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On the τ-Closedness of n-Multiply σ-Local Formations

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Abstract

All groups under consideration are finite. Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, G be a group, $\mathfrak{F}$ be a class of groups, $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \neq \emptyset\}$, and $\sigma(\mathfrak{F}) = \cup_{G\in\mathfrak{F}}\sigma(G)$. A function f of the form $f : \sigma \to \{\text{formations of groups}\}$ is called a *formation* σ*-function*. For any formation σ-function f the class $LF_\sigma(f)$ is defined as follows:

$$LF_\sigma(f) = \big(G \mid G = 1 \text{ or } \ G \neq 1 \text{ and } \ G/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i) \ \text{ for all } \sigma_i \in \sigma(G)\big).$$

If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then the class $\mathfrak{F}$ is called σ*-local* and f is called a σ*-local definition* of $\mathfrak{F}$. Every formation is called 0*-multiply* σ*-local*. For $n > 0$, a formation $\mathfrak{F}$ is called n*-multiply* σ*-local* if either $\mathfrak{F} = (1)$ is the class of all identity groups or $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply σ-local for all $\sigma_i \in \sigma(\mathfrak{F})$.

Let $\tau(G)$ be a set of subgroups of G such that $G \in \tau(G)$. Then τ is called a *subgroup functor* if for every epimorphism $\varphi : A \to B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$. A class of groups $\mathfrak{F}$ is called τ*-closed* if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$.

In this paper, we describe some properties of τ-closed n-multiply σ-local formations. In particular, we prove that n-multiply σ-local formation is τ-closed if and only if all non-empty values of the canonical definition of it are τ-closed.

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Keywords: n-multiply σ-local formation; τ-closed formation

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group; $\mathbb{P}$ is the set of all primes and $\emptyset \neq \pi \subseteq \mathbb{P}$; σ is some partition of $\mathbb{P}$ (see [12, p.206]), i.e. $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \neq \emptyset\}$ and if $\mathfrak{F}$ is a formation of groups, then $\sigma(\mathfrak{F}) = \bigcup_{G\in\mathfrak{F}} \sigma(G)$; $\mathfrak{G}_\pi$ denotes the formation of all π-groups. Recall that a *formation* is a class of groups $\mathfrak{F}$ which is closed under taking homomorphic images and subdirect products.

A σ-*property* of a group [14]–[16],[18] is a property of this group which does not depend on the choice of σ. In other words, in the theory of σ-properties of a group, we do not impose any restrictions on the partition σ of $\mathbb{P}$. Among the approaches that were found and developed on the way of constructing such a theory, some new aspects of the theory of formations, which were first proposed in Skiba's work [17], turned out to be very useful.

Now, recall the main concepts in [17].

Every function f of the form $f : \sigma \to \{\text{formations}\}$ is called *a formation* σ-*function*. For any formation σ-function f the class $LF_\sigma(f)$ defined as follows:

$$LF_\sigma(f) = \big(G \mid G = 1 \text{ or } G \neq 1 \text{ and } G/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i)\ \forall \sigma_i \in \sigma(G)\big).$$

If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then the class $\mathfrak{F}$ is called σ-*local*, and f called σ-*local definition* of $\mathfrak{F}$.

Formations of these type were introduced by A.N. Skiba [17] as a tool for studying the σ-properties of a group and they have already found successful applications not only in the theory of σ-properties of a group [1],[3],[4],[17] but also in the study of algebra classes [2],[6]–[10],[19],[21], and in the theory of formal languages [19, 20]. These observations make the task of further study of σ-local formations quite relevant and interesting.

Recall that every formation is called *o-multiply* σ-*local* (see [17]). For $n > 0$, a formation $\mathfrak{F}$ is called n-*multiply* σ-*local* provided that either $\mathfrak{F} = (1)$ is the class of all identity groups or $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply σ-local for all $\sigma_i \in \sigma(\mathfrak{F})$. A formation is called *totally* σ-*local* if it is n-multiply σ-local for all non-negative integer n.

When studying different classes of groups, it is often necessary to

consider classes of groups closed with respect to certain systems of subgroups which consistent with homomorphisms. The most general approach in this direction was proposed by A.N. Skiba [13].

Let $\tau(G)$ be a set of subgroups of G such that $G \in \tau(G)$. Then τ is called a *subgroup functor* [13] if for every epimorphism $\varphi : A \to B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$. A class of groups $\mathfrak{F}$ is called τ-*closed* if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$. A formation is called τ-*closed*, if it is a τ-closed class of groups. The collection of all τ-closed n-multiply σ-local formations denote by $l^{\tau}_{\sigma_n}$. Formations from $l^{\tau}_{\sigma_n}$ we call $l^{\tau}_{\sigma_n}$-*formations.*

In [7, 8] the author began the study of $l^{\tau}_{\sigma_n}$-formations, where τ is a subgroup functor in the sense of A.N. Skiba [13]. In this paper, we describe some properties of τ-closed n-multiply σ-local formations of groups. We introduce the concept σ-locality index of a formation and study the relationship between the σ-locality index of formation and σ-locality indexes of the values of its σ-local definition.

Following [13, p.14], we denote by the symbol $(l^{\tau}_{\sigma_{n-1}})^{\sigma}$ the collection of all σ-local formations which have at least one $l^{\tau}_{\sigma_{n-1}}$-valued σ-local definition. In this paper, we prove that $(l^{\tau}_{\sigma_{n-1}})^{\sigma}$ coincides with the collection $l^{\tau}_{\sigma_n}$, that is, the following theorem is true.

Theorem 1 $\left(l^{\tau}_{\sigma_{n-1}}\right)^{\sigma} = l^{\tau}_{\sigma_n}$ *for any positive integer* n.

