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Join-Distributive Elements in the Lattice of Closed Subgroups of a Profinite Group

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Abstract

In [8], the lattice of closed (and open) subgroups of a profinite group has extensively been examined from many classical lattice-theoretical point of views, such as distributivity, modularity and decomposability. The aim of this paper is to give a further push forward to the study of these lattices by characterizing its join-distributive, standard and neutral elements in terms of the profinite structure.

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1 Introduction

Let G be a group. Although the lattice $\mathcal{L}(G)$ of all subgroups of G does not not provide information about the abelianity or nilpotency of G (think of the Klein 4-group and the symmetric group of degree 3), it is still an extremely relevant source of information about its group-theoretical structure. For example, a well-known theorem of Ore shows that G is locally cyclic if and only if $\mathcal{L}(G)$ is distributive; while a theorem of Suzuki shows that $\mathcal{L}(G)$ can be decomposed into the direct product of two non-trivial lattices if and only if G is periodic and is the direct product of two non-trivial subgroups H and K such that $\pi(H) \cap \pi(K) = \{1\}$. These and many other classical results

can be found in the monograph [19], but we also refer the interest reader to [1],[2],[3],[5],[6],[9],[10] and their bibliographies for more recent results and research lines on the topic.

Lately, a systematic approach to the study of abstract group theoretical properties in topological groups et simila is being carried out by some authors (see e.g. [4],[7],[11],[12],[13],[14],[15],[16],[17]), and it was just a matter of time before someone would have starting looking at topological groups (and more in particular at profinite groups) from the point of view of the lattice of their closed (or open) subgroups. In fact, [8] laid the foundation of a lattice-theoretical theory of profinite groups, providing for example analogues of the abovementioned results of Ore and Suzuki.

The aim of this paper is to give a further contribution to this foundational study by giving a profinite description of join-distributive, standard and neutral elements of the lattice $\mathcal{L}_c(G)$ (resp., $\mathcal{L}_o(G)$) of all closed (resp., open) subgroups of a profinite group G. These are relevant lattice-theoretical concepts that were introduced by Ore, Grätzer and Birkhoff in arbitrary lattices, and studied in the subgroup lattice of a group by many authors such as Curzio, Ivanov, Napolitani, Suzuki and Zappa (see Section 3.4 of [19]). Our main results provide characterizations of join-distributive elements (see Theorem 12) and neutral elements (see Theorems 15 and 17, and Corollary 18) of the closed (and open) subgroup lattice of a profinite group; note also that join-distributive elements are shown to coincide with the standard ones (see Corollary 9). In our study, we give results about meet-distributive, uniquely complemented and join-quasi-distributive elements that can be of an independent interest (see Theorems 10, 19 and Lemma 14).

Our notation is standard, and can for instance be found in [18] (for what concerns profinite groups) and in [19] (for what concerns the subgroup lattice of a group). Also, since we deal with both group-theoretical concepts and their profinite analogues, we stick here to the rule that whenever we work with a topological concept, we add the adjective "topological". For example, if p is a prime, then a topological Sylow p-subgroup of a profinite group G is a maximal closed pro-p subgroup of G, while a Sylow p-subgroup is a maximal p-subgroup of G. Further terminology we will frequently use is the following:

- If G is a profinite group, then $N \trianglelefteq_o G$ means that N is an open normal subgroup of G.

- If G is a group (resp., profinite group), then we denote by $\pi(G)$ (resp., $\pi^*(G)$) the set of all primes p for which G has non-trivial elements of order p (resp., has non-trivial procyclic pro-p subgroups).
- If X is a subgroup of a profinite group G, then its topological closure in G will be denoted by $\overline{X}$.

2 Results

We start by recalling the basic lattice-theoretical definitions we need. Let L be a lattice. An element a of L is *neutral* if every triple $a, x, y \in L$ generates a distributive sublattice of L. It follows from Theorem 3.4.1 of [19] that a is a neutral element if and only if the following three conditions hold for every $x, y \in L$:

(1) $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$;

(2) $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$;

(3) if $a \vee x = a \vee y$ and $a \wedge x = a \wedge y$, then $x = y$.

