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On the Autocentral Series of a Group *

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In memoriam of Francesco de Giovanni

Abstract

Let G be a group. Following Ellis [10], a group M is said to be a G-group if there exists a homomorphism $\varphi : G \to \mathrm{Aut}(M)$ such that $\mathrm{Inn}(M) \leqslant \mathrm{Im}(\varphi)$. In the present article we study the upper and lower G-central series of a G-group M and derive, among other results, a generalization of a famous theorem by Hegarty on the autocentre of a group.

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1 Introduction

Let G be a group. In 1994, P.V. Hegarty, following the same definition given in 1966 by A.P. Šapiro for abelian groups, introduced the *absolute centre* or *autocentre* of G as the set

$$L(G) = \{g \in G \mid g^{\alpha} = g, \forall \alpha \in \mathrm{Aut}(G)\}$$

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of all the elements of G which are fixed by every automorphism of G. In the same paper, Hegarty introduced the *autocommutator subgroup* of G as the set

$$G^* = \{g^{-1}g^{\alpha} \mid g \in G, \forall \alpha \in \mathrm{Aut}(G)\}.$$

In recent years, the properties of these relevant subgroups and the relations between them have been investigated by many authors focusing on finiteness conditions, on the study of $\mathrm{Aut}(G)$, and on extended central series of groups. Here we just mention a few. Regarding finiteness conditions, in [14] and [15] the author studied the case in which the factor group $G/L(G)$ is finite and the relation with the finiteness of G^*, while in [7], among other things, the authors studied in greater generality the groups G in which $G/L(G)$ is locally finite. Following [11], different authors studied, under many aspects, the automorphisms of G acting trivially on $G/L(G)$; see, for instance, [6], [16], [19], [25], [26] and [28]. Finally, in [20] the authors introduced the so-called *autocentral series* of a group, which has been investigated in deeper details and generality in [7].

One of the most important results in the area shows that, for any group G, if $G/L(G)$ is finite, then so is G^* (see [14]). This is the autocentral analogue of a famous theorem of Schur (1902) stating that a group whose centre has finite index has also a finite commutator subgroup. Actually, Hegarty shows that in this case also $\mathrm{Aut}(G)$ and

$$\mathrm{Var}(G) = \{\alpha \in \mathrm{Aut}(G) \mid [g, \alpha] \in L(G) \text{ for all } g \in G\}$$

are finite. (Groups with finitely many automorphisms have been studied for instance in [23].)

The aim of this paper is to give a further contribution to the study of the autocentral series of a group, in the more general setting of G-groups introduced by Ellis in [10]. Recall that, given a group G, a group M is a G-group if there is a homomorphism $\varphi : G \to \mathrm{Aut}(M)$ such that $\mathrm{Inn}(M) \leqslant \mathrm{Im}(\varphi)$. We focus in particular on providing a meaningful generalization of Hegarty's theorem, in the same way the theorem of Schur has been generalized by Baer in 1952 for the upper and lower central series of groups (see [1]). We conclude the work with a couple examples.

Most of our notation is standard and can be found in [22].

2 Preliminaries

Let G be a group and let M be a G-group. The G-*centre* $\zeta_G(M)$ of M is the subgroup of all elements $m \in M$ such that $m^g = m$ for every $g \in G$. In particular, $\zeta_M(M)$ is the centre of M and $\zeta_{\mathrm{Aut}(M)}(M)$ is the autocentre of M. Now, the *upper* G-*central series* of M

$$\{1\} = \zeta_{G,0}(M) \leqslant \zeta_{G,1}(M) \leqslant \ldots \leqslant \zeta_{G,n}(M) \leqslant \ldots$$

is defined by $\zeta_{G,i+1}(M)/\zeta_{G,i}(M) = \zeta_G(M/\zeta_{G,i}(M))$, while the *lower* G-*central series* of G

$$M = \gamma_{G,1}(M) \geqslant \gamma_{G,2}(M) \geqslant \ldots \geqslant \gamma_{G,n}(M) \geqslant \ldots$$

is defined by $\gamma_{G,i+1}(M) = [\gamma_{G,i}(M), G]$. We also put

$$\zeta_{G,\infty}(M) = \bigcup_i \zeta_{G,i}(M) \quad \text{and} \quad \gamma_{G,\infty}(M) = \bigcap_i \gamma_{G,i}(M).$$

Clearly, if we put $G = \mathrm{Aut}(M)$, then $\gamma_{G,2}(M)$ is the autocommutator of G and the lower G-central series is the lower autocentral series as defined in [20] for abelian groups. We remark that each term of the upper and of the lower G-central series of M are G-invariant and hence also M-invariant.