Recall that a formation σ-function f is called: $l^{\tau}_{\sigma_n}$-*valued*, if any non-empty value $f(\sigma_i)$ of f is a τ-closed n-multiply σ-local formation; *integrated* if $f(\sigma_i) \subseteq LF_{\sigma}(f)$ for all i; *full* if $f(\sigma_i) = \mathfrak{G}_{\sigma_i} f(\sigma_i)$ for all i. We use $\mathfrak{G}_{\sigma_i} f(\sigma_i)$ to denote the class of all groups which are extension of some σ_i-group be a group in $f(\sigma_i)$.

If F is a full integrated formation σ-function and $\mathfrak{F} = LF_{\sigma}(F)$, then F is called the *canonical* σ-*local definition* of $\mathfrak{F}$. It is known [4, Corollary 2.1] that every σ-local formation has a unique canonical definition.

The next result of our work is the criterion for τ-closedness of an n-multiply σ-local formation.

Theorem 2 *Let* $\mathfrak{F}$ *be an* n*-multiply* σ*-local formation* ($n \geqslant 1$) *and let* F *be the canonical* σ*-local definition of* $\mathfrak{F}$*. Then* $\mathfrak{F}$ *is* τ*-closed if and only if* $F(\sigma_i)$ *is a* τ*-closed formation for all* $\sigma_i \in \sigma(\mathfrak{F})$.

We will prove these theorems in Section 3, and also present some of the most important corollaries of the results obtained.

2 Definitions, notations and auxiliary statements

Following [12], $\sigma = \{\sigma_i \mid i \in I\}$ will always denote a partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Let n be an integer, G be a group, and let $\mathfrak{F}$ be a class of groups. Then the symbol $\sigma(n)$ denotes the set $\{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset\}$, $\sigma(G) = \sigma\big(|G|\big)$, and $\sigma(\mathfrak{F}) = \cup_{G\in\mathfrak{F}}\sigma(G)$.

A group G is called (cf. [15]): σ-*primary* if G is a σ_i-group for some i; σ-*nilpotent* if every chief factor H/K from G is σ-central in G, that is, the semidirect product $(H/K)\rtimes(G/C_G(H/K))$ is σ-primary; σ-*soluble* if $G = 1$ or $G \neq 1$ and each chief factor from G is σ-primary. The symbol $\mathfrak{S}_\sigma$ denotes the class of all σ-soluble groups and $\mathfrak{N}_\sigma$ denotes the class of all σ-nilpotent groups.

Let $\mathfrak{F}$ be a non-empty formation. Then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-*residual* of a group G, that is, the intersection of all normal subgroups K of G with $G/K \in \mathfrak{F}$. If $\mathfrak{F}$ and $\mathfrak{H}$ are non-empty formations, then the formation $\mathfrak{F}\mathfrak{H} = (G \mid G^{\mathfrak{H}} \in \mathfrak{F})$ is called the *Gaschütz product* of the formations $\mathfrak{F}$ and $\mathfrak{H}$.

The subgroup functor τ is called [13]: *closed* if for any two groups G and $H \in \tau(G)$ we have $\tau(H) \subseteq \tau(G)$; *trivial*, if for any group G we have that $\tau(G) = \{G\}$; *identity*, if for any group G we have $\tau(G)$ is the collection of all subgroups of G.

A formation $\mathfrak{F}$ is called τ-*closed*, if $\tau(G) \subseteq \mathfrak{F}$ for any group $G \in \mathfrak{F}$. In particular, a formation is called: *hereditary*, if it is τ-closed, where τ is a identity subgroup functor; *normally hereditary*, if its is τ-closed, where $\tau(G) = S_n(G)$ is the collection of all normal subgroups of G for any group G.

We will need a number of known results, which we formulate in the form of the following lemmas

Lemma 3 (see [4]) *If* $\mathfrak{F} = LF_\sigma(f)$, *then* $\mathfrak{F} = LF_\sigma(t)$, *where* $t(\sigma_i) = f(\sigma_i) \cap \mathfrak{F}$ *for all* $\sigma_i \in \sigma$.

Lemma 4 (see [2]) *If the class of groups* $\mathfrak{F}_j$ *is an* n*-multiply* σ*-local formation for all* $j \in J$, *then the class* $\cap_{j\in J}\mathfrak{F}_j$ *is also an* n*-multiply* σ*-local formation.*

If f is a formation σ-function, then the symbol $\mathrm{Supp}(f)$ denotes the support of f, that is, the set of all σ_i such that $f(\sigma_i) \neq \emptyset$. We denote subsets from σ with the capital letters of the Greek alphabet. In particular, if $\Pi \subseteq \sigma$, then $\Pi' = \sigma \setminus \Pi$.

Lemma 5 (see [4]) *Let* f, h *be formation* σ*-functions and let* $\Pi = \mathrm{Supp}(f)$. *Suppose* $\mathfrak{F} = LF_\sigma(f) = LF_\sigma(h)$.

(1) $\Pi = \sigma(\mathfrak{F})$.

(2) $\mathfrak{F} = \left(\cap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right) \cap \mathfrak{G}_\Pi$. *Therefore,* $\mathfrak{F}$ *is a saturated formation.*

(3) *If every group from* $\mathfrak{F}$ *is a* σ*-soluble, then*

$$\mathfrak{F} = \left(\cap_{\sigma_i \in \Pi} \mathfrak{S}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right) \cap \mathfrak{S}_\Pi.$$

(4) *If* $\sigma_i \in \Pi$, *then* $\mathfrak{G}_{\sigma_i}\big(f(\sigma_i) \cap \mathfrak{F}\big) = \mathfrak{G}_{\sigma_i}\big(h(\sigma_i) \cap \mathfrak{F}\big) \subseteq \mathfrak{F}$.