An element $a \in L$ is called *join-distributive* if it satisfies (1), *meet-distributive* if it satisfies (2), *uniquely complemented* if it satisfies (3), *join-quasi-distributive* if (1) holds for all $x, y \in L$ such that $a \vee x = a \vee y$, *meet-quasi-distributive* if (2) holds for all $x, y \in L$ with $a \wedge x = a \wedge y$, and *standard* if it is join-distributive and uniquely complemented.

In arbitrary lattices there are only the obvious relations between the above concepts, but in subgroup lattices there is more to say. Let G be a group. It turns out for example that every join- or meet-distributive element of $\mathcal{L}(G)$ is a normal subgroup of G and is uniquely complemented. Also, the properties of being join-quasi-distributive and join-distributive are equivalent in $\mathcal{L}(G)$, although there are finite groups having subgroups that are meet-quasi-distributive elements of their subgroup lattices but not meet-distributive elements.

Let G be a profinite group. The set of all closed subgroups of G will be denoted by $\mathcal{L}_c(G)$, while the set of all open subgroups of G will be denoted by $\mathcal{L}_o(G)$. If X and Y are subgroups of G, then $X \vee_c Y$ denotes the *closed join* of X and Y, that is, the closure of $X \vee Y = \langle X, Y \rangle$. It is clear that $\big(\mathcal{L}_c(G), \vee_c, \cap\big)$ is a lattice of which

$$\big(\mathcal{L}_o(G), \vee_c, \cap\big) = \big(\mathcal{L}_o(G), \vee, \cap\big)$$

is a sublattice; note also that if $Y \leqslant X$ are closed subgroups of G, then $[X/Y]_c$ will denote the sublattice of $\mathcal{L}_c(G)$ made by all closed subgroups of G contained between Y and X. Of course, the previously explained lattice-theoretical concepts can be specialized to the lattices $\mathcal{L}_c(G)$ and $\mathcal{L}_o(G)$, and in order to simplify notation we give the following definitions in which we speak of a "closed subgroup having a given property" instead of a "closed subgroup that is an element of a certain lattice and has some given property as such".

Definition 1 Let G be a profinite group, and let X be a closed subgroup of G. We say that X is *c-uniquely complemented* in G if X is a uniquely complemented element of $\mathcal{L}_c(G)$, or in other words, if whenever $Y, Z \in \mathcal{L}_c(G)$ satisfy

$$X \vee_c Y = X \vee_c Z \quad \text{and} \quad X \cap Y = X \cap Z,$$

then $Y = Z$.

Similarly, if X is an open subgroup of G, we say that X is *o-uniquely complemented* if X is an o-uniquely complemented element of $\mathcal{L}_o(G)$.

Definition 2 Let G be a profinite group. We say that a closed subgroup X of G is *c-join-distributive* (resp., *o-join-distributive*) if

$$X \vee_c (U \cap V) = (X \vee_c U) \cap (X \vee_c V) \tag{2.1}$$

for all $U, V \in \mathcal{L}_c(G)$ (resp., $U, V \in \mathcal{L}_o(G)$).

Remark 3 Note that if $U, V \in \mathcal{L}_o(G)$, then Equation (2.1) becomes

$$X \vee (U \cap V) = (X \vee U) \cap (X \vee V)$$

because the join of an open subgroup and any other subgroup is always open and so closed.

Obviously, every c-join-distributive subgroup of a profinite group is also o-join-distributive, but as we shall soon see, also the converse of this fact holds; this will allow us to generally speak of *join-distributive* subgroups of a profinite group (see Corollary 7).

Lemma 4 *Let* G *be a profinite group. If* X *is an o-join-distributive subgroup of* G, *and* $N \trianglelefteq_o G$, *then* XN/N *is a join-distributive element of the lattice* $\mathcal{L}(G/N)$.

