}^{\beta_1}\mathbf{x}_{1,j_1})\otimes\ldots\otimes(\lambda_{n,j_n}^{\beta_n}\mathbf{x}_{n,j_n})\\ &\quad=\Big(\sum_{\beta\in\mathcal{B}}\lambda_{1,j_1}^{\beta_1}\ldots\lambda_{n,j_n}^{\beta_n}\Big)(\mathbf{x}_{1,j_1}\otimes\ldots\otimes\mathbf{x}_{n,j_n}).\end{aligned}$$

This is true for $\prod_{i=1}^n n_i\deg\pi_i$ eigenvalues, so all possible values in

the spectrum of $\Pi(A_\Psi)$ are determined. Moreover the multiplicity of $\sum_{\beta\in\mathcal{B}} \lambda_{1,j_1}^{\beta_1}\dots\lambda_{n,j_n}^{\beta_n}$ as an eigenvalue is the product of the multiplicities of the λ_{i,j_i}s. □

Corollary 3.11 *Let (Γ_i, G_i, ψ_i) and $(\Gamma_i', G_i, \psi_i')$ be two G_i-gain graphs. Let π_i be a representation of G_i and let $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0,\dots,0)\}$. Then the following hold.*

1. *If, for each $i=1,\dots,n$, (Γ_i, G_i, ψ_i) and $(\Gamma_i', G_i, \psi_i')$ are π_i-cospectral, then the corresponding NEPS are Π-cospectral, with $\Pi=\pi_1\otimes\dots\otimes\pi_n$;*

2. *Suppose G_i finite, for each $i=1,\dots,n$. If (Γ_i, G_i, ψ_i) and $(\Gamma_i', G_i, \psi_i')$ are G_i-cospectral, then the corresponding NEPS are $(G_1\times\dots\times G_n)$-cospectral.*

Proof — The first point is a consequence of Theorem 3.10.

The second point is a consequence of Theorem 3.10 and Theorem 2.3. More precisely, a complete system of irreducible representations for $G_1 \times \dots \times G_n$ is given by

$$\{\pi_{1,j_1} \otimes \dots \otimes \pi_{n,j_n}\},$$

where $\{\pi_{i,1}, \pi_{i,2}, \dots\}$ is a complete system of irreducible representations of G_i. Therefore, by Theorem 2.3, if (Γ_i, G_i, ψ_i) and $(\Gamma_i', G_i, \psi_i')$ are G_i-cospectral, they are also π_{i,j_i}-cospectral, for each π_{i,j_i}. By Theorem 3.10, this implies that the associated NEPS are Π-cospectral, with $\Pi = \pi_{1,j_1} \otimes \dots \otimes \pi_{n,j_n}$. Then we conclude by applying again Theorem 2.3. □

Notice that the result just presented is original also with respect to the classical graph products obtained by NEPS, i.e., the Cartesian, the direct, and the strong product, because as far as we know, they have never been investigated for gain graphs at this level of generality. This gives us a method to effortlessly create pairs of G-cospectral gain graphs, whose paradigm is described with the next example.

Example 3.12 Consider the gain graphs $(\Gamma_1, \{1\}, \psi_1)$ and $(\Gamma_2, \{1\}, \psi_2)$, where $\{1\}$ is the trivial group, depicted in Fig. 8. Since the gain group is trivial, they can be identified with their underlying graphs Γ_1 and Γ_2, respectively.

The graphs Γ_1 and Γ_2 constitute an example of two cospectral nonisomorphic graphs. More precisely, they share the same characteristic



Figure 8: The gain graphs $(\Gamma_1,\{1\},\psi_1)$ and $(\Gamma_2,\{1\},\psi_2)$.

polynomial, which is equal to

$$(x-1)(x^3-x^2-5x+1)(x+1)^2,$$

although they are not isomorphic, since Γ_1 has two pendant vertices, but Γ_2 has only one pendant vertex.

Consider now the gain graph (T, Q_8, ψ_T) on the Quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

depicted in Fig. 9, whose underlying graph is a triangle, where the oriented edge from the vertex 2 to the vertex 3 has gain k, and the remaining edges have a trivial gain.

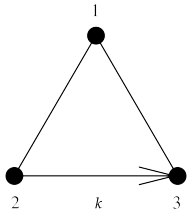


Figure 9: The gain graph (T, Q_8, ψ_T).

Recall now that Q_8 admits (up to equivalence) five pairwise non-equivalent unitary irreducible representations:

1. the trivial representation π_0;
2. the 1-dimensional representation π_i whose kernel is the cyclic group generated by i, and such that $\pi_i(\pm j) = \pi_i(\pm k) = -1$;
3. the 1-dimensional representation π_j whose kernel is the cyclic group generated by j, and such that $\pi_i(\pm i) = \pi_i(\pm k) = -1$;
4. the 1-dimensional representation π_k whose kernel is the cyclic group generated by k, and such that $\pi_i(\pm i) = \pi_i(\pm j) = -1$;

5. the 2-dimensional representation π_2 such that

$$\pi_2(\pm 1) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \pi_2(\pm i) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

$$\pi_2(\pm j) = \pm \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \qquad \pi_2(\pm k) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Let us fix the basis $\mathcal{B} = \{(1,0),(0,1)\}$ and consider the gain graphs

$$(\mathrm{NEPS}(\Gamma_1, T; \mathcal{B}), \{1\} \times Q_8, \Psi) \quad \text{and} \quad (\mathrm{NEPS}(\Gamma_2, T; \mathcal{B}), \{1\} \times Q_8, \Psi).$$

In particular, such NEPS graphs can be regarded as gain graphs over the group Q_8. They are represented in Fig. 10, where the gain of the oriented edges represented by an arrow are all equal to $k \in Q_8$, while the gain of all the other edges is trivial. Notice that our choice for the basis $\mathcal{B}$ corresponds to consider the Cartesian product of two gain graphs, and then we can also write $(\Gamma_i, \{1\}, \psi_i) \square (T, Q_8, \psi_T)$ instead of $(\mathrm{NEPS}(\Gamma_i, T; \mathcal{B}), \{1\} \times Q_8, \Psi)$, for $i = 1, 2$.