(5) $\mathfrak{F} = LF_\sigma(F)$, *where* F *is the unique formation* σ*-function such that* $F(\sigma_i) = \mathfrak{G}_{\sigma_i} F(\sigma_i) \subseteq \mathfrak{F}$ *for all* $\sigma_i \in \Pi$ *and* $F(\sigma_i) = \emptyset$ *for all* $\sigma_i \in \Pi'$. *Besides,* $F(\sigma_i) = \mathfrak{G}_{\sigma_i}\big(f(\sigma_i) \cap \mathfrak{F}\big)$ *for all* i.

Lemma 6 (see [2]) *The set* $\mathcal{S}_n^\sigma$, *of all* n*-multiply* σ*-local formations, forms a subsemigroup of the semigroup of all formations* $\mathfrak{GG}$. *Moreover,* $|\mathcal{S}_n^\sigma| = 2^{\aleph_0}$ *for every* σ *for* $|\sigma| > 1$, *and* $\mathfrak{G}_{\sigma_i}$ *is the minimal idempotent in* $\mathcal{S}_n^\sigma$ *for all* $n > 0$ *and* $i \in I$.

Lemma 7 (see [10]) *If* $\mathfrak{F} = LF_\sigma(f)$ *and there exists* σ_i *in* $\sigma(G)$ *such that* $G/O_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{F}$, *then* $G \in \mathfrak{F}$.

Recall [5, p.28] that for any nonempty sets of primes π and ν the symbol $O_{\pi,\nu}(G)$ denotes the characteristic subgroup of the group G, defined by the relation $O_{\pi,\nu}(G)/O_\pi(G) = O_\nu(G/O_\pi(G))$.

We need the following special case of Lemma 14 from [11].

Lemma 8 (see [11]) *Let* A *be a non-identity* p*-group for some prime* p, π *and* ν *be nonempty sets of primes such that* $p \in \nu$, $\pi \subseteq p'$. *If*

$$W_1 = K \cdot B_1 \leqslant W = A \wr B = K \rtimes B,$$

where K *is the base group of the regular wreath product* W, $B \neq 1$ *and* B_1 *is a subgroup of* B, *then*

$$O_\pi(W_1) = 1, \quad O_\nu(W_1) = K \cdot O_\nu(B_1),$$
$$O_{\nu,\pi}(W_1) = K \cdot O_{\nu,\pi}(B_1), \quad O_{\pi,\nu}(W_1) = O_\nu(W_1) = K \cdot O_\nu(B_1).$$

3 Main results

Following [13, p.27], we introduce the concept of the σ-locality index of a formation. Let n be a non-negative integer. We say that the formation $\mathfrak{F}$ has a σ-*locality index* $\mathrm{Ind}_\sigma(\mathfrak{F})$ equal to n, if $\mathfrak{F}$ is an n-multiply σ-local formation, but is not $(n+1)$-multiply σ-local. If the formation $\mathfrak{F}$ is n-multiply σ-local for all natural numbers n, then we say that $\mathfrak{F}$ has *infinite* σ-*locality index* and write $\mathrm{Ind}_\sigma(\mathfrak{F}) = \infty$. If the formation $\mathfrak{F}$ is not σ-local, then $\mathrm{Ind}_\sigma(\mathfrak{F}) = 0$.

It is clear that for a totally σ-local formation $\mathfrak{F}$ we have $\mathrm{Ind}_\sigma(\mathfrak{F}) = \infty$. In particular, as shown in [2], the formation (1) of all identity groups, the formation of $\mathfrak{N}_\sigma$ of all σ-nilpotent groups, and the formation $\mathfrak{S}_\sigma$ of all σ-soluble groups are totally σ-local. Therefore $\mathrm{Ind}_\sigma\big((1)\big) = \infty$, $\mathrm{Ind}_\sigma(\mathfrak{N}_\sigma) = \infty$ and $\mathrm{Ind}_\sigma(\mathfrak{S}_\sigma) = \infty$.

In particular, in the classical case, when $\sigma = \sigma^1 = \big\{\{2\},\{3\},\dots\big\}$ a σ^1-locality index of a formation, is obviously, coincides with the formation locality index, i.e. $\mathrm{Ind}_{\sigma^1}(\mathfrak{F}) = \mathrm{Ind}_l(\mathfrak{F})$.

Proposition 9 *Let* F *be the canonical* σ*-local definition of* $\mathfrak{F}$*. Then* $\mathrm{Ind}_\sigma(\mathfrak{F}) = n$ *if and only if there is* $\sigma_i \in \sigma$ *such that* $\mathrm{Ind}_\sigma\big(F(\sigma_i)\big) = n-1$*, and* $\mathrm{Ind}_\sigma\big(F(\sigma_j)\big) \geqslant n-1$ *for all* $\sigma_j \in \sigma\big(\mathfrak{F}\big) \setminus \{\sigma_i\}$.