Lemma 1 *Let* G *be a group,* M *a* G*-group, and* N *a* G*-invariant subgroup of* M*.*

(i) $\gamma_{G,n}(M/N) = \gamma_{G,n}(M)N/N$.

(ii) $\zeta_{G,m}(M/\zeta_{G,n}(M)) = \zeta_{G,m+n}(M)/\zeta_{G,n}(M)$.

(iii) $[\gamma_{G,m}(M), \zeta_{G,n}(M)] \leqslant \zeta_{G,n-m}(M)$ *if* $m \leqslant n$.

In particular, $[\gamma_{G,m}(M), \zeta_{G,m}(M)] = 1$ *for all* $m \geqslant 1$.

Proof — By way of example, we prove the last point. We proceed by induction on m. If $m = 1$, the result is obvious from the definition of the upper G-central series of M. Assume now that $2 \leqslant m \leqslant n$, and that the inequality holds for $m-1$. Let H be the natural semidirect product $G \ltimes M$. Moreover, by induction hypothesis we have that $[G, \zeta_{G,n}(M), \gamma_{G,m-1}(M)]$ and $[\zeta_{G,n}(M), \gamma_{G,m-1}(M), G]$ are subgroups of $\zeta_{G,n-m}(M)$, so we may apply the Three Subgroup Lemma

to $\gamma_{G,m-1}(M)$, G and $\zeta_{G,i+1}(M)$ as subgroups of H, to get

$$[\gamma_{G,m}(M), \zeta_{G,n}(M)] = [\gamma_{G,m-1}(M), G, \zeta_{G,n}(M)] \leqslant \zeta_{G,n-m}(M)$$

The thesis follows. □

Let M be a group and let G be a group acting on M. Then the sets $\{[m, g] \mid m \in M, g \in G\}$ and $\{[m, g] \mid m \in M, g \in G/C_G(M)\}$ are clearly the same. This observation and a theorem of Baer (see for example [22, Theorem 4.15]) gives the following result.

Proposition 2 *Let M be a group, and G a group acting on M. Then the finiteness of any two between $\{[m, g] \mid m \in M, g \in G\}$, $|G : C_G(M)|$, and $|M : C_M(G)|$ implies the finiteness of the third.*

Moreover, we need the following result, which immediately follows from Theorem 2.1 in [7] and induction.

Theorem 3 *Let G be a group, and M a G-group such that $G/\zeta_{G,n}(M)$ is locally finite of exponent e. Then $\gamma_{G,n+1}(M)$ is locally finite of e-bounded exponent.*

3 Main results

In this section we provide a generalisation of the quoted theorem of Hegarty for G-groups. Following [7], let

$$\mathrm{Var}_G(M) = \{g \in G \mid [m, g] \in \zeta_G(M),\ m \in M\}.$$

Firstly, we point out that imposing the finiteness of $M/\zeta_{G,n}(M)$ is not sufficient to guarantee that $\gamma_{G,n+1}(M)$ is finite, even in the strong case in which $G = \mathrm{Aut}(M)$. This is shown in Example 10.

However, if we require that $M/\zeta_{G,n}(M)$ and $\mathrm{Var}_G(M/\zeta_{G,n-1}(M))$ are finite, then things are better. Note that $\mathrm{Var}_G(M/\zeta_{G,n-1}(M))$ is a normal subgroup of G and that $G/\mathrm{Var}_G(M/\zeta_{G,n-1}(M))$ is isomorphic to a subgroup of $\mathrm{Aut}(M/\zeta_{G,n}(M))$.