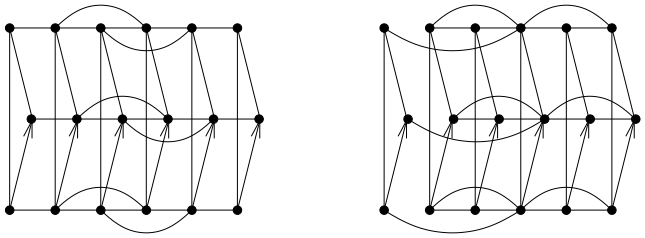


Figure 10: The gain graphs $(\mathrm{NEPS}(\Gamma_1, T; \mathcal{B}), \{1\} \times Q_8, \Psi)$ and $(\mathrm{NEPS}(\Gamma_2, T; \mathcal{B}), \{1\} \times Q_8, \Psi)$.

Observe that the underlying graphs are not isomorphic, since it is not difficult to see that the degree sequence for $\mathrm{NEPS}(\Gamma_1, T; \mathcal{B})$ is $\{3^6, 5^{12}\}$, while the degree sequence for $\mathrm{NEPS}(\Gamma_2, T; \mathcal{B})$ is $\{3^3, 4^{12}, 7^3\}$. Let A_1 (resp. A_2) be the adjacency matrix of the graph Γ_1 (resp. Γ_2), and let

$$A_{(T,\psi_T)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & k \\ 1 & -k & 0 \end{pmatrix}$$

be the adjacency matrix of (T, Q_8, ψ_T). Then, with a direct computa-

tion, it is possible to see that:

- the represented adjacency matrices of the two NEPS graphs with respect to π_0 and π_k are

  $$A_1 \otimes I_3 + I_6 \otimes \pi_0(A_{(T,\psi_T)}) \quad \text{and} \quad A_2 \otimes I_3 + I_6 \otimes \pi_0(A_{(T,\psi_T)})$$

  and

  $$A_1 \otimes I_3 + I_6 \otimes \pi_k(A_{(T,\psi_T)}) \quad \text{and} \quad A_2 \otimes I_3 + I_6 \otimes \pi_k(A_{(T,\psi_T)})$$

  and they share the same characteristic polynomial

  $$x^2(x-3)(x^3-7x^2+11x-1)(x-1)^2(x^3+2x^2-4x-4)^2(x+2)^4;$$

- the represented adjacency matrices of the two NEPS graphs with respect to π_i and π_j are

  $$A_1 \otimes I_3 + I_6 \otimes \pi_i(A_{(T,\psi_T)}) \quad \text{and} \quad A_2 \otimes I_3 + I_6 \otimes \pi_i(A_{(T,\psi_T)})$$

  and

  $$A_1 \otimes I_3 + I_6 \otimes \pi_j(A_{(T,\psi_T)}) \quad \text{and} \quad A_2 \otimes I_3 + I_6 \otimes \pi_j(A_{(T,\psi_T)})$$

  and they share the same characteristic polynomial

  $$x^4(x+1)(x^3+5x^2+3x-5)(x-2)^2(x+3)^2(x^3-4x^2+4)^2;$$

- the represented adjacency matrices of the two NEPS graphs with respect to π_2 are

  $$A_1 \otimes I_6 + I_6 \otimes \pi_2(A_{(T,\psi_T)}) \quad \text{and} \quad A_2 \otimes I_6 + I_6 \otimes \pi_2(A_{(T,\psi_T)})$$

  and they share the same characteristic polynomial

  $$(x-1)^2(x^2-2x-2)^2(x^3-x^2-5x+1)^2(x+1)^4 \cdot$$
  $$\cdot(x^6-2x^5-18x^4+24x^3+44x^2-40x-8)^2(x^2+2x-2)^4.$$

It follows that the gain graphs

$$(\mathrm{NEPS}(\Gamma_1, T; \mathcal{B}), \{1\} \times Q_8, \Psi) \quad \text{and} \quad (\mathrm{NEPS}(\Gamma_2, T; \mathcal{B}), \{1\} \times Q_8, \Psi)$$

are an example of a pair of gain graphs over Q_8 that are Q_8-cospectral but not switching isomorphic. This is precisely what we expected, thanks to Corollary 3.11, since $(\Gamma_1, \{1\}, \psi_1)$ and $(\Gamma_2, \{1\}, \psi_1)$ are $\{1\}$-cospectral, and clearly, (T, Q_8, ψ_T) is Q_8-cospectral with itself.

Remark 3.13 The previous Example 3.12 shows that the NEPS construction is a powerful tool in order to construct pairs of G-cospectral switching nonisomorphic gain graphs. Indeed, starting from any pair of cospectral nonisomorphic graphs (which can be considered as the underlying graphs of two gain graphs over the trivial group), then the NEPS composition with a gain graph over a group G with respect to any basis $\mathcal{B}$ produces a pair of G-gain graphs which are cospectral with respect to every representation of the group G, so that the G-cospectrality is ensured. Actually, this is a result that is somewhat new, or at least not explicitly stated, as far as we know, even for signed graphs.

Even though we have stated that the NEPS of signed graphs with our definition is not a signed graph, in the particular case where only one of the factor graphs is actually signed (while the others are balanced, i.e., indistinguishable from classical graphs or gain graphs on the trivial group), the result of our construction can be considered as a signed graph.

Corollary 3.14 *Let $(\Gamma_1, \mathbb{T}_2, \sigma)$ be a signed graph, Γ_i and Γ_i' be two cospectral graphs, for each $i = 2, \dots, n$. Then the signed graphs*

$$(\mathrm{NEPS}(\Gamma_1, \Gamma_2 \dots, \Gamma_n; \mathcal{B}), \mathbb{T}_2, \Psi) \quad \text{and} \quad (\mathrm{NEPS}(\Gamma_1, \Gamma_2', \dots, \Gamma_n'; \mathcal{B}), \mathbb{T}_2, \Psi')$$

are cospectral with cospectral underlying graphs.

Proof — It is a combination of Corollary 3.11, Theorem 2.3, and the fact that $\mathbb{T}_2$ admits only two irreducible representations, one associated to the spectrum of the underlying graph (the trivial representation), and one to the classical spectrum of a signed graph (the identical representation). □

Example 3.15 We can consider again the gain graphs (Γ_1, ψ_1) and (Γ_2, ψ_2) from Example 3.12, and a signed graph (T, σ) constructed as follows: we observe Fig. 9, and we interpret the arrow as a negative edge, thus obtaining an unbalanced triangle. By consistently maintaining this way of interpreting the arrows, the Fig. 10 can also represent the signed graphs $(\Gamma_1, \psi_1)\square(T, \sigma)$ and $(\Gamma_2, \psi_2)\square(T, \sigma)$,

which, according to Corollary 3.14, are cospectral with cospectral underlying graphs as well. It is worth noting that they are not balanced: in the case of balance instead, cospectrality of the underlying graphs would be automatically implied by signed cospectrality.