Proof — *Necessity* Let $\mathrm{Ind}_\sigma(\mathfrak{F}) = n$. Then $\mathfrak{F}$ has a σ-local definition f such that $\mathrm{Ind}_\sigma\big(f(\sigma_i)\big) = n-1$ for some $\sigma_i \in \sigma\big(\mathfrak{F}\big)$ and $\mathrm{Ind}_\sigma\big(f(\sigma_j)\big) \geqslant n-1$ for all $\sigma_j \in \sigma(\mathfrak{F}) \setminus \{\sigma_i\}$. In addition, for any other σ-local definition k of $\mathfrak{F}$ there exists $\sigma_s \in \sigma(\mathfrak{F})$ such that $\mathrm{Ind}_\sigma\big(k(\sigma_s)\big) \leqslant n-1$. Let h be a formation σ-function such that $h(\sigma_j) = f(\sigma_j) \cap \mathfrak{F}$ for all $\sigma_j \in \sigma$. Then $\mathfrak{F} = LF_\sigma(h)$ by Lemma 3. Since $\mathrm{Ind}_\sigma(\mathfrak{F}) = n$, in view of Lemma 4 we have

$$\mathrm{Ind}_\sigma\big(f(\sigma_j) \cap \mathfrak{F}\big) = \mathrm{Ind}_\sigma\big(h(\sigma_j)\big) \geqslant n-1$$

for all $\sigma_j \in \sigma(\mathfrak{F})$. Note also that $F(\sigma_j) = \mathfrak{G}_{\sigma_j} h(\sigma_j)$ for all $\sigma_j \in \sigma$ by Lemma 5 (5). In view of Lemma 6, the formation $\mathfrak{G}_{\sigma_j}$ is $(n-1)$-multiply σ-local and also the product $\mathfrak{G}_{\sigma_j} h(\sigma_j) = F(\sigma_j)$ is a $(n-1)$-multiply σ-local formation for all $\sigma_j \in \sigma(\mathfrak{F})$. Since $\mathrm{Ind}_\sigma(\mathfrak{F}) = n$, there exists $\sigma_j \in \sigma(\mathfrak{F})$ such that $\mathrm{Ind}_\sigma\big(F(\sigma_j)\big) = n-1$.

Sufficiency Suppose there is $\sigma_i \in \sigma$ such that $\mathrm{Ind}_\sigma\big(F(\sigma_i)\big) = n-1$, and for all $\sigma_j \in \sigma(\mathfrak{F}) \setminus \{\sigma_i\}$ we have $\mathrm{Ind}_\sigma\big(F(\sigma_j)\big) \geqslant n-1$. Suppose that $\mathrm{Ind}_\sigma(\mathfrak{F}) > n$. Then there exists a σ-local definition h of $\mathfrak{F}$ such that $h(\sigma_i)$ is n-multiply σ-local. Then $h(\sigma_i) \cap \mathfrak{F}$ is n-multiply σ-local by Lemma 4. Now, applying Lemma 3 and Lemma 5 (5), we have

that $F(\sigma_i) = \mathfrak{G}_{\sigma_i}\big(h(\sigma_i) \cap \mathfrak{F}\big)$. It follows from Lemma 6 that the formation $F(\sigma_i) = \mathfrak{G}_{\sigma_i}\big(h(\sigma_i) \cap \mathfrak{F}\big)$ is n-multiply σ-local. This contradiction shows that $\mathrm{Ind}_\sigma(\mathfrak{F}) = n$. The proposition is proved. □

Example 10 Let $\mathfrak{A}$ be the formation of all abelian groups. Then $\mathfrak{A}$ is not σ-local for any partition σ. Indeed, in view of Lemma 5 (2) every σ-local formation is local. But it is well known that $\mathfrak{A}$ is not a local formation. Hence $\mathfrak{A}$ is not σ-local for any partition σ and also $\mathrm{Ind}_\sigma(\mathfrak{A}) = 0$.

Example 11 Let $\mathfrak{F} = \mathfrak{N}_\sigma^n\mathfrak{A}$, where $\mathfrak{A}$ is the formation of all abelian groups. Using induction on n, we show that $\mathrm{Ind}_\sigma(\mathfrak{F}) = n$. Let $n = 1$. Then $\mathfrak{F} = \mathfrak{N}_\sigma\mathfrak{A}$ is the class of all groups G with σ-nilpotent derived subgroups G'. In view of Lemma 2.7 of [2],

$$\mathfrak{N}_\sigma\mathfrak{A} = \mathrm{LF}_\sigma(f),$$

where $f(\sigma_i) = \mathfrak{A}$ for all i, so $\mathrm{Ind}_\sigma(\mathfrak{F}) \geqslant 1$. Suppose that $\mathrm{Ind}_\sigma(\mathfrak{F}) = k$, where $k \geqslant 2$. Let F be the canonical σ-local definition of $\mathfrak{N}_\sigma\mathfrak{A}$. By Lemma 5 (5), we have $F(\sigma_i) = \mathfrak{G}_{\sigma_i}\mathfrak{A}$ for all i. In view of Proposition 9 there is $\sigma_s \in \sigma$ such that $\mathrm{Ind}_\sigma\big(F(\sigma_s)\big) = k-1$, and $\mathrm{Ind}_\sigma\big(F(\sigma_j)\big) \geqslant k-1$ for all $\sigma_j \in \sigma(\mathfrak{F}) \setminus \{\sigma_s\}$. Then $F(\sigma_s) = \mathfrak{G}_{\sigma_s}\mathfrak{A}$ and $F(\sigma_j) = \mathfrak{G}_{\sigma_j}\mathfrak{A}$ are $(k-1)$-multiply σ-local formations, where $s \neq j$. Therefore, in wiev of Lemma 4

$$F(\sigma_s) \cap F(\sigma_j) = \mathfrak{G}_{\sigma_s}\mathfrak{A} \cap \mathfrak{G}_{\sigma_j}\mathfrak{A} = (\mathfrak{G}_{\sigma_s} \cap \mathfrak{G}_{\sigma_j})\mathfrak{A} = \mathfrak{A}$$

is a σ-local formation. The latter contradicts Example 10. Therefore $\mathrm{Ind}_\sigma(\mathfrak{F}) = 1$.