Proof — Let U/N and V/N be subgroups of G/N. Then U and V are open in G, so

$$X \vee (U \cap V) = (X \vee U) \cap (X \vee V)$$

by Remark 3. Now,

$$\begin{aligned}\langle XN/N, (U/N) \cap (V/N)\rangle &= \langle XN/N, (U \cap V)/N\rangle = \langle X, U \cap V\rangle/N \\ &= (\langle X, U\rangle \cap \langle X, V\rangle)/N = (\langle X, U\rangle/N) \cap (\langle X, V\rangle/N).\end{aligned}$$

Therefore XN/N is a join-distributive element of $\mathcal{L}(G/N)$. □

Lemma 5 *Let X be a closed subgroup of a profinite group G. If XN/N is a join-distributive element of $\mathcal{L}(G/N)$ for all $N \trianglelefteq_o G$, then X is normal in G.*

Proof — Let $N \trianglelefteq_o G$. Since XN/N is a join-distributive element of $\mathcal{L}(G/N)$, it follows from Theorem 3.4.2 of [19] that $XN/N \trianglelefteq G/N$. Now, X is the intersection of all subgroups XN, where $N \trianglelefteq_o G$, and so X is normal in G. □

Corollary 6 *Let G be a profinite group. If X is an o-join-distributive subgroup of G, then X is normal in G.*

Proof — If $N \trianglelefteq_o G$, then XN/N is a join-distributive element of $\mathcal{L}(G/N)$ by Lemma 4. Thus, it follows from Lemma 5 that $X \trianglelefteq G$ and we are done. □

An immediate consequence of the above result is that we can replace the closed joins in Equation (2.1) with joins, even in the case of a c-join-distributive subgroup: in fact, an easy application of Tychonoff theorem (see [18], Chapter 1, Theorem 3) yields that the product of two closed subgroups is always closed. This will be tacitly employed in the following result.

Theorem 7 *Let X be a closed subgroup of a profinite group G. Then the following are equivalent:*

(1) *X is c-join-distributive.*

(2) *X is o-join-distributive.*

(3) *XN/N is a join distributive element of $\mathcal{L}(G/N)$ for all $N \trianglelefteq_o G$.*

Proof — Clearly, (1) implies (2), while (2) implies (3) by Lemma 4. Assume (3), and let $U, V \in \mathcal{L}_c(G)$. Let $\mathcal{U}$ be the family of all open subgroups of G containing U, and let $\mathcal{V}$ be the set of all open subgroups of G containing V. Then

$$U = \bigcap_{H \in \mathcal{U}} H \quad \text{and} \quad V = \bigcap_{K \in \mathcal{V}} K.$$

It follows from Lemma 5 that $X \trianglelefteq G$. Let

$$\mathcal{V} \cap \mathcal{U} = \{H \cap K : H \in \mathcal{U},\ K \in \mathcal{V}\}.$$

Now, Proposition 2.1.4 of [18] yields that

$$X(U \cap V) = X \cdot \bigcap_{H \in \mathcal{U},\, K \in \mathcal{V}} (H \cap K) = \bigcap_{H \in \mathcal{U},\, K \in \mathcal{V}} \big(X \cdot (H \cap K)\big)$$

$$= \bigcap_{H \in \mathcal{U},\, K \in \mathcal{V}} (XH \cap XK) = \bigcap_{H \in \mathcal{U}} (XH) \cap \bigcap_{K \in \mathcal{V}} (XK)$$

$$= \left(X \bigcap_{H \in \mathcal{U}} H\right) \cap \left(X \bigcap_{K \in \mathcal{V}} K\right) = XU \cap XV.$$

Thus, (1) holds and the statement is proved. □

As we have already mentioned, the previous result allows us to freely speak of a join-distributive subgroup of a profinite group, with no distinction as to which lattice we are considering.