4 NEPS with a gain function induced by a homomorphism

The aim of this section is to introduce a new gain function on a NEPS of gain graphs. More precisely, given a group homomorphism

$$\phi\colon G_1 \times \ldots \times G_n \to H,$$

we want to define an H-gain function Ψ_ϕ, induced by ϕ, on the NEPS of these graphs. In Section 4.1 we present the most general construction of a gain function on a NEPS in this paper. In the next subsections we investigate some special cases, where it will be possible to investigate and understand some spectral properties:

- the case $G_1 = \ldots = G_n = H$, where H is an Abelian group, and ϕ is the multiplication of the group (Section 4.2);
- the case $G_i = S_{m_i}$ and $H = S_{m_1 \ldots m_n}$, where S_{m_i} denotes the symmetric group on m_i elements (Section 4.3);
- the case when G_i is simple or $U(\mathbb{H})$ (Section 4.4).

4.1 The gain function Ψ_ϕ

Let (Γ_i, G_i, ψ_i) be a G_i-gain graph, for each $i = 1, \ldots, n$, and let

$$\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0,\ldots,0)\}.$$

Suppose we are given a group H and a group homomorphism

$$\phi\colon G_1 \times \ldots \times G_n \to H.$$

We can define an H-gain function on $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$ in the following way.

Definition 4.1 The H-*gain function induced by* ϕ on the graph

$$\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$$

is the gain function Ψ_ϕ defined by

$$\Psi_\phi(e) := \phi(\Psi(e))$$

on every oriented edge $e \in \overrightarrow{E}(\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B}))$, where Ψ is the gain function of Definition 3.1.

Remark 4.2 Notice that, since ϕ is a homomorphism, Ψ_ϕ is in fact a gain function. More precisely, if e_- is the edge with the opposite orientation with respect to e, one has:

$$\Psi_\phi(e_-) = \phi(\Psi(e_-)) = \phi(\Psi(e)^{-1}) = \phi(\Psi(e))^{-1} = \Psi_\phi(e)^{-1}.$$

Moreover, given a second group homomorphism $\tau\colon H \to K$, we have that $\tau \circ \Psi_\phi = \Psi_{\tau\circ\psi}$. In particular, $\Psi = \Psi_{id}$.

Remark 4.3 Observe that a group homomorphism

$$\phi\colon G_1 \times \ldots \times G_n \to H$$

can be also identified as a family of homomorphisms $\phi_i\colon G_i \to H$ such that, for each $i, j = 1, \ldots, n$, one has

$$\phi_i(x_i) \cdot \phi_j(x_j) = \phi_j(x_j) \cdot \phi_i(x_i) \tag{4.1}$$

for all $x_i \in G_i$ and $x_j \in G_j$. More precisely, given a homomorphism ϕ, one can define the homomorphism ϕ_i by setting

$$\phi_i(x_i) := \phi(1_{G_1}, \ldots, 1_{G_{i-1}}, x_i, 1_{G_{i+1}}, \ldots, 1_{G_n}), \qquad \text{for each } x_i \in G_i.$$

The inverse construction holds: given a family $\{\phi_i\}_{i=1,\ldots,n}$ satisfying Eq. (4.1), it is possible to define a homomorphism

$$\phi\colon G_1 \times \ldots \times G_n \to H$$

as

$$\phi(x_1, \ldots, x_n) := \phi_1(x_1) \ldots \phi_n(x_n)$$

whose i-th restriction coincides with ϕ_i.

Theorem 4.4 *Let* (Γ_i, G_i, ψ_i) *and* $(\Gamma_i', G_i, \psi_i')$ *be two switching isomorphic* G_i*-gain graphs, for each* $i = 1, \ldots, n$*, and let*

$$\phi\colon G_1 \times \ldots \times G_n \to H$$

be a group homomorphism. Then the corresponding gain graphs

$$(\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B}), H, \Psi_\phi) \quad \text{and} \quad (\mathrm{NEPS}(\Gamma_1', \ldots, \Gamma_n'; \mathcal{B}), H, \Psi_\phi')$$

are switching isomorphic. Moreover if (Γ_i, G_i, ψ_i') *is switching equivalent to* (Γ_i, G_i, ψ_i) *for each* i*, then the corresponding NEPS are switching equivalent.*

Proof — In Theorem 3.5 we proved, under the same hypothesis, that the NEPS with gain functions Ψ and Ψ' on $G_1 \times \ldots \times G_n$ are switching isomorphic. More precisely if (Γ_i, G_i, ψ_i) and $(\Gamma_i', G_i, \psi_i')$ are switching isomorphic, then there exist a graph isomorphism

$$\theta_i\colon V_{\Gamma_i} \to V_{\Gamma_i'}$$

and a map

$$f_i\colon V_{\Gamma_i} \to G_i$$

such that

$$\psi_i'(\theta_i(x_i), \theta_i(y_i)) = f_i(x_i)^{-1}\psi_i(x_i, y_i)f_i(y_i),$$

for each pair of adjacent vertices x_i, y_i in Γ_i. In the proof of Theorem 3.5 we defined $\Theta := (\theta_1, \ldots, \theta_n)$ and $F := (f_1, \ldots, f_n)$ and proved that

$$\Psi'(\Theta(\overline{x}), \Theta(\overline{y})) = F(\overline{x})^{-1}\Psi(\overline{x}, \overline{y})F(\overline{y}),$$

for each pair of adjacent vertices $\overline{x}, \overline{y}$ in $(\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B}), H, \Psi_\phi)$. Now, this implies that

$$\Psi_\phi'(\Theta(\overline{x}), \Theta(\overline{y})) = \phi(\Psi'(\Theta(\overline{x}), \Theta(\overline{y}))) = \phi(F(\overline{x}))^{-1}\Psi_\phi(\overline{x}, \overline{y})\phi(F(\overline{y}))$$

and then also

$$(\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B}), H, \Psi_\phi) \quad \text{and} \quad (\mathrm{NEPS}(\Gamma_1', \ldots, \Gamma_n'; \mathcal{B}), H, \Psi_\phi')$$

are switching isomorphic. In particular, if (Γ_i, G_i, ψ_i) and (Γ_i, G_i, ψ_i') are switching equivalent, the homomorphism Θ is exactly the iden-

tity map on $V(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B}))$ and then the two NEPS are switching equivalent. □

Corollary 4.5 *Let* (Γ_i, G_i, ψ_i) *be a balanced* G_i*-gain graph, for each* $i = 1, \dots, n$. *Then* $(\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B}), H, \Psi_\phi)$ *is balanced.*

Proof — It is sufficient to use Theorem 4.4 and the characterization of balance in terms of switching equivalence with the underlying graph. □

In order to describe the adjacency matrix with respect to the H-gain function Ψ_ϕ, we need to introduce another type of Kronecker product.