Let $n > 1$ and suppose $\mathrm{Ind}_\sigma(\mathfrak{N}_\sigma^{n-t}\mathfrak{A}) = n-t$ for all $t \in \{1, \ldots, n-1\}$. Since the Gaschütz product of formations is associative, we have

$$\mathfrak{F} = \mathfrak{N}_\sigma^n\mathfrak{A} = (\mathfrak{N}_\sigma\mathfrak{N}_\sigma^{n-1})\mathfrak{A} = \mathfrak{N}_\sigma(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}).$$

In view of Lemma 2.7 of [2],

$$\mathfrak{N}_\sigma(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}) = \mathrm{LF}_\sigma(f),$$

where $f(\sigma_i) = \mathfrak{N}_\sigma^{n-1}\mathfrak{A}$ for all i. By induction $\mathrm{Ind}_\sigma(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}) = n-1$. Let F be canonical σ-local definition of $\mathfrak{F} = \mathfrak{N}_\sigma^n\mathfrak{A}$. By Lemma 5 (5), we have $F(\sigma_i) = \mathfrak{G}_{\sigma_i}(\mathfrak{N}_\sigma^{n-1}\mathfrak{A})$ for all i. In view of Lemma 6 for all i the formation $\mathfrak{G}_{\sigma_i}$ is totally σ-local. Since $F(\sigma_i) = \mathfrak{G}_{\sigma_i}(\mathfrak{N}_\sigma^{n-1}\mathfrak{A})$ is a

product of two $(n-1)$-multiply σ-local formations $\mathfrak{G}_{\sigma_i}$ and $\mathfrak{N}_\sigma^{n-1}\mathfrak{A}$, we have $F(\sigma_i)$ is $(n-1)$-multiply σ-local for all i, by Lemma 6. Therefore, $\mathrm{Ind}_\sigma(\mathfrak{F}) \geqslant n$.

Assume that $\mathrm{Ind}_\sigma(\mathfrak{F}) = k$, where $k > n$. Then by Proposition 9 there is $\sigma_s \in \sigma$ such that $\mathrm{Ind}_\sigma\big(F(\sigma_s)\big) = k-1$, and $\mathrm{Ind}_\sigma(F(\sigma_j)) \geqslant k-1$ for all $\sigma_j \in \sigma(\mathfrak{F}) \setminus \{\sigma_s\}$. Hence

$$F(\sigma_s) = \mathfrak{G}_{\sigma_s}(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}) \quad \text{and} \quad F(\sigma_j) = \mathfrak{G}_{\sigma_j}(\mathfrak{N}_\sigma^{n-1}\mathfrak{A})$$

are $(k-1)$-multiply σ-local formations, where $s \neq j$. By Lemma 4

$$F(\sigma_s) \cap F(\sigma_j) = \mathfrak{G}_{\sigma_s}(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}) \cap \mathfrak{G}_{\sigma_j}(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}) =$$

$$(\mathfrak{G}_{\sigma_s} \cap \mathfrak{G}_{\sigma_j})(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}) = \mathfrak{N}_\sigma^{n-1}\mathfrak{A}$$

is a $(k-1)$-multiply σ-local formation. Therefore,

$$\mathrm{Ind}_\sigma(\mathfrak{N}_\sigma^{n-1}\mathfrak{A}) = k-1 \geqslant n.$$

The latter contradicts the inductive assumption. Hence $\mathrm{Ind}_\sigma(\mathfrak{F}) = n$.

Example 12 Let $\mathfrak{F}$ be an n-multiply σ-local formation ($n \geqslant 1$) such that $\mathfrak{F} \subseteq (\mathfrak{N}_\sigma)^m$ for some $m \in \mathbb{N}$. If $\mathrm{Ind}_\sigma(\mathfrak{F}) \geqslant m$, then $\mathrm{Ind}_\sigma(\mathfrak{F}) = \infty$. We will use induction on m. For $m = 1$ the statement obviously. Let $m > 1$ and the statement be true for $m-1$. Let f be the smallest l^σ_{n-1}-valued σ-local definition of $\mathfrak{F}$. Then, Theorem 1.1 of [8] shows that $f(\sigma_i) \subseteq (\mathfrak{N}_\sigma)^{m-1}$ for all i. Byy induction, $\mathrm{Ind}_\sigma\big(f(\sigma_i)\big) = \infty$ for all i. But then $\mathrm{Ind}_\sigma(\mathfrak{F}) = \infty$.

In the classical case, when $\sigma = \sigma^1$ from Proposition 9 we obtain

Corollary 13 (see [13], p.27) *Let f be the canonical screen of $\mathfrak{F}$. Then $\mathrm{Ind}_l(\mathfrak{F}) = n$ if and only if there is $p \in \mathbb{P}$ with $\mathrm{Ind}_l\big(f(p)\big) = n-1$, and $\mathrm{Ind}_l\big(f(q)\big) \geqslant n-1$ for all $q \in \pi(\mathfrak{F}) \setminus \{p\}$.*

Recall that a formation σ-function f is called $l^\tau_{\sigma_{n-1}}$-*valued*, if $f(\sigma_i)$ is a τ-closed $(n-1)$-multiply σ-local formation for all $\sigma_i \in \mathrm{Supp}(f)$.

Following [13, p.14], we denote by $(l^\tau_{\sigma_{n-1}})^\sigma$ the collection of all σ-local formations which have at least one $l^\tau_{\sigma_{n-1}}$-valued σ-local definition. We will assume that, by definition, the class (1) of all identity groups belongs to $(l^\tau_{\sigma_{n-1}})^\sigma$.