Theorem 8 *Let G be a profinite group. If X is any join-distributive subgroup of G, then X is c-uniquely complemented.*

Proof — Let U and V be closed subgroups of G such that $XU = XV$ (recall that by Lemma 5, $X \trianglelefteq G$) and $X \cap U = X \cap V$. Then

$$XU = XV = X(U \cap V)$$

and therefore

$$U = XU \cap U = \big(X(U \cap V)\big) \cap U = (U \cap V)(X \cap U)$$
$$= (U \cap V)(X \cap V) = \big(X(U \cap V)\big) \cap V = XV \cap V = V$$

and we are done. □

Corollary 9 *Let* G *be a profinite group. Then the join-distributive subgroups of* G *are precisely the standard elements of* $\mathcal{L}_c(G)$*. Moreover, the join-distributive open subgroups of* G *are precisely the standard elements of* $\mathcal{L}_o(G)$*.*

Our next result shows that similarly to what happen in arbitrary subgroup lattices, also in our situation the join-distributive elements coincide with the join-quasi-distributive ones.

Theorem 10 *Let* G *be a profinite group, and let* X *be a closed subgroup of* G*. The following conditions are equivalent:*

(1) X *is join-distributive.*

(2) $X \vee_c (U \cap V) = (X \vee_c U) \cap (X \vee_c V)$ *for all* $U, V \in \mathcal{L}_c(G)$ *with* $U \vee_c X = V \vee_c X$*.*

(3) $X \vee (U \cap V) = (X \vee U) \cap (X \vee V)$ *for all* $U, V \in \mathcal{L}_o(G)$ *with* $U \vee X = V \vee X$*.*

(4) XN/N *is a join-quasi-distributive element of* $\mathcal{L}(G/N)$ *for all* $N \trianglelefteq_o G$*.*

Proof — Clearly, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Assume (4), and let $N \trianglelefteq_o G$. Then XN/N is a join-distributive element of $\mathcal{L}(G/N)$ by Theorem 3.4.3 of [19], so X is join-distributive by Theorem 7, and we obtain (1).

Assume (3) and let $N \trianglelefteq_o G$. Suppose that U/N and V/N are subgroups of G/N satisfying $\langle U, X\rangle/N = \langle V, X\rangle/N$; in particular, U, V, $\langle U, X\rangle$ and $\langle V, X\rangle$ are open in G. Then

$$X \vee (U \cap V) = (X \vee U) \cap (X \vee V)$$

by (3), and hence

$$\begin{aligned}\big\langle XN/N, (U/N) \cap (V/N)\big\rangle &= \big\langle XN/N, (U \cap V)/N\big\rangle = \big\langle X, U \cap V\big\rangle/N \\ &= \big(\langle X, U\rangle \cap \langle X, V\rangle\big)/N = \big(\langle X, U\rangle/N\big) \cap \big(\langle X, V\rangle/N\big).\end{aligned}$$

Thus XN/N is a join-quasi-distributive element of $\mathcal{L}(G/N)$ and we have proved (4). □

Before stating and proving our first main result, we provide an example showing that join-distributive elements in the closed (or, open) subgroup lattice of a profinite group do not necessarily coincide with the join-distributive elements in the subgroup lattice.

Example 11 *There is a profinite group* G *having a join-distributive subgroup that is not a join-distributive element of* $\mathcal{L}(G)$.

Proof — Let G be a pro-cyclic pro-p group for some prime p, and let $X = G^p$. Since $\mathcal{L}_c(G)$ is distributive, we have that all closed subgroups of G are join-distributive, so X is join-distributive. Now, let $g \in G \setminus X$; so g is non-trivial and non-periodic. Since X is an abelian group of infinite rank, there is a non-trivial aperiodic element $x \in X$ such that $\langle g\rangle \cap \langle x\rangle = \{1\}$. Then gx is an element of $G \setminus X$ such that $\langle gx\rangle \cap \langle g\rangle = \{1\}$, so

$$\langle X, gx\rangle = G = \langle X, g\rangle \quad \text{and} \quad X \vee \big(\langle g\rangle \cap \langle gx\rangle\big) = X.$$

Therefore X is not a join-distributive element of $\mathcal{L}(G)$. □

Theorem 12 *Let* X *be a closed subgroup of a profinite group* G. *Then* X *is join-distributive if and only if there exists closed subgroups* L *and* K *of* G *with the following properties:*

(a) $G = L \times K$ *and* $\pi^*(L) \cap \pi^*(K) = \emptyset$.