Theorem 4 *Let G be a group, and M a G-group such that $M/\zeta_{G,n}(M)$ is finite for some positive integer n. If $\mathrm{Var}_G(M/\zeta_{G,n-1}(M))/C_G(M)$ is finite, then $\gamma_{G,n+1}(M)$ is finite.*

PROOF — Since

$$G/\operatorname{Var}_G(M/\zeta_{G,n-1}(M)) \quad \text{and} \quad \operatorname{Var}_G(M/\zeta_{G,n-1}(M))/C_G(M)$$

are finite, we have that $G/C_G(M)$ is finite. If $n = 1$, then $M/C_M(G)$ is finite too, so Proposition 2 yields that the set $\{[m, g] \mid m \in M, g \in G\}$ is finite. Since $\gamma_{G,2}(M)$ is locally finite by Theorem 3, it follows that it is finite. Let now $n > 1$, and suppose that the thesis holds for $n-1$. Then, Lemma 1 gives

$$\zeta_{G,n-1}(M/\zeta_{G,1}(M)) = \zeta_{G,n}(M)/\zeta_{G,1}(M)$$

and this has finite index in $M/\zeta_{G,1}(M)$. Moreover, by the same lemma, we get that

$$\operatorname{Var}_G\big((M/\zeta_{G,1}(M))/\zeta_{G,n-2}(M/\zeta_{G,1}(M))\big) = \operatorname{Var}_G(M/\zeta_{G,n-1}(M))$$

and this is finite modulo $C_G(M)$, so we may apply the induction hypothesis to get the finiteness of $\gamma_{G,n}(M/\zeta_{G,1}(M))$. Again by Lemma 1, this means that $\zeta_{G,1}(M)$ has finite index in $\gamma_{G,n}(M)\zeta_{G,1}(M)$ and in particular that $|\gamma_{G,n}(M) : C_{\gamma_{G,n}(M)}(G)|$ is finite. On the other hand, since $G/C_G(M)$ is finite, also $|G : C_G(\gamma_{G,n}(M))|$ is finite, so another application of Proposition 2 and of Theorem 3 gives that

$$\gamma_{G,n+1}(M) = [\gamma_{G,n}(M), G]$$

is finite. □

The following result shows that Theorem 4 is in fact a generalization of Theorem 1.1 in [9] and hence of Hegarty's theorem. Recall that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the smallest positive integer with such a property. Otherwise, the group is said to have *infinite rank*. Rank conditions are often very useful in group theory (see for instance [4] and [8]) and we use them in the following result.

Theorem 5 *Let M be a group and let G be a subgroup of $\operatorname{Aut}(M)$ containing $\operatorname{Inn}(M)$. If $M/\zeta_G(M)$ is finite and $\operatorname{Aut}(M)$ contains no uncountable elementary abelian subgroup with trivial intersection with G, then $[M, \operatorname{Var}_G(M)]$ is finite.*

PROOF — Assume that $[M, \operatorname{Var}_G(M)]$ is infinite. Since $[M, \operatorname{Var}_G(M)]$

is clearly a subgroup of $\gamma_{G,2}(M)$, so that it is an infinite G-central subgroup of bounded exponent by Theorem 3. In particular, we may find a prime p such that the p-component P of $\zeta_G(M)$ has infinite rank. Assume that P contains a divisible subgroup D of infinite rank. As M is central-by-finite, M' is finite, so we may assume that D intersects M' trivially. Since M/M' splits over DM'/M', there is a subgroup C containing M' such that $M = C \times D$. Now, the cartesian product of the subgroups generated by the inversion automorphisms of infinitely many Prüfer p-subgroups of D is an uncountable elementary abelian 2-subgroup of $\mathrm{Aut}(D)$ and hence it can also be seen as a subgroup of $\mathrm{Aut}(M)$. Moreover, this subgroup clearly has trivial intersection with G, since D is a subgroup of $\zeta_G(M)$, thus contradicting the hypothesis. It follows that D must have finite rank and we can write P as $D \times R$, where R is a reduced subgroup of P of infinite p-rank. In particular, R/R^p and $S = \mathrm{Soc}(R)$ are infinite elementary abelian p-groups. Denote with a bar the natural epimorphism of M over M'. As M' is finite, $|\overline{R}/\overline{R}^p| = |R/R^p|$. Moreover, since $\overline{R}/\overline{R}^p$ is a pure bounded subgroup of $\overline{\zeta_G(M)}/\overline{R}^p$, it is a direct factor of the latter group. On the other hand, $M/\zeta_G(M)$ is finite, so we may find a subgroup R_1 of R containing R^p such that $|R_1| = |R|$ and that $\overline{R_1}/\overline{R}^p$ is a direct factor of $\overline{M}/\overline{R}^p$. Then, we may find a subgroup L of M containing R^pM' such that $\overline{M}/\overline{R}^p = \overline{L}/\overline{R}^p \times \overline{R_1}/\overline{R}^p$. If we now let $S = S_0 \times S_1$, where $|S| = |S_0| = |S_1|$, and put $N = LS_0$, then the elementary abelian p-group $\mathrm{Hom}(M/N, S_0)$ is uncountable. Notice now that the mapping