Proof of Theorem 1 — Let us show first that $(l^\tau_{\sigma_{n-1}})^\sigma \subseteq l^\tau_{\sigma_n}$. Let $\mathfrak{F} \in (l^\tau_{\sigma_{n-1}})^\sigma$. Then $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i) \in l^\tau_{\sigma_{n-1}}$ for

all $\sigma_i \in \mathrm{Supp}(f)$. The latter means that $\mathfrak{F}$ is n-multiply σ-local by definition.

Now we prove that $\mathfrak{F}$ is a τ-closed formation. Let $G \in \mathfrak{F}$ and $H \in \tau(G)$. It is necessary to show that $H \in \mathfrak{F}$. Let $\sigma_i \in \sigma(H)$. Since $G \in \mathfrak{F}$, we have $G/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i)$. But the formation $f(\sigma_i)$ is τ-closed and $H \in \tau(G)$, hence

$$HO_{\sigma_i',\sigma_i}(G)/O_{\sigma_i',\sigma_i}(G) \in \tau(G/O_{\sigma_i',\sigma_i}(G)) \subseteq f(\sigma_i).$$

But then

$$H/O_{\sigma_i',\sigma_i}(G) \cap H \simeq HO_{\sigma_i',\sigma_i}(G)/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i).$$

Since

$$O_{\sigma_i',\sigma_i}(G) \cap H \subseteq O_{\sigma_i',\sigma_i}(H),$$

we have $H/O_{\sigma_i',\sigma_i}(H) \in f(\sigma_i)$. Hence,

$$H/O_{\sigma_i',\sigma_i}(H) \in f(\sigma_i)$$

for all $\sigma_i \in \sigma(H)$. The latter implies $H \in \mathfrak{F}$. Hence, the formation $\mathfrak{F}$ is τ-closed. Thus, $\mathfrak{F} \in l^\tau_{\sigma_n}$. Therefore $(l^\tau_{\sigma_{n-1}})^\sigma \subseteq l^\tau_{\sigma_n}$.

Now let $\mathfrak{F} \in l^\tau_{\sigma_n}$ and let F be the canonical σ-local definition of $\mathfrak{F}$. We show that $F(\sigma_i) \in l^\tau_{\sigma_{n-1}}$ for all $\sigma_i \in \sigma(\mathfrak{F})$. Using Proposition 9 we have $F(\sigma_i) \in l^\sigma_{n-1}$. We prove that $F(\sigma_i)$ is τ-closed. Let $G \in F(\sigma_i)$ and $H \in \tau(G)$. By induction on $|\,G\,|$, we show that $H \in F(\sigma_i)$.

Suppose that G has two different minimal normal subgroups K_1 and K_2. Then

$$HK_j/K_j \in \tau(G/K_j), \quad j = 1, 2.$$

Hence, by induction $H/K_j \cap H \simeq HK_j/K_j \in F(\sigma_i)$, $j = 1, 2$. But then we have

$$H \simeq H/1 = H/(K_1 \cap H) \cap (K_2 \cap H) \in F(\sigma_i).$$

Now let K be the unique minimal normal subgroup of G. Suppose K is a σ_i-group. Then

$$H^{F(\sigma_i)} \subseteq K \cap H \in \mathfrak{G}_{\sigma_i}$$

since $H/K \cap H \simeq HK/K \in F(\sigma_i)$. Hence, $H \in \mathfrak{G}_{\sigma_i}F(\sigma_i)$. On the other hand, $\mathfrak{G}_{\sigma_i}F(\sigma_i) = F(\sigma_i)$ in view of Lemma 5 (5). Therefore $H \in F(\sigma_i)$.

Suppose now that $O_{\sigma_i}(G) = 1$ and let $p \in \sigma_i$ and V be a group of

order p. We denote by W the regular wreath product of the groups V and G. Then $W = V \wr G = P \rtimes G$, where P is the base group of W. Since $O_{\sigma_i}(G) = 1$ and $P \in \mathfrak{G}_{\sigma_i}$ we have

$$O_{\sigma_i}(W) = O_{\sigma_i}(W) \cap PG = P(O_{\sigma_i}(W) \cap G) = P.$$

But then $W/P \simeq G \in F(\sigma_i) \subseteq \mathfrak{F}$. Hence, $W \in \mathfrak{F}$ by Lemma 7. Now if $\varphi : W \to G$ is a natural epimorphism of W on G, then, obviously, $H^{\varphi^{-1}} = PH$. Hence, $PH \in \tau(W)$ and $PH \in \mathfrak{F}$. Therefore,

$$PH/O_{\sigma_i', \sigma_i}(PH) \in F(\sigma_i).$$

Now applying Lemma 8 in the case when $\nu = \sigma_i$, $\pi = \sigma_i'$, we get that

$$O_{\sigma_i', \sigma_i}(PH) = O_{\sigma_i}(PH) = PO_{\sigma_i}(H).$$

Consequently, we have

$$PH/O_{\sigma_i', \sigma_i}(PH) = PH/PO_{\sigma_i}(H) \simeq$$
$$H/O_{\sigma_i}(H)(P \cap H) = H/O_{\sigma_i}(H) \in F(\sigma_i).$$

But then $H \in \mathfrak{G}_{\sigma_i} F(\sigma_i)$ and again we have $H \in F(\sigma_i) = \mathfrak{G}_{\sigma_i} F(\sigma_i)$ in view of Lemma 5 (5). Thus, $F(\sigma_i)$ is a τ-closed formation. Consequently, $F(\sigma_i) \in \mathfrak{l}^{\tau}_{\sigma_{n-1}}$. Hence $\mathfrak{F} \in (\mathfrak{l}^{\tau}_{\sigma_{n-1}})^{\sigma}$ and so $\mathfrak{l}^{\tau}_{\sigma_n} \subseteq (\mathfrak{l}^{\tau}_{\sigma_{n-1}})^{\sigma}$. The theorem is proved. □