Definition 4.6 Let H be a group. Let

$$A \in M_{m\times n}(\mathbb{C}H) \quad \text{and} \quad B \in M_{p\times q}(\mathbb{C}H).$$

The *Kronecker product* $A \otimes B$ is the matrix in $M_{mp\times nq}(\mathbb{C}H)$ defined as the block matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

where $a_{ij}B$ is the $p \times q$ matrix whose entry (r, s) is $a_{ij}b_{rs}$.

Observe that, considering $A \otimes B \in M_{mp\times nq}(\mathbb{C}H)$ as a map from H to $M_{mp\times nq}(\mathbb{C})$, one has

$$\begin{aligned} (A\otimes B)(h) &= \sum_{\substack{x,y\in H:\\ xy=h}} A(x) \otimes B(y) \\ &= \sum_{g\in H} A(hg^{-1}) \otimes B(g), \qquad \forall h \in H. \end{aligned} \tag{4.2}$$

Remark 4.7 Note that the Kronecker product of Definition 4.6 differs from the mixed Kronecker product of Definition 3.6. In fact, $A\otimes B$ is a matrix in $M_{mp\times nq}(\mathbb{C}H)$, while $A \otimes_M B \in M_{mp\times nq}(\mathbb{C}(H \times H))$. However, these two elements are somehow related, as Eq. (4.3) shows.

Clearly, a group homomorphism $\phi\colon G \to H$ can be extended by linearity to a homomorphism $\phi\colon \mathbb{C}G \to \mathbb{C}H$, and also to a map

$$\phi\colon M_{m\times n}(\mathbb{C}G) \to M_{m\times n}(\mathbb{C}H).$$

By interpreting an element A of $M_{m\times n}(\mathbb{C}G)$ as a map from G to $M_{m\times n}(\mathbb{C})$, and so $\phi(A)$ as a map from H to $M_{m\times n}(\mathbb{C})$, one has

$$\phi(A)(h) = \sum_{\substack{g\in G:\\ \phi(g)=h}} A(g).$$

As we said, the homomorphism ϕ can be factorized by $\phi_1, \ldots, \phi_n$. Let $B_i \in M_{m_i\times n_i}(\mathbb{C}G_i)$ for $i = 1, \ldots, n$. Then we have

$$\phi(B_1 \otimes_M \ldots \otimes_M B_n) = \phi_1(B_1) \otimes \ldots \otimes \phi_n(B_n). \tag{4.3}$$

Using this equation, we are going to describe the adjacency matrix of this new type of gain function on $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$. Recall that fixed orderings on each V_i induce a lexicographic order on the set of vertices of $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$.

Theorem 4.8 *Let $\phi\colon G_1 \times \ldots \times G_n \to H$ be a group homomorphism. The adjacency matrix A_{Ψ_ϕ} of the graph $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$ with respect to the gain function Ψ_ϕ is the $\mathbb{C}H$ valued matrix given by*

$$A_{\Psi_\phi} = \sum_{\beta\in\mathcal{B}} \phi_1(A_1)^{\beta_1} \otimes \ldots \otimes \phi_n(A_n)^{\beta_n},$$

where A_i is the adjacency matrix of (Γ_i, G_i, ψ_i), and the tensor product $\otimes$ is as in Definition 4.6.

Proof — By combining Theorem 3.7 and Eq. (4.3), together with the linearity of the extension of ϕ to group algebra valued matrices, one has:

$$A_{\Psi_\phi} = \phi(A_\Psi) = \phi\Big(\sum_{\beta=(\beta_1,\ldots,\beta_n)\in\mathcal{B}} A_1^{\beta_1} \otimes_M \ldots \otimes_M A_n^{\beta_n} \Big)$$

$$= \sum_{\beta=(\beta_1,\ldots,\beta_n)\in\mathcal{B}} \phi_1(A_1^{\beta_1}) \otimes \ldots \otimes \phi_n(A_n^{\beta_n}).$$

The thesis follows by noting that $\phi_i(A_i^{\beta_i}) = \phi_i(A_i)^{\beta_i}$ for every i. □

As in Section 3, we will use this description of the adjacency matrix in order to deduce some information about spectra. At first, notice that if

$$\phi = (\phi_1, \dots, \phi_n) \colon G_1 \times \dots \times G_n \to H$$

is a homomorphism and $\pi \colon H \to \mathrm{GL}_{\deg \pi}(\mathbb{C})$ is a representation of H, the composition $\pi \circ \phi_i$ is a representation of G_i and the $(\pi \circ \phi_i)$-spectrum of (Γ_i, G_i, ψ_i) is the spectrum of the complex matrix $\pi(\phi_i(A_i))$.

Theorem 4.9 *Let (Γ_i, G_i, ψ_i) be a G_i-gain graph on n_i vertices, for $i = 1, \dots, n$, and let $\phi \colon G_1 \times \dots \times G_n \to H$ be a group homomorphism. Let π be a representation of H of degree 1. Then the π-spectrum associated to the gain function Ψ_ϕ consists of the complex numbers*

$$\Lambda_{j_1,\dots,j_n} := \sum_{\beta \in \mathcal{B}} \lambda_{1,j_1}^{\beta_1} \dots \lambda_{n,j_n}^{\beta_n}$$

where $\{\lambda_{i,1}, \dots, \lambda_{i,n_i}\}$ is the $(\pi \circ \phi_i)$-spectrum of (Γ_i, G_i, ψ_i).