(b) $X = L \times (X \cap K)$ *and* $X \cap K \leqslant Z(G)$.

(c) *For every* $p \in \pi^*(X \cap K)$, *the topological Sylow* p*-subgroups of* G *are procyclic or* (*finite*) *generalized quaternion groups.*

Proof — Suppose first that G has subgroups L and K satisfying (a), (b) and (c). Let $N \trianglelefteq_o G$. By Theorem 7, we only need to show that XN/N is a join-distributive element of $\mathcal{L}(G/N)$.

Clearly, $G/N = LN/N \times KN/N$ so

$$XN/N = XN/N \cap G/N = LN/N \cdot (XN/N \cap KN/N).$$

Now, write $N = N_1 \times N_2$, where N_1 is a closed subgroup of L, and N_2 is a closed subgroup of K. Then

$$XN \cap KN = \Big(L \cdot \big((X \cap K)N_2\big)\Big) \cap \big(N_1 \cdot (KN_2)\big) = N_1 \cdot \big((X \cap K)N_2\big)$$

and consequently

$$XN/N = LN/N \cdot (X \cap K)N/N.$$

Thus, if p is a prime dividing the order of $XN/N \cap KN/N$, then $p \in \pi^*(X \cap K)$, so the topological Sylow p-subgroups of G are either procyclic or (finite) generalized quaternion groups. However, in

the latter case the Sylow 2-subgroup of $X \cap K$ coincides with the centre of the Sylow 2-subgroup of G, so the Sylow 2-subgroups of G/N are generalized quaternion groups as well. Therefore, Theorem 3.4.8 of [19] yields that XN/N is a join-distributive element of $\mathcal{L}(G)$, as we claimed.

Assume conversely that X is a join-distributive subgroup of G. Let π be the set of all primes p such that the topological Sylow p-subgroups of X are not central in G. Now, if N is any open normal subgroup of G, we claim that $G/N = U/N \times V/N$, where

$$\pi(U/N) \subseteq \pi, \quad \pi(V/N) \subseteq \pi' \quad \text{and} \quad U/N \leqslant XN/N.$$

In fact, XN/N is a join-distributive element of $\mathcal{L}(G/N)$, so it follows from Theorem 3.4.8 of [19] that there are subgroups $L_N, K_N \geqslant N$ such that:

(1) $G/N = L_N/N \times K_N/N$ and $\pi(L_N/N) \cap \pi(K_N/N) = \emptyset$;

(2) $XN/N = L_N/N \times (X \cap K_N)N/N$ and $(X \cap K_N)N/N \leqslant Z(G/N)$;

(3) no Sylow p-subgroup of L_N/N is central in G.

Obviously, $\pi(L_N/N) \subseteq \pi$. Suppose $q \in \pi\big((X \cap K_N)N/N\big) \cap \pi$. Then there exists an open normal subgroup M of G contained in N such that the Sylow q-subgroups of XM/M are not central in G. However, this means that the Sylow q-subgroups of XM/M are Sylow q-subgroups of G/M, so also the Sylow q-subgroups of XN/N are Sylow q-subgroups of G/N. Consequently, the Sylow q-subgroup of G/N is central, and so is a direct factor of G/N. This means that G/N decomposes as required.

Therefore there exist closed subgroups L and K of G such that

$$G = L \times K,\ X = L \times (X \cap K),\ \pi^*(L) \cap \pi^*(X \cap K) = \emptyset,\ \text{and}\ X \cap K \leqslant Z(G).$$

Also, we may require that every prime $p \in \pi^*(X \cap K)$ is contained in $\pi^*\big(G/(X \cap K)\big)$ since otherwise the topological Sylow p-subgroup of $X \cap K$ would be a direct factor of K and so of G.

In order to complete the proof, we need to show that (c) holds. Let $p \in \pi^*(X \cap K)$. Then there exists an open normal subgroup N of G such that, for every $M \trianglelefteq_o G$ with $M \leqslant N$, we have

$$p \in \pi(XM/M) \cap \pi(G/XM),$$

by Theorem 3.4.8 of [19], this yields that the Sylow p-subgroups of G/M are either cyclic or generalized quaternion. Since this holds for all open normal subgroups of G contained in N, we have that the topological Sylow p-subgroups of G are procyclic or (finite) and generalized quaternion. The statement is proved. □

Now, we turn to our second main result, namely, to the characterization of neutral elements in the closed (and open) subgroup lattices of a profinite group.