Corollary 14 *Let $\mathfrak{F} = LF_{\sigma}(F)$ be a τ-closed n-multiply σ-local formation, where $n \geqslant 1$, and let F be the canonical σ-local definition of $\mathfrak{F}$. Then $F(\sigma_i) \in \mathfrak{l}^{\tau}_{\sigma_{n-1}}$ for all $\sigma_i \in \sigma(\mathfrak{F})$.*

Corollary 15 $(\mathfrak{l}^{S}_{\sigma_{n-1}})^{\sigma} = \mathfrak{l}^{S}_{\sigma_n}$ *for any positive integer n.*

Corollary 16 *Let $\mathfrak{F} = LF_{\sigma}(F)$ be a hereditary n-multiply σ-local formation, where $n \geqslant 1$, and let F be the canonical σ-local definition of $\mathfrak{F}$. Then $F(\sigma_i)$ is a hereditary n-multiply σ-local formation for all $\sigma_i \in \sigma(\mathfrak{F})$.*

Let $\tau(G) = S_n(G)$ be the collection of all normal subgroups of G for any group G (see [13, p.17]). Then Theorem 1 has the following consequence.

Corollary 17 $(\mathfrak{l}^{S_n}_{\sigma_{m-1}})^{\sigma} = \mathfrak{l}^{S_n}_{\sigma_m}$ *for any positive integer m.*

Corollary 18 *Let* $\mathfrak{F} = \mathsf{LF}_\sigma(F)$ *be a normally hereditary* m*-multiply* σ*-local formation, where* $m \geqslant 1$*, and let* F *be the canonical* σ*-local definition of* $\mathfrak{F}$*. Then* $F(\sigma_i)$ *is a normally hereditary* m*-multiply* σ*-local formation for all* $\sigma_i \in \sigma(\mathfrak{F})$*.*

If τ is a trivial subgroup functor from Theorem 1 we obtain

Corollary 19 $(l^\sigma_{n-1})^\sigma = l^\sigma_n$ *for any positive integer* n*.*

Corollary 20 *Let* $\mathfrak{F} = \mathsf{LF}_\sigma(F)$ *be an* n*-multiply* σ*-local formation, where* $n \geqslant 1$*, and let* F *be the canonical* σ*-local definition of* $\mathfrak{F}$*. Then* $F(\sigma_i) \in l^\sigma_{n-1}$ *for all* $\sigma_i \in \sigma(\mathfrak{F})$*.*

In the classical case, when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \dots, \}$ from Theorem 1 we obtain the following well-known result.

Corollary 21 (see [13], p.29) $(l^\tau_{n-1})^l = l^\tau_n$ *for any positive integer* n*.*

Lemma 22 *Let* $\mathfrak{M} = \mathsf{LF}_\sigma(m)$*, where* m *is a formation* σ*-function such that* $m(\sigma_i)$ *is a* τ*-closed formation for all* $\sigma_i \in \mathrm{Supp}(m)$*. Then* $\mathfrak{M}$ *is* τ*-closed.*

Proof — Let $G \in \mathfrak{M}$ and $H \in \tau(G)$. Let us show that $H \in \mathfrak{M}$. Note that $G/O_{\sigma'_i,\sigma_i}(G) \in m(\sigma_i)$ for all $\sigma_i \in \sigma(G)$ since $G \in \mathfrak{M}$. Hence

$$\tau\big(G/O_{\sigma'_i,\sigma_i}(G)\big) \subseteq m(\sigma_i)$$

for all $\sigma_i \in \sigma(G)$, since $m(\sigma_i)$ is τ-closed by the condition of the lemma.

Now we prove that $H/O_{\sigma'_i,\sigma_i}(H) \in m(\sigma_i)$ for every $\sigma_i \in \sigma(H)$. Let $\sigma_i \in \sigma(H)$. Then, clearly,

$$\sigma_i \in \sigma(G) \subseteq \mathrm{Supp}(m).$$

If $H \subseteq O_{\sigma'_i,\sigma_i}(G)$, then

$$H = O_{\sigma'_i,\sigma_i}(H) \quad \text{and} \quad H/O_{\sigma'_i,\sigma_i}(H) \simeq 1 \in m(\sigma_i) \neq \emptyset.$$

Assume $H \not\subseteq O_{\sigma'_i,\sigma_i}(G)$. Then since

$$HO_{\sigma'_i,\sigma_i}(G)/O_{\sigma'_i,\sigma_i}(G) \in \tau(G/O_{\sigma'_i,\sigma_i}(G))$$

and taking into account the following isomorphism, we have

$$H/O_{\sigma'_i,\sigma_i}(G) \cap H \simeq HO_{\sigma'_i,\sigma_i}(G)/O_{\sigma'_i,\sigma_i}(G) \in m(\sigma_i).$$

But $O_{\sigma_i',\sigma_i}(G) \cap H \subseteq O_{\sigma_i',\sigma_i}(H)$. Hence we have

$$H/O_{\sigma_i',\sigma_i}(H) \simeq (H/O_{\sigma_i',\sigma_i}(G) \cap H)/(O_{\sigma_i',\sigma_i}(H)/O_{\sigma_i',\sigma_i}(G) \cap H) \in \mathfrak{m}(\sigma_i).$$

Thus, $H/O_{\sigma_i',\sigma_i}(H) \in \mathfrak{m}(\sigma_i)$ for all $\sigma_i \in \sigma(H)$, so $H \in \mathfrak{M}$. Consequently, $\mathfrak{M}$ is a τ-closed formation. The lemma is proved. □

Proof of Theorem 2 — Let $\mathfrak{F}$ be a τ-closed n-multiply σ-local formation, where $n \geqslant 1$. By Corollary 14, $F(\sigma_i)$ is a τ-closed formation for all $\sigma_i \in \sigma(\mathfrak{F})$. On the other hand, if $F(\sigma_i)$ is a τ-closed formation for all $\sigma_i \in \sigma(\mathfrak{F})$, then, in view of Lemma 22, $\mathfrak{F}$ is a τ-closed formation as well. The theorem is proved. □

Let τ be an identity subgroup functor, that is, $\tau(G)$ is the collection of all subgroups of G for any group G. Then from Theorem 2 we obtain the following result.