Proof — By gluing together Theorem 4.8, Eq. (2.5) and Eq. (4.2), we obtain

$$\begin{aligned}
&\pi(A_{\Psi_\phi}) \\
&= \sum_{\beta \in \mathcal{B}} \pi\left(\phi_1(A_1)^{\beta_1} \otimes \dots \otimes \phi_n(A_n)^{\beta_n}\right) && \text{Th. 4.8} \\
&= \sum_{\beta \in \mathcal{B}} \sum_{h \in H} \left(\phi_1(A_1)^{\beta_1} \otimes \dots \otimes \phi_n(A_n)^{\beta_n}\right)(h) \otimes \pi(h) && \text{Eq. (2.5)} \\
&= \sum_{\beta \in \mathcal{B}} \sum_{h \in H} \sum_{\substack{h_1,\dots,h_n \in H: \\ h_1 \dots h_n = h}} \left(\phi_1(A_1)^{\beta_1}\right)(h_1) \otimes \dots \otimes \left(\phi_n(A_n)^{\beta_n}\right)(h_n) \otimes \pi(h) && \text{Eq. (4.2)} \\
&= \sum_{\beta \in \mathcal{B}} \sum_{h_1,\dots,h_n \in H} \left(\phi_1(A_1)^{\beta_1}\right)(h_1) \otimes \dots \otimes \left(\phi_n(A_n)^{\beta_n}\right)(h_n) \otimes \pi(h_1 \dots h_n) \\
&= \sum_{\beta \in \mathcal{B}} \left(\sum_{h_1 \in H} \left(\phi_1(A_1)^{\beta_1}\right)(h_1) \otimes \pi(h_1)\right) \otimes \dots \otimes \left(\sum_{h_n \in H} \left(\phi_n(A_n)^{\beta_n}\right)(h_n) \otimes \pi(h_n)\right) && \deg \pi = 1 \\
&= \sum_{\beta \in \mathcal{B}} \pi(\phi_1(A_1)^{\beta_1}) \otimes \dots \otimes \pi(\phi_n(A_n)^{\beta_n}) && \text{Eq. (2.5)} \\
&= \sum_{\beta \in \mathcal{B}} \pi(\phi_1(A_1))^{\beta_1} \otimes \dots \otimes \pi(\phi_n(A_n))^{\beta_n}.
\end{aligned}$$

The description of the eigenvalues as in the statement follows. □

Remark 4.10 The hypothesis that π is a 1-dimensional representation is crucial in the proof of Theorem 4.9; under this assumption, $\pi(h_1) \dots \pi(h_n)$ equals the Kronecker product $\pi(h_1) \otimes \dots \otimes \pi(h_n)$, and in general this assumption allows to prove that

$$\pi(A \otimes B) = \pi(A) \otimes \pi(B), \tag{4.4}$$

where the symbol $\otimes$ on the left-hand side is that of Definition 4.6 and on the right-hand side is the classical Kronecker product of matrices. When $\deg\pi \neq 1$ Eq. (4.4) does not hold anymore. To see this, suppose for instance that A is an $n \times n$-matrix and B is an $m \times m$-matrix. Then, if $\deg\pi = k$, the matrix $\pi(A \otimes B)$ is an mnk-matrix, whereas $\pi(A) \otimes \pi(B)$ is an mnk^2-matrix.

Remark 4.11 In Theorem 4.9, we essentially proved that the π-spectrum associated with Ψ_ϕ equals the Π_ϕ-spectrum of Ψ, where Ψ is the general gain on $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$ defined in Definition 3.1, and $\Pi_\phi = \bigotimes_i^n (\pi \circ \phi_i)$. Indeed, $\pi(A_{\Psi_\phi}) = \Pi_\phi(A_\Psi)$.

Corollary 4.12 *Let* (Γ_i, G_i, ψ_i) *and* $(\Gamma_i', G_i, \psi_i')$ *be* G_i*-cospectral gain graphs, for* $i = 1, \ldots, n$*, let* H *be a finite Abelian group and*

$$\phi\colon G_1 \times \ldots \times G_n \to H$$

be a group homomorphism. Then the gain graphs

$$(\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B}), H, \Psi_\phi) \quad \text{and} \quad (\mathrm{NEPS}(\Gamma_1', \ldots, \Gamma_n'; \mathcal{B}), H, \Psi_\phi')$$

are H*-cospectral.*

Proof — Le $\pi_1, \ldots, \pi_k$ be a complete system of irreducible representations of H. Since H is Abelian, $\deg\pi_j = 1$, for each $j = 1, \ldots, k$. The hypothesis that (Γ_i, G_i, ψ_i) and $(\Gamma_i', G_i, \psi_i')$ are G_i-cospectral implies that they are cospectral with respect to any representation of G_i, in particular they are $\pi_j \circ \phi_i$-cospectral. By virtue of Theorem 4.9 the two NEPS gain graphs are π_j-cospectral, for each $j = 1, \ldots, k$. Now the thesis follows from Theorem 2.3. □

4.2 Gain functions on a NEPS of Abelian gain graphs

In this section we consider a special case of gain function Ψ_ϕ on a NEPS, namely when the gain graphs (Γ_i, G, ψ_i) are all over the same Abelian group G. In this case we have a canonical homomorphism:

$$\begin{aligned} \phi\colon G \times \ldots \times G &\longrightarrow G \\ (g_1, \ldots, g_n) &\longmapsto g_1 \ldots g_n. \end{aligned}$$

Given the importance and uniqueness of the case, we have assigned the symbol $\Psi_{Ab} = \Psi_\phi$ to such a gain function. More explicitly, if

$e = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ is an edge in $\mathrm{NEPS}(\Gamma_1, \dots, \Gamma_n; \mathcal{B})$, then

$$\Psi_{Ab}(e) = z_1 \dots z_n \quad \text{with:} \quad z_i = \begin{cases} \psi_i(x_i, y_i) & \text{if } \beta_i(e) = 1 \\ 1_G & \text{if } \beta_i(e) = 0. \end{cases} \tag{4.5}$$

Notice that if G is not Abelian in general $(g_1 \dots g_n)^{-1} \neq g_1^{-1} \dots g_n^{-1}$ (multiplication is not a homomorphism): therefore Eq. (4.5) does not define a gain function.

Remark 4.13 Actually any gain function Ψ_ϕ associated to the group homomorphism

$$\phi = (\phi_1, \dots, \phi_n) \colon G_1 \times \dots \times G_n \to H$$

can be interpreted as a kind of H-gain function Ψ_{Ab}, but associated to the gain graphs $(\Gamma_i, H, \phi_i \circ \psi_i)$. Observe that in that case Eq. (4.5) defines indeed a gain function, even if $\phi_i(G_i)$ is not Abelian, thank to commutativity of ϕ_i and ϕ_j.

When the Abelian group G is the group $\mathbb{T}$ of complex units or the group of order two, the gain map Ψ_{Ab} recovers the definition of NEPS of complex unit graphs and NEPS of signed graphs [5, 18] respectively.

