

Advances in Group Theory and Applications
© 2024 AGTA - www.advgrouptheory.com/journal
18 (2024), pp. 3–40 ISSN: 2499-1287
DOI: 10.32037/agta-2024-001

Beauville Structures for Quotients of Generalised GGS-groups

Elena Di Domenico — Şükran Gül
Anitha Thillaisundaram

(*Received Jan 16, 2023; Accepted June 12, 2023 — Communicated by N. Mazza*)

Abstract

A finite group with a Beauville structure gives rise to a certain compact complex surface called a Beauville surface. Gül and Uria-Albizuri showed that quotients of the periodic Grigorchuk–Gupta–Sidki (GGS-)groups that act on the p-adic tree, for p an odd prime, admit Beauville structures. We extend their result by showing that quotients of infinite periodic GGS-groups acting on p^n-adic trees, for p any prime and $n \geqslant 2$, also admit Beauville structures.

Mathematics Subject Classification (2020): 20E08, 20D15, 14J29

Keywords: groups acting on rooted trees; finite p-group; Beauville structure

1 Introduction

A *Beauville surface* is a compact complex surface that is isomorphic to $(C_1 \times C_2)/G$, where

- C_1 and C_2 are algebraic curves of genus at least 2, and G is a finite group acting freely on $C_1 \times C_2$ by holomorphic transformations,

- the group G acts faithfully on the curves C_i with $C_i/G \simeq \mathbb{P}_1(\mathbb{C})$ and the covering map $C_i \to C_i/G$ is ramified over three points, for $i \in \{1, 2\}$.

The group G is then said to be a *Beauville group*.

One can reformulate the condition for a finite group G to be a Beauville group purely in group-theoretical terms: for $x, y \in G$, let

$$\Sigma(x, y) = \bigcup_{g \in G} \left(\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g \right),$$

that is, the union of all conjugates of the cyclic subgroups generated by x, y and xy. Then G is a Beauville group if and only if G is 2-generated and there exist generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{1\}$. The sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are then called a *Beauville structure* for G.

Beauville groups have been intensely studied in recent times; see surveys [1],[8],[9],[18]. For example, the abelian Beauville groups were classified by Catanese [4]: a finite abelian group G is a Beauville group if and only if $G \simeq C_n \times C_n$ for $n > 1$ with $\gcd(n, 6) = 1$. After abelian groups, the most natural class of finite groups to consider are nilpotent groups. The determination of nilpotent Beauville groups is easily reduced to the case of p-groups.

In [1], it was shown that there are non-abelian Beauville p-groups of order p^n for every $p \geqslant 5$ and every $n \geqslant 3$. The first explicit infinite family of Beauville 2-groups was constructed in [2]. In [22], Stix and Vdovina constructed an infinite family of Beauville p-groups, for every prime p, by considering quotients of ordinary triangle groups. In [11], Fernández-Alcober and Gül extended Catanese's criterion for abelian Beauville groups to finite p-groups satisfying certain conditions which are much weaker than commutativity. They also give the first explicit infinite family of Beauville 3-groups, and they show that there are Beauville 3-groups of order 3^n for every $n \geqslant 5$.

In [17], Gül and Uria-Albizuri showed that the quotients of periodic Grigorchuk–Gupta–Sidki (GGS-)groups admit Beauville structures, giving another infinite family of Beauville p-groups, for p an odd prime. The GGS-groups were some of the early examples of groups acting on rooted trees, which first arose as easily describable examples of infinite finitely generated periodic groups. Since then, groups acting on rooted trees have provided many other interesting and exotic examples, such as infinite finitely generated groups of in-

termediate word growth, infinite finitely generated amenable but not elementary amenable groups, etc; see for instance [16],[3].

The GGS-groups are infinite groups acting faithfully on the p-adic tree. The natural quotients of a GGS-group G are $G/\mathrm{St}_G(n)$, for $n \in \mathbb{N}$, where $\mathrm{St}_G(n)$ is the normal subgroup of the elements of G that pointwise fix the vertices at the nth level of the tree. The quotient $G/\mathrm{St}_G(n)$ acts on the finite tree consisting of the first n layers of the full p-adic tree.

We extend the result of [17] to a generalisation of the GGS-groups to the p^n-adic tree, for any prime p and $n \geqslant 2$, by showing that infinitely many quotients of infinite periodic GGS-groups acting on the p^n-adic tree do admit Beauville structures.