$$\varphi \in \mathrm{Hom}(M/N, S_0) \mapsto (\alpha_\varphi : m \in M \mapsto m(mN)^\varphi \in M) \in \mathrm{Aut}(M)$$

is an embedding of $\mathrm{Hom}(M/N, S_0)$ into $\mathrm{Aut}(M)$. However, if we call H the image of this embedding, each non-trivial element of H obviously acts non-trivially on some element of R and, since R lies in $\zeta_G(M)$, we have that H has trivial intersection with G and this contradiction proves the statement. □

Using this theorem together with Theorem 4, we can drop the normality hypothesis on Γ in Theorem 1.1 in [9], taking also into account a non-faithful action on M.

Theorem 6 *Let* G *be a group, let* M *be a* G*-group and let* Γ *be the natural image of* G *inside* $\mathrm{Aut}(M)$. *If* $M/\zeta_G(M)$ *is finite and* $\mathrm{Aut}(M)$ *contains no uncountable abelian subgroups of prime exponent with trivial intersection with* Γ, *then* $\gamma_{G,2}(M)$ *is finite.*

From this, for $\Gamma = \mathrm{Aut}(M)$, it immediately follows Hegarty's theorem.

4 Some Examples

Note that, just for this section, the composition will be on the left.

First, we need to recall some of the theory developed in [2] concerning automorphisms of direct products. For the reader's convenience, we report here a partial result which will be used especially in the final section. For the general results see [2, Theorem B and Proposition 4.5]. Recall that two groups H and K are *totally incompatible*, if for any $\sigma \in \mathrm{Hom}(H,K)$, $\tau \in \mathrm{Hom}(K,H)$, $h \in H$ and $k \in K$, when $\sigma\tau$ and $\tau\sigma$ are normal endomorphisms there exists a positive integer n (depending on σ, τ, h and k) such that one of the following equivalent conditions hold:

(1) $(\sigma\tau)^n(k) = 1$;

(2) $(\tau\sigma)^n(h) = 1$.

Lemma 7 *Let* H *and* K *be totally incompatible groups and let*

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Aut}(H \times K).$$

Then $\alpha \in \mathrm{Aut}(H)$, $\delta \in \mathrm{Aut}(K)$, $\beta \in \mathrm{Hom}(K, \zeta(H))$ *and* $\gamma \in \mathrm{Hom}(H, \zeta(K))$.

Proof — Denote the inverse of φ with

$$\varphi' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

put $\theta = \beta'\gamma$ and note that, since $\varphi'\varphi = 1$, we have that $\alpha'\alpha + \theta = 1$. We first claim that θ is a normal endomorphism of H. To this aim, let h be an element of H and x be an element of $\mathrm{Im}(\beta')$. Note here that the fact that φ' is a homomorphism implies that $[\mathrm{Im}(\alpha'), \mathrm{Im}(\beta')] = 1$. Since clearly $\mathrm{Im}(\alpha'\alpha) \subseteq \mathrm{Im}(\alpha')$, we have $x^h = x^{\theta(h)}$. This holds for every element of $\mathrm{Im}(\beta')$ and in particular for every element of $\mathrm{Im}(\theta)$, so that, if we take y to be an element of H, we get that

$$\theta(y^h) = \theta(h^{-1})\theta(y)\theta(h) = \theta(y)^{\theta(h)} = \theta(y)^h.$$

Hence θ is a normal endomorphism of H and our claim is proved.