Remark 4.14 Since Ψ_{Ab} is a specific instance of Ψ_ϕ, we immediately deduce from Theorem 4.4 and Corollary 4.5 that:

- if (Γ_i, G, ψ_i) is balanced for each i, then the gain function Ψ_{Ab} is balanced;
- if (Γ_i, G, ψ_i') is switching equivalent to (Γ_i, G, ψ_i), for each i, then also the corresponding NEPS are switching equivalent;
- if (Γ_i', G, ψ_i') is switching isomorphic to (Γ_i, G, ψ_i), for each i, then also the corresponding NEPS are switching isomorphic.

In particular, we recover the analogous results for signed and complex unit gain graphs in [5, 18].

Remark 4.15 By Theorem 4.8, the adjacency matrix $A_{\Psi_{Ab}}$ associated with the gain function Ψ_{Ab} is the $\mathbb{C}G$ valued matrix given by

$$A_{\Psi_{Ab}} = \sum_{\sigma \in \mathcal{B}} A_1^{\sigma_1} \otimes \dots \otimes A_n^{\sigma_n},$$

where A_i is the adjacency matrix of (Γ_i, G, ψ_i). Moreover, if one fixes now an irreducible representation π of G, as a consequence of Theorem 4.9, the π-spectrum of Ψ_{Ab} is given by the complex numbers

$$\Lambda_{j_1,\ldots,j_n} := \sum_{\beta \in \mathcal{B}} \lambda_{1,j_1}^{\beta_1} \ldots \lambda_{n,j_n}^{\beta_n}$$

where $\{\lambda_{i,1}, \ldots, \lambda_{i,n_i}\}$ is the π-spectrum of (Γ_i, G, ψ_i). Observe also that the π-spectrum of Ψ_{Ab} coincides with the $\pi^{\otimes^n}$-spectrum of the $(G \times \ldots \times G)$-gain function Ψ defined in Section 3.

This implies that we also recover the analogous spectral results for signed and complex unit gain graphs by choosing the representation π_{id} of $\mathbb{T}_2$ and $\mathbb{T}$ respectively.

Example 4.16 Let us consider again the $\mathbb{T}_2$-gain graphs

$$(K_3, \mathbb{T}_2, \psi_3) \quad \text{and} \quad (K_2, \mathbb{T}_2, \psi_2)$$

of Example 3.8 depicted in Fig. 6. We have shown that the gain function Ψ of Definition 3.1 for the graph $\mathrm{NEPS}(K_3, K_2; \mathcal{B})$ depicted in Fig. 7 does not take value in $\mathbb{T}_2$, and thus it is not consistent with [5, 18]. But now we have instead that Ψ_{Ab} is the gain function on $\mathbb{T}_2$ as specified by the definition in [5, 18], and whose adjacency matrix is

$$\begin{aligned} A_{\Psi_{Ab}} &= A_{(K_3,\psi_3)} \otimes I_2 + A_{(K_3,\psi_3)} \otimes A_{(K_2,\psi_2)} \\ &= \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ \hline 1 & -1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ \hline -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{array}\right). \end{aligned}$$

4.3 Gain functions on a NEPS of S_n-gain graphs

Another special instance of the gain function Ψ_ϕ occurs when we deal with symmetric gain groups, with the homomorphism ϕ given by the inclusion of their direct product into a larger symmetric group. Let S_n be the group of permutations of n elements. We consider the

following map:

$$\begin{aligned} \phi\colon S_n \times S_m &\longrightarrow S_{nm} \\ (\sigma,\tau) &\longmapsto \sigma\otimes\tau \end{aligned}$$

where

$$(\sigma\otimes\tau)(qm+r) = \sigma(q)m + \tau(r),$$

for all $q \in \{0,1,\dots,n-1\}, r \in \{0,1,\dots,m-1\}$. If we arrange the numbers from 0 to $nm-1$ in an $n\times m$ matrix in this way

$$\begin{pmatrix} 0 & 1 & \cdots & m-1 \\ m & m+1 & \cdots & 2m-1 \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)m & (n-1)m+1 & \cdots & nm-1 \end{pmatrix},$$

then the permutation $\sigma\otimes\tau$ is the one obtained by permuting the n rows with σ and the m columns with τ. It is well known that ϕ is an injective homomorphism.

Suppose now that $(\Gamma_i, S_{m_i}, \psi_i)$ is an S_{m_i}-gain graph, for each $i = 1,\dots,n$, and let $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0,\dots,0)\}$. Put $N = m_1 \dots m_n$. Let ϕ be the above-mentioned homomorphism

$$\begin{aligned} \phi\colon\ S_{m_1}\times\ldots\times S_{m_n} &\longrightarrow S_N \\ (\sigma_1,\dots,\sigma_n) &\longmapsto \sigma_1\otimes\ldots\otimes\sigma_n. \end{aligned}$$

We use the symbol Ψ_S to denote the S_N-gain function Ψ_ϕ of Definition 4.1 associated to this homomorphism ϕ. Explicitly, if

$$e = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

is an edge in $\mathrm{NEPS}(\Gamma_1,\dots,\Gamma_n;\mathcal{B})$, then

$$\Psi_S(e) = \sigma_1\otimes\ldots\otimes\sigma_n \quad \text{with:} \quad \sigma_i = \begin{cases} \psi_i(x_i,y_i) & \text{if } \beta_i(e)=1 \\ \mathrm{id} & \text{if } \beta_i(e)=0. \end{cases}$$

This implies, in particular, that the properties of switching stability are satisfied:

- If $(\Gamma_i, S_{m_i}, \psi_i)$ and $(\Gamma_i, S_{m_i}, \psi_i')$ are switching equivalent, then

the gain functions Ψ_S and Ψ'_S are switching equivalent;

- If $(\Gamma_i, S_{m_i}, \psi_i)$ and $(\Gamma'_i, S_{m_i}, \psi'_i)$ are switching isomorphic, then the gain functions Ψ_S and Ψ'_S are switching isomorphic.

This case involving symmetric groups is notable not only for the homomorphism we have discussed, but also because the group S_n has a special representation, that we denote by $\mathcal{P}_n$, which assigns to each element of S_n its corresponding permutation matrix of size n. Indeed, even though this representation is clearly not one-dimensional, we will see that similar assertions of Theorem 4.9 hold true. However, before that, we need to recall some properties that can be found, for instance, in [33].