Briefly, a GGS-group acting on the p^n-adic tree is a group $G = \langle a, b \rangle$, where $a = (1\,2 \ldots p^n)$ permutes the first-level vertices, whereas b fixes the first-level vertices and is defined recursively by

$$b = (a^{e_1}, \ldots, a^{e_{p^n-1}}, b),$$

for $e_1, \ldots, e_{p^n-1} \in \mathbb{Z}/p^n\mathbb{Z}$ with not all e_i being zero. Here

$$(a^{e_1}, \ldots, a^{e_{p^n-1}}, b)$$

refers to the respective independent actions at the p^n maximal subtrees. We write $\mathbf{e} = (e_1, \ldots, e_{p^n-1})$ and call it the *defining vector* of G. We refer the reader to Section 2 for background material and precise definitions. These groups were constructed by Vovkivsky [23]. A prominent group in this family, that was defined earlier in [15], is the second Grigorchuk group, which acts on the 4-adic tree with defining vector $\mathbf{e} = (1, 0, 1)$.

For an infinite periodic GGS-group G acting on the p^n-adic tree, we associate to G a number $m_G \in \mathbb{N}$ that is related to the order of certain group elements; we refer the reader to Section 4 for precise details. The main result of this paper is as follows.

Theorem 1.1 *Let G be an infinite periodic GGS-group acting on the p^n-adic tree for p a prime and $n \geqslant 2$. Then $G/\mathrm{St}_G(k)$ admits a Beauville structure for $k \geqslant m_G$.*

In particular, these quotients of infinite periodic GGS-groups acting on the 2^n-adic tree yield infinite families of Beauville 2-groups, and infinite periodic GGS-groups acting on the 3^n-adic tree give many more infinite families of Beauville 3-groups. Note that Beau-

ville 2-groups and 3-groups are somewhat rare, precisely because of the small number of maximal subgroups. Indeed, for a time it was unclear whether such groups even existed, until the first examples were given in [13]. We also want to emphasise that for the GGS-groups acting on the p-adic tree, if one were to consider the case $p = 2$ there is only one group, which is the infinite dihedral group, hence its quotients do not admit Beauville structures, and for $p = 3$ only one out of the three isomorphism classes of such groups has quotients admitting Beauville structures; see [17] and [21].

For finite 2-groups, one can further ask the question of whether the group gives rise to a mixed Beauville surface, where a mixed Beauville surface is a Beauville surface where the action of the group interchanges the two curves C_1 and C_2 defining the surface. It is quite difficult to find examples of finite 2-groups with mixed Beauville structures; see [10] and references therein. As clarified in Remark 4.2, for G an infinite periodic GGS-group acting on the 2^n-adic tree for $n \geqslant 2$, the quotient $G/St_G(k)$ does not admit a mixed Beauville structure for $k \geqslant m_G$.

Gül and Uria-Albizuri also showed in [17] that for G a GGS-group acting on the p-adic tree, there are quotients of G admitting Beauville structures if and only if G is periodic. The situation is not so clear cut for GGS-groups acting on the p^n-adic tree, for $n \geqslant 2$. In particular, there are non-periodic GGS-groups acting on the p^n-adic tree which have quotients admitting Beauville structures; see Remark 4.3. Furthermore, our main result above deals only with infinite periodic GGS-groups. For finite GGS-groups, some quotients admit Beauville structures; see Remark 4.4. However a large class $\mathcal{E}$ of finite GGS-groups have no level-stabiliser quotients admitting Beauville structures. We refer the reader to Section 5 for the definition of $\mathcal{E}$. We have the following result.

Theorem 1.2 *Let $G \in \mathcal{E}$ be a finite GGS-group acting on the p^n-adic tree for p a prime and $n \geqslant 2$. Then $G/St_G(k)$ is not a Beauville group for all $k \in \mathbb{N}$.*

To prove our theorems, we largely make use of the subgroup structure of the respective groups.

We note that a generalisation of a Beauville group is a group admitting a ramification structure. These are groups associated to surfaces isogenous to a higher product of curves. We refer the reader to [20] for an overview of groups with ramification structures. Only recently

were some groups acting on rooted trees shown to admit (non-Beauville) ramification structures. These include quotients of the Grigorchuk groups and quotients of periodic multi-EGS groups acting on the p-adic tree, for p an odd prime; see [20] and [7] respectively.

Layout of the paper Section 2 of this paper consists of background material for groups acting on rooted trees and here we also recall the generalisation of the GGS-groups. In Section 3 we establish key properties of these groups. In Section 4 we prove Theorem 1.1 and lastly in Section 5 we prove Theorem 1.2.

2 Preliminaries

All trees considered here will be rooted, meaning that there is a distinguished vertex called the root, with one degree less than all other vertices. For $d \in \mathbb{N}_{\geqslant 2}$, let T be the d-adic tree, meaning all vertices have d children. Using the alphabet $