Corollary 23 *Let $\mathfrak{F}$ be an n-multiply σ-local formation ($n \geqslant 1$) and let F be the canonical σ-local definition of $\mathfrak{F}$. Then $\mathfrak{F}$ is hereditary if and only if $F(\sigma_i)$ is a hereditary formation for all $\sigma_i \in \sigma(\mathfrak{F})$.*

Let $\tau(G) = S_n(G)$. Then from Theorem 2 we have the following result.

Corollary 24 *Let $\mathfrak{F}$ be an n-multiply σ-local formation ($n \geqslant 1$) and let F be the canonical σ-local definition of $\mathfrak{F}$. Then $\mathfrak{F}$ is normally hereditary if and only if $F(\sigma_i)$ is a normally hereditary formation for all $\sigma_i \in \sigma(\mathfrak{F})$.*

In the classical case, when $\sigma = \sigma^1$ from Theorem 2 we obtain the following result.

Corollary 25 *Let f be the maximal integrated local screen of a formation $\mathfrak{F}$. The formation $\mathfrak{F}$ is τ-closed if and only if for any prime p the formation $f(p)$ is τ-closed.*

In particular, if $\tau = S$ ($\tau = S_n$) from Theorem 2 we obtain the following well-known results.

Corollary 26 (see [12], p.41) *Let f be the maximal integrated local screen of a formation $\mathfrak{F}$. The formation $\mathfrak{F}$ is S-closed (S_n-closed) if and only if for any prime p the formation $f(p)$ is S-closed (S_n-closed).*

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REFERENCES

[1] Z. Chi – V.G. Safonov – A.N. Skiba: "On one application of the theory of n-multiply σ-local formations of finite groups", *Probl. Fiz. Mat. Tekh.* 2 (2018), 85–88.

[2] Z. Chi – V.G. Safonov – A.N. Skiba: "On n-multiply σ-local formations of finite groups", *Comm. Algebra* 47 (2019), 957–968.

[3] Z. Chi – A.N. Skiba: "A generalization of Kramer's theory", *Acta Math. Hungar.* 158 (2019), 87–99.

[4] Z. Chi – A.N. Skiba: "On Σ_t^σ-closed classes of finite groups", *Ukranian Math. J.* 70 (2019), 1966–1977.

[5] K. Doerk – T. Hawkes: "Finite Soluble Groups", *de Gruyter*, Berlin (1992).

[6] I.N. Safonova: "On minimal σ-local non-$\mathfrak{H}$-formations of finite groups", *Probl. Fiz. Mat. Tekh.* 4 (2020), 105–112.

[7] I.N. Safonova: "On τ-closed n-multiply σ-local formations of finite groups" (2021); *arXiv*:2105.00430.

[8] I.N. Safonova: "Some properties of n-multiply σ-local formations of finite groups", *Asian-Eur. J. Math.* 15 (2022), 2250138, 12pp.

[9] I.N. Safonova: "A criterion for σ-locality of a non-empty formation", *Comm. Algebra* 50 (2022), 2366–2376.

[10] I.N. Safonova – V.G. Safonov: "On some properties of the lattice of totally σ-local formations of finite groups", *Zh. Belruss. Gos. Univ. Mat. Inform.* 3 (2020), 6–16.

[11] V.V. Shcherbina – V.G. Safonov: "On sublattices of the lattice of partially totally saturated formations of finite groups", *Belgorod State Univ. Sci. Bull. Math. Phys.* 51 (2019), 64–87.

[12] L.A. Shemetkov: "Formations of Finite Groups", *Nauka*, Moscow (1978).

[13] A.N. Skiba: "Algebra of Formations", *Belarus. Navuka*, Minsk (1997).

[14] A.N. Skiba: "On σ-properties of finite groups I", *Probl. Fiz. Mat. Tekh.* 4 (2014), 89–96.

[15] A.N. Skiba: "On σ-subnormal and σ-permutable subgroups of finite groups", *J. Algebra* 436 (2015), 1–16.

[16] A.N. Skiba: "On some results in the theory of finite partially soluble groups", *Commun. Math. Stat.* 4 (2016), 281–309.

[17] A.N. Skiba: "On one generalization of the local formations", *Probl. Fiz. Mat. Tekh.* 1 (2018), 79–82.

[18] A.N. Skiba: "On sublattices of the subgroup lattice defined by formation Fitting sets", *J. Algebra* 550 (2020), 69–85.

[19] A. Tsarev: "Laws of the lattices of σ-local formations of finite groups", *Mediterr. J. Math.* 17 (2020), Paper No. 75, 13pp.

[20] A. Tsarev – A. Kukharev: "Classes of monoids with applications: formations of languages and multiply local formations of finite groups", *Rend. Circ. Mat. Palermo* (2) 70 (2021), 1257–1268.

[21] N.N. Vorobiev – I.I. Staselko – A.O. Khodzhagulyev: "Separated lattices of multiply σ-local formations", *Sib. Math. J.* 62 (2021), 586–597.

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