Lemma 4.17 *The following properties hold:*

1. $\sigma_1 \otimes (\sigma_2 \otimes \sigma_3) = (\sigma_1 \otimes \sigma_2) \otimes \sigma_3$, *for all* $\sigma_i \in S_{m_i}$;
2. $(\sigma_1 \otimes \sigma_2)(\tau_1 \otimes \tau_2) = (\sigma_1\tau_1) \otimes (\sigma_2\tau_2)$, *for all* $\sigma_i, \tau_i \in S_{m_i}$;
3. $\mathcal{P}_N(\sigma_1 \otimes \ldots \otimes \sigma_n) = \mathcal{P}_{m_1}(\sigma_1) \otimes \ldots \otimes \mathcal{P}_{m_n}(\sigma_n)$.

We are now in a position to state the following corollary.

Corollary 4.18 *The* $\mathcal{P}_N$*-spectrum of the NEPS of the* S_{m_i}*-gain graphs* $(\Gamma_i, S_{m_i}, \psi_i)$ *with gain function* Ψ_S *consists of the values*

$$\Lambda_{j_1,\ldots,j_n} = \sum_{\beta \in \mathcal{B}} \lambda_{1,j_1}^{\beta_1} \ldots \lambda_{n,j_n}^{\beta_n}$$

where $\{\lambda_{i,1}, \ldots, \lambda_{i,n_i m_i}\}$ *is the* $\mathcal{P}_{m_i}$*-spectrum of* $(\Gamma_i, S_{m_i}, \psi_i)$.

Proof — The property (3) of Lemma 4.17 ensures that, if we restrict the representation $\mathcal{P}_N$ to the range of ϕ, we obtain the tensor product of the representations $\mathcal{P}_{m_i}$:

$$\mathcal{P}_N|_{\phi(S_{m_1} \times \ldots \times S_{m_n})} = \bigotimes_{i=1}^{n} \mathcal{P}_{m_i}.$$

The entries of the adjacency matrix A_{Ψ_S} actually belong, by definition, to $\phi(S_{m_1} \times \ldots \times S_{m_n})$, or equivalently, the support of the map A_{Ψ_S} is a subset of $\phi(S_{m_1} \times \ldots \times S_{m_n})$. By Lemma 3.9, this yields

that the complex matrices

$$\mathcal{P}_N(A_{\Psi_S}) \quad \text{and} \quad \bigotimes_{i=1}^{n} \mathcal{P}_{m_i}(A_\Psi)$$

are similar, where A_Ψ is the adjacency matrix associated to the $(S_{m_1} \times \ldots \times S_{m_n})$-gain function Ψ on $\mathrm{NEPS}(\Gamma_1, \ldots, \Gamma_n; \mathcal{B})$ constructed according to Definition 3.1. The thesis follows from Theorem 3.10. □

4.4 The case of simple groups and $U(\mathbb{H})$

The aim of this section is to investigate the spectral properties of the gain function Ψ_ϕ when G_i is a simple group for each i, and in some other special cases where we assume that some suitable algebraic conditions are satisfied.

In general, the computation of the spectrum of Ψ_ϕ is far from being trivial, and it requires a study *ad hoc* for each case. However, there are some algebraic tools that help in the investigation of the possible groups H such that the NEPS admits an H-gain function Ψ_ϕ. As an example, at the end of this section, we give a complete classification of the NEPS of two $U(\mathbb{H})$-gain graphs.

From now on, we assume that $\phi\colon G_1 \times \ldots \times G_n \to H$ is a surjective group homomorphism, what is not restrictive from a group theoretical point of view, nor from a spectral point of view.

Proposition 4.19 *For each $i = 1, \ldots, n$, let (Γ_i, G_i, ψ_i) be a G_i-gain graph. Fix $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0, \ldots, 0)\}$ and a surjective group homomorphism*

$$\phi\colon G_1 \times \ldots \times G_n \to H,$$

with $\phi = (\phi_1, \ldots, \phi_n)$. Then:

1. *if, for each $i, j = 1, \ldots, n$, we have $\phi_i(G_i) \cap \phi_j(G_j) = \{1_H\}$, then*

$$H \simeq \phi_1(G_1) \times \ldots \times \phi_n(G_n)$$

   and Ψ_ϕ is the general gain function Ψ defined on the NEPS of the gain graphs $(\Gamma_i, \phi_i(G_i), \phi_i \circ \psi_i)$;

2. *if there exist $i \neq j$ such that $\phi_i(G_i) = \phi_j(G_j) = H$, then H is Abelian and so the spectrum of Ψ_ϕ is described by* Theorem 4.9.

Proof — The first claim can be proved for $n = 2$ and then it immediately follows for each n. Since ϕ is surjective, then $H \simeq \frac{G_1 \times G_2}{\ker \phi}$.

Notice that

$$\ker\phi = \{(g_1, g_2) \text{ s.t. } \phi_1(g_1)\phi_2(g_2) = 1_H\}.$$

Therefore, if $(g_1, g_2) \in \ker\phi$, it must be $\phi_1(g_1) = \phi_2(g_2)^{-1}$ and the hypothesis implies

$$\phi_1(g_1) = \phi_2(g_2) = 1_H.$$

Thus $\ker\phi = \ker\phi_1 \times \ker\phi_2$ and so

$$H \simeq \frac{G_1 \times G_2}{\ker\phi_1 \times \ker\phi_2} \simeq \phi_1(G_1) \times \phi_2(G_2).$$

Now we focus on the second claim. Let $h_1, h_2 \in H$. Then by hypothesis there exist $g_i \in G_i$ and $g_j \in G_j$ such that

$$h_1 \cdot h_2 = \phi_i(g_i) \cdot \phi_j(g_j) = \phi_j(g_j) \cdot \phi_i(g_i) = h_2 \cdot h_1$$

and so H is Abelian. The statement follows. □

Lemma 4.20 *Let $\phi\colon G_1 \times \ldots \times G_n \to H$ be a surjective group homomorphism. Then, for each $i, j = 1, \ldots, n$, one has that $\phi_i(G_i) \cap \phi_j(G_j)$ is an Abelian, normal subgroup of $\phi_i(G_i)$, of $\phi_j(G_j)$ and of H.*

Proof — The fact that $\phi_i(G_i) \cap \phi_j(G_j)$ is Abelian follows from Remark 4.3. Now by surjectivity of ϕ it is sufficient to notice that, for each h in $\phi_i(G_i) \cap \phi_j(G_j)$, and for each $\phi_k(g_k)$, we have:

$$\begin{aligned}\phi_k(g_k) \cdot h \cdot \phi_k(g_k)^{-1} &= \phi_k(g_k) \cdot \phi_s(g_s) \cdot \phi_k(g_k)^{-1} \\ &= \phi_k(g_k) \cdot \phi_k(g_k)^{-1} \cdot \phi_s(g_s) = h,\end{aligned}$$

with $s \in \{i, j\}$ such that $s \neq k$. □

We now focus on non-Abelian simple groups, since we already know how to manage Abelian groups (see Section 4.2).