Let now h be an element of H such that $\alpha(h) = 1$. Since θ is a normal endomorphism of H, we can use the total incompatibility of H and K to get $h = 1$, so that α is injective. Symmetrically, the same can be proved for α', δ and δ'. In particular, $\alpha'\alpha$ is injective. Let a and b be elements of H. Since $1 - \theta = \alpha'\alpha$ is a homomorphism, we have that

$$a^{-1}b\theta(b^{-1})\theta(a) = a^{-1}\theta(a)b\theta(b^{-1}),$$

from which we get

$$b\theta(b^{-1}) = (b\theta(b^{-1}))^{\theta(a)} = b^{\theta(a)}\theta(b^{-\theta(a)}),$$

using that θ is a normal endomorphism and hence

$$(1-\theta)(b) = (1-\theta)(b^{\theta(a)}).$$

However, $1 - \theta$ is injective, so $b = b^{\theta(a)}$, which, by the generality of a and b, yields that $\mathrm{Im}(\theta)$ is central in H. Let now h be any element of H, let n be the least positive integer such that $\theta^n(h) = 1$ and put

$$s = h\theta(h^{-1}) \dots \theta^{n-1}(h^{(-1)^{n-1}}).$$

It is easy to see that $(1 - \theta)(s) = h$, so $\alpha'\alpha = 1 - \theta$ is also surjective. Hence α is bijective and, in the same way, one shows that the same holds for δ. Finally, since both α and δ are surjective and since $[\mathrm{Im}(\alpha), \mathrm{Im}(\beta)] = 1$ and $[\mathrm{Im}(\gamma), \mathrm{Im}(\delta)] = 1$ because φ is a homomorphism, we get that $\mathrm{Im}(\beta)$ and $\mathrm{Im}(\gamma)$ are central in H and K, respectively, thus proving the result. □

We recall a well-known lemma due to Stammbach (see [29, Proposition II.4.3]) which will be needed in the following.

Lemma 8 *Let* $E : A \rightarrowtail G \twoheadrightarrow Q$ *and* $E_1 : A_1 \rightarrowtail G_1 \twoheadrightarrow Q_1$ *be central extensions and let* Δ *and* Δ_1 *be the respective cohomology classes. If*

$$\alpha : A \to A_1 \quad \textit{and} \quad \beta : Q \to Q_1$$

are homomorphims, then there exists a homomorphism $\gamma : G \to G_1$ *such*

that the following diagram

$$\begin{array}{ccccc} A & \rightarrowtail & G & \twoheadrightarrow & Q \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\gamma}\vdots & & \downarrow{\scriptstyle\beta} \\ A_1 & \rightarrowtail & G_1 & \twoheadrightarrow & Q_1 \end{array}$$

is commutative if and only if $\alpha_*(\Delta) = \beta^*(\Delta)$, *where*

$$\alpha_* : H^2(Q, A) \longrightarrow H^2(Q_1, A_1) \quad \text{and} \quad \beta^* : H^2(Q, A) \longrightarrow H^2(Q_1, A_1)$$

are the natural maps induced by α *and* β, *respectively.*

In the notation of this lemma, let $E = E_1$. If we now endow $H^2(Q, A)$ with the usual structure of $\mathrm{Aut}(Q) \times \mathrm{Aut}(A)$-module, it is immediate to rephrase the lemma when α and β are automorphisms: in this case the prescribed automorphism γ of G exists if and only if $(\beta^{-1}, \alpha)(\Delta) = \Delta$.

Example 9 *Let* n *be a non-negative integer. Then there is a group* G *such that* $G/\zeta_{\mathrm{Aut}(G),n}(G)$ *is finite,* $\zeta_{\mathrm{Aut}(G),n-1}(G) \neq \zeta_{\mathrm{Aut}(G),n}(G)$ *and* $\zeta_{\mathrm{Aut}(G),n}(G)$ *is abelian and torsion-free.*

Proof — Let $\pi_1, \pi_2, \ldots$ be disjoint infinite sets of primes, and let A_k be the subgroup of $(\mathbb{Q}, +)$ generated by the set

$$\left\{ \frac{1}{p} \mid p \in \pi_1 \cup \pi_2 \cup \ldots \cup \pi_k \right\}.$$

Let n be a positive integer and let $\{q_1, q_2, \ldots, q_n\}$ and $\{r_1, r_2, \ldots, r_n\}$ be sets of odd primes such that $q_i \notin \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n$ and $r_i \equiv_{q_i} 1$ for every i. For each i, let G_i be the semidirect product of A_i acting on a cyclic group of order r_i with kernel $q_i A_i$. Clearly, G_i is a central extension of $q_i A_i \simeq A_i$ by a non-abelian group Q_i of order $q_i r_i$. Note that $\mathrm{Aut}(Q_i)$ acts trivially on Q_i/Q_i'. Since $\mathrm{Aut}(A_i)$ has order 2, it easily follows that $q_i A_i$ is the autocentre of G_i. Set $G = \bigoplus_{i=1}^{n} G_i$.