Definition 13 Let G be a profinite group. We say that a closed subgroup X of G is c-*neutral* if it is a neutral element of $\mathcal{L}_c(G)$. Similarly, an open subgroup of G is o-*neutral* if it is a neutral element of $\mathcal{L}_o(G)$.

Before proving Theorem 15, we need the following relevant properties of c-uniquely complements subgroups of a profinite group.

Lemma 14 *Let G be a profinite group. If X is a c-uniquely complemented subgroup of G, then the following properties hold:*

(1) $N_X(X \cap Y) \leqslant N_G(Y)$ *for all* $Y \in \mathcal{L}_c(G)$.

(2) *Let* $p \in \pi^*(X)$. *If Y is a procyclic pro-p subgroup of G, then* $Y \cap X \neq \{1\}$.

Proof — (1) Let $z \in N_X(X \cap Y)$. Then $X \cap Y = (X \cap Y)^z = X \cap Y^z$ and $X \vee_c Y = (X \vee_c Y)^z = X \vee_c Y^z$. Since X is uniquely complemented, we obtain that $Y = Y^z$, thus proving (1).

(2) Suppose $Y \cap X = \{1\}$. Thus, we may further assume that $|Y| = p$, when Y is finite. Now, $X \leqslant N_G(Y)$ by (1). Let P be a procyclic pro-p subgroup of X that is either infinite or cyclic of order p, and that centralizes Y. In any case, we can find two distinct (non-necessarily procyclic) pro-p subgroups L and M of G such that

$$X \vee_c L = X \vee_c M = XY \quad \text{and} \quad X \cap L = X \cap M.$$

This contradiction completes the proof of (2). □

Theorem 15 *Let G be a profinite group and let X be a closed subgroup of G. Then X is c-neutral if and only if there exist closed subgroups L, M and N of G such that:*

(a) $G = L \times HM$, $M \trianglelefteq G$, $\pi^*(L) \cap \pi^*(H) = \pi^*(L) \cap \pi^*(M) = \emptyset$ *and* $\pi^*(H) \cap \pi^*(M) = \emptyset$.

(b) $X = L \times (X \cap H)$ *and* $X \cap H \leqslant Z(G)$.

(c) H *is pronilpotent and the topological Sylow subgroups are either procyclic or (finite) generalized quaternion groups.*

Proof — First, suppose X is c-neutral, so Theorem 3.4.1 of [19] yields that X is join-distributive. By Theorem 12 there are subgroups L and K satisfying:

(i) $G = L \times K$ and $\pi^*(L) \cap \pi^*(K) = \emptyset$.

(ii) $X = L \times (X \cap K)$ and $X \cap K \leqslant Z(G)$.

(iii) For every $p \in \pi^*(X \cap K)$, the topological Sylow p-subgroups of G are procyclic or (finite) generalized quaternion groups.

Let $\pi = \pi^*(X)$, and let M be the set of all topological π'-elements of G. Let x and y be topological π'-elements. Then

$$X \cap \overline{\langle x \rangle} = \{1\} = X \cap \overline{\langle y \rangle};$$

but X is a meet-distributive element of $\mathcal{L}_c(G)$, and so

$$X \cap \overline{\langle x, y \rangle} = \{1\}.$$

By Lemma 14 (2), $\pi^*(X) \cap \pi^*\big(\overline{\langle x, y \rangle}\big) = \emptyset$, so xy^{-1} is a topological π'-element of G. Then M is a subgroup of G, and so is also a closed normal subgroup of G that is contained in K. The topological version of the Schur–Zassenhaus theorem (see [18], Theorem 2.3.15) yields that there is a complement H to M in K. Since $X \trianglelefteq G$ and LH is a topological Hall subgroup of G, we have that $X \leqslant LH$, so $X \cap K = X \cap H$. Thus, (a) and (b) are satisfied. Since every prime in $\pi^*(H)$ belongs to $\pi^*(X \cap H)$, it remains to prove that H is pronilpotent.