Corollary 4.21 *For each $i = 1, \ldots, n$, let (Γ_i, G_i, ψ_i) be a gain graph over a simple non-Abelian group G_i. For any $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0, \ldots, 0)\}$ and any homomorphism $\phi\colon G_1 \times \ldots \times G_n \to H$, with $\phi = (\phi_1, \ldots \phi_n)$, the gain function Ψ_ϕ is described by* (1) *of* Proposition 4.19.

Proof — We know from Lemma 4.20 that $\phi_i(G_i) \cap \phi_j(G_j)$ is a normal subgroup of both $\phi_j(G_j)$ and $\phi_i(G_i)$. Moreover, by simplicity

of G_i, we have that $\phi_i(G_i)$ is either trivial or isomorphic to G_i. Therefore, the only chance for $\phi_i(G_i) \cap \phi_j(G_j)$ of being non-trivial is that

$$\phi_i(G_i) \cap \phi_j(G_j) = \phi_i(G_i) = \phi_j(G_j),$$

but in this case

$$G_i \simeq \phi_i(G_i) = \phi_j(G_j) \simeq G_j$$

should be Abelian by Lemma 4.20, what contradicts our hypothesis. Then it must be $\phi_i(G_i) \cap \phi_j(G_j) = \{1_H\}$ and the claim follows. □

Next, we prove a lemma which helps to discover which are the candidate groups H such that the NEPS of two gain graphs (Γ_1, G_1, ψ_1) and (Γ_2, G_2, ψ_2) admits an H-gain function Ψ_ϕ. In Theorem 4.23 we will focus on the case of quaternion unit gain graphs.

Lemma 4.22 *Let $\phi = (\phi_1, \phi_2)$ be a surjective group homomorphism of $G_1 \times G_2$ onto H. Then*

$$H \simeq \frac{\phi_1(G_1) \times \phi_2(G_2)}{\phi_1(G_1) \cap \phi_2(G_2)}.$$

Proof — First we need an embedding

$$\iota : \phi_1(G_1) \cap \phi_2(G_2) \hookrightarrow \phi_1(G_1) \times \phi_2(G_2).$$

This embedding is defined as

$$\iota(h) := (h, h^{-1}).$$

Observe that ι is in fact a homomorphism, since $\phi_1(G_1) \cap \phi_2(G_2)$ is an Abelian group by Lemma 4.20. Then we conclude by observing that $\iota(\phi_1(G_1) \cap \phi_2(G_2))$ is the kernel of the surjective homomorphism $R\colon \phi_1(G_1) \times \phi_2(G_2) \to H$ defined by

$$R(\phi_1(g_1), \phi_2(g_2)) := \phi_1(g_1) \cdot \phi_2(g_2).$$

The statement is proved. □

Let $(\Gamma_1, U(\mathbb{H}), \psi_1)$ and $(\Gamma_2, U(\mathbb{H}), \psi_2)$ be two quaternion unit gain graphs. The following proposition holds.

Theorem 4.23 *Consider a NEPS of two quaternion unit gain graphs*

$$(\Gamma_1, U(\mathbb{H}), \psi_1) \quad \text{and} \quad (\Gamma_2, U(\mathbb{H}), \psi_2).$$

Let

$$\phi : U(\mathbb{H}) \times U(\mathbb{H}) \to H$$

be a surjective group homomorphism. Then the gain function Ψ_ϕ *either is described by* (1) *of* Proposition 4.19*, or it is an* $SO(4)$*-gain function.*

Proof — Since the only normal subgroups of the group $U(\mathbb{H})$ are $U(\mathbb{H})$, $\{\pm 1_{U(\mathbb{H})}\}$ and $\{1_{U(\mathbb{H})}\}$, then $\phi_i(U(\mathbb{H}))$ may be isomorphic to either $\{1_{U(\mathbb{H})}\}$, or $SO(3)$ or $U(\mathbb{H})$. Moreover, we know from Lemma 4.20 that $\phi_1(U(\mathbb{H})) \cap \phi_2(U(\mathbb{H}))$ is an Abelian, normal subgroup of both $\phi_1(U(\mathbb{H}))$ and $\phi_2(U(\mathbb{H}))$. Then the following three possibilities may occur:

- at least one among $\phi_1(U(\mathbb{H}))$ and $\phi_2(U(\mathbb{H}))$ is trivial. In this case $\phi_1(U(\mathbb{H})) \cap \phi_2(U(\mathbb{H}))$ is trivial and so Ψ_ϕ is described by (1) of Proposition 4.19.
- $\phi_1(U(\mathbb{H}))$ and $\phi_2(U(\mathbb{H}))$ are both nontrivial and at least one of them is isomorphic to $SO(3)$. In this case, if the intersection $\phi_1(U(\mathbb{H})) \cap \phi_2(U(\mathbb{H}))$ is not trivial, then it is isomorphic to $SO(3)$, since $SO(3)$ is simple. But this is not possible, since the group $\phi_1(U(\mathbb{H})) \cap \phi_2(U(\mathbb{H}))$ must be Abelian by Lemma 4.20.
- $\phi_1(U(\mathbb{H})) \simeq \phi_2(U(\mathbb{H})) \simeq U(\mathbb{H})$. In this case, since the intersection $\phi_1(U(\mathbb{H})) \cap \phi_2(U(\mathbb{H}))$ must be Abelian by Lemma 4.20, the only non trivial possibility is that

  $$\phi_1(U(\mathbb{H})) \cap \phi_2(U(\mathbb{H})) \simeq \{\pm 1_{U(\mathbb{H})}\}.$$

  So, by Lemma 4.22, we obtain

  $$H \simeq \frac{U(\mathbb{H}) \times U(\mathbb{H})}{\{(1_{U(\mathbb{H})}, 1_{U(\mathbb{H})}), (-1_{U(\mathbb{H})}, -1_{U(\mathbb{H})})\}} \simeq SO(4).$$

The statement is proved. □

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