For each i, denote the type of A_i with t_i (see, for instance, [24]). Clearly, $t_i < t_j$ and $\mathrm{Hom}(A_i, A_j) \neq 0$ if and only if $i < j$. It follows that, for any $\sigma \in \mathrm{Hom}(G_i, G_j)$ and $\tau \in \mathrm{Hom}(G_j, G_i)$, $\mathrm{Im}(\sigma\tau) \leqslant G_j'$ and $\mathrm{Im}(\tau\sigma) \leqslant G_i'$, so in particular G_i and G_j are totally incompatible. Then, Lemma 7 yields that each $\bigoplus_{i=j}^{n} G_i$ for each $j \in \{1, \ldots n\}$ is

characteristic. This, together with the above information about the automorphisms groups of the G_is, yields that

$$\zeta_{\mathrm{Aut}(G),n-j+1}(G) = \bigoplus_{i=j}^{n} q_i A_i$$

for each $j \in \{1, \dots n\}$ and G is the group we were looking for. □

We construct now an example which is mainly based on Example 3.1 in [5].

Example 10 *There is a group* G *such that* $\zeta_{\mathrm{Aut}(G),2}(G) = \zeta_{\mathrm{Aut}(G),\infty}(G)$ *has finite index in* G *and* $\gamma_{\mathrm{Aut}(G),3}(G) = \gamma_{\mathrm{Aut}(G),\infty}(G)$ *is infinite.*

Proof — Let p be an odd prime and let H be a nilpotent p-group of class 2 generated by set of elements $\{x_i\}_{i\in\mathbb{Z}}$ with the following defining relations:

$$[x_1, x_5] = 1, \qquad x_{2i}^p = [x_{2i}, x_{2i+1}], \qquad x_{2i+1}^p = [x_{2i-1}, x_{2i+2}]$$

for every $i \in \mathbb{Z}$. Since

$$x_{2i}^{p^2} = [x_{2i}, x_{2i+1}]^p = [x_{2i}, x_{2i+1}^p] = [x_{2i}, [x_{2i-1}, x_{2i+2}]] = 1,$$

and similarly $x_{2i+1}^{p^2} = 1$ for each $i \in \mathbb{Z}$, then H has exponent p^2. Moreover, $\zeta(H) = H' = H^{[p]}$. From the relations it is immediate to get that $\zeta(H)$ is generated by the independent set

$$\mathcal{B} = \big\{ [x_i, x_j] \,|\, (i,j) \in \mathbb{Z}\times\mathbb{Z} \setminus \{(1,5)\}, i < j \big\}.$$

In [5], it is proved that an automorphism of H acts trivially on $H/\zeta(H)$. From this, we also find that $\mathrm{Aut}(H)$ acts trivially also on $\zeta(H)$, for $\mathcal{B}$ is a set of generators of $\zeta(H)$ and, if α is an automorphism of H, then we have that $\alpha([x_i, x_j]) = [\alpha(x_i), \alpha(x_j)] = [x_i, x_j]$. Let now $D = A \ltimes B$ be a dihedral group of order $2p$ acting irreducibly on a suitable elementary abelian 2-group E. Put $K = B \ltimes E$ and $G = H \times K$. Since $H = O_p(G)$ is a characteristic subgroup of G, $\zeta(H)$ is the autocentre of G itself and H is the second autocentre of G. Since $\mathrm{Aut}(K)$ contains D, we obtain that $\gamma_{\mathrm{Aut}(K),2}(K) = K$. Moreover, the structure of $\mathrm{Aut}(G)$ yields that

$$\gamma_{\mathrm{Aut}(G),3}(G) = \gamma_{\mathrm{Aut}(G),\infty}(G) = K \times \zeta(H)$$

is infinite. □

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