Suppose S and T are different topological Sylow p-subgroups of H for a given prime p. Since $X \trianglelefteq G$, it follows that $X \cap S$ and $X \cap T$ are topological Sylow p-subgroups of $X \cap H$.

Since $X \cap H$ is (pro)abelian, we have that $X \cap S = X \cap T$. Moreover, the fact that X is meet-distributive element of $\mathcal{L}_c(G)$ implies that

$$X \cap \overline{\langle S, T \rangle} = \overline{\langle X \cap S, X \cap T \rangle} = X \cap S$$

is a pro-p group. Since $S \neq T$, we have that $R = \overline{\langle S, T \rangle}$ is not a pro-p group, so there is $q \in \pi^*(R) \setminus \{p\}$. But $R \leqslant H$, so $q \in \pi^*(X)$,

so $q \in \pi^*(X \cap \overline{\langle S, T\rangle})$ by Lemma 14, a contradiction. Then H has only one topological Sylow p-subgroup for every prime p and is therefore pronilpotent.

Suppose conversely that G contains closed subgroup L, H and M satisfying (a)–(c) and let $U, V \in \mathcal{L}_c(G)$. We only have to show that

$$X \cap (U \vee_c V) = (X \cap U) \vee_c (X \cap V). \tag{2.2}$$

By [8], Theorem 4.1, we may assume $L = \{1\}$, so $X \leqslant Z(G)$. Also, since the topological Sylow subgroups of H are either procyclic or generalized quaternion groups, we have that (2.2) is satisfied whenever U and V are subgroups of H. Furthermore, $G/M \simeq H$, so MX is a meet-distributive element of $[G/M]_c$.

Now, if Z is any closed subgroup of G, we have

$$M(X \cap Z) = M\big((M \cap Z) \times (X \cap Z)\big) = M(MX \cap Z) = MX \cap MZ.$$

Therefore,

$$\begin{aligned} M\big(X \cap (U \vee_c V)\big) &= MX \cap M(U \vee_c V) = MX \cap (M \vee_c U \vee_c V) \\ &= MX \cap (MU \vee_c MV) = (MX \cap MU) \vee_c (MX \cap MV) \\ &= M(X \cap U) \vee_c M(X \cap V) = M\big((X \cap U) \vee_c (X \cap V)\big). \end{aligned}$$

It follows that Equation (2.2) is satisfied and we are done. □

Corollary 16 *Let G be a profinite group. If X is any c-neutral subgroup of G, then XN/N is c-neutral in G/N for all closed normal subgroups N of G.*

Theorem 17 *Let G be a profinite group and let X be a closed subgroup of G. Then X is c-neutral if and only if XN/N is a neutral element of $\mathcal{L}(G/N)$ for all $N \trianglelefteq_o G$.*

Proof — Suppose first that XN/N is a neutral element of $\mathcal{L}(G/N)$ for all $N \trianglelefteq_o G$. Then X is join-distributive by Theorem 7, so we may assume by Theorem 12 that $X \leqslant Z(G)$ and that the topological Sylow p-subgroups of G are either procyclic or (finite) generalized quaternion groups for all $p \in \pi = \pi^*(X)$.

Let $N \trianglelefteq_o (G)$. Then XN/N is a neutral element of $\mathcal{L}(G/N)$, so we can write

$$G/N = H_N/N \leftthreetimes M_N/N,$$

where $\pi(H_N/N) = \pi(XN/N)$, $\pi(H_N/N) \cap \pi(M_N/N) = \emptyset$, H_N/N is nilpotent.

Suppose that there is a prime $p \in \pi(M_N/N) \cap \pi$. Then we can find an open normal subgroup L contained in N such that $p \in \pi(XL/L)$, so $p \notin \pi(M_L/L)$. This is easily seen to imply that the set of all π'-elements of G/N is a subgroup. The arbitrariness of N shows that G has a unique topological Hall π'-subgroup M. Then the topological version of the Schur–Zassenhaus theorem (see [18], Theorem 2.3.15) yields that M has a complement H in G and that any two complements of M are conjugate. Since $X \trianglelefteq G$, we have that $X \leqslant H$. Thus, $\pi^*(X) = \pi^*(H)$.

Let $N \trianglelefteq_o G$. Since $\pi^*(X) = \pi^*(H)$, there is an open normal subgroup L contained in N such that $\pi(HN/N) \subseteq \pi(XL/L)$. Let m be the largest $\pi(XL/L)$-number dividing $|G/N|$. Since the nilpotent subgroup H_L/L is a Hall $\pi(XL/L)$-subgroup of G/L complemented by the normal subgroup M_L/L, so H_LN/N is a nilpotent Hall $\pi(XL/L)$-subgroup of G/N complemented by the normal subgroup M_LN/N, and so $|H_LN/N| = m$. Also, HN/N is a Hall $\pi(HN/N)$-subgroup of G/N complemented by the π'-subgroup MN/N. Thus, $|HN/N| = m$, and consequently HN/N and H_LN/N are both complements to M_LN/N; in particular, they are conjugate by the usual Schur–Zassenhaus theorem. Therefore HN/N is nilpotent, and the arbitrariness of N shows that H is pronilpotent. Therefore X is c-neutral by Theorem 15.

Conversely, if X is c-neutral, then Corollary 16 implies that XN/N is a neutral element of $\mathcal{L}(G/N)$ for all $N \trianglelefteq_o G$. □

Corollary 18 *Let G be a profinite group and let X be an open subgroup of G. Then X is o-neutral if and only if X is c-neutral.*

We end this paper by proving an analogue of Corollary 6 and Theorem 8 for meet-distributive elements of the closed (resp., open) subgroup lattice of a profinite group; these will be referred to simply as *c-meet-distributive* (resp., *o-meet-distributive*) subgroups.

Theorem 19 *Let G be a profinite group.*

1) *If X is c-meet-distributive, then $X \trianglelefteq G$ and X is c-uniquely complemented.*

2) *If X is o-meet-distributive, then $X \trianglelefteq G$ and X is o-uniquely complemented.*

Proof — 1) Let $y \in G$ and put $H = \overline{\langle y \rangle}$. Then

$$X^y \cap H = (X \cap H)^y = X \cap H$$

since H is abelian. Thus,

$$X = X \cap (X^y \vee_c H) = (X \cap X^y) \vee_c (X \cap H) = X \cap X^y$$

and hence $X \leqslant X^y$. The arbitrariness of $y \in G$ shows that $X \trianglelefteq G$.

Now, let U, V be closed subgroups of G such that

$$X \vee_c U = X \vee_c V \quad \text{and} \quad X \cap U = X \cap V.$$

Then

$$X \cap (U \vee_c V) = X \cap V = X \cap U \leqslant U \cap V$$

and therefore since $X \trianglelefteq G$,

$$\begin{aligned} U &= \big(X \cap (U \vee_c V)\big)U = XU \cap (U \vee_c V) \\ &= XV \cap (U \vee_c V) = \big(X \cap (U \vee_c V)\big)V = V. \end{aligned}$$

Thus, X is uniquely complemented.

2) If N be any open normal subgroup of G contained in X, then X/N is a meet-distributive element of $\mathcal{L}(G/N)$, so $X/N \trianglelefteq G/N$ by [19], Theorem 3.4.2, and hence $X \trianglelefteq G$. Now, if U and V are open subgroups of G such that

$$XU = XV \quad \text{and} \quad X \cap U = X \cap V,$$

and N is an open normal subgroup of G contained in $X \cap U \cap V$, then X/N is a meet-distributive element of $\mathcal{L}(G/N)$ and so is uniquely complement too (see again [19], Theorem 3.4.2). Since

$$X/N \cdot U/N = X/N \cdot V/N \quad \text{and} \quad X/N \cap U/N = X/N \cap V/N,$$

we have that $U/N = V/N$ and so $U = V$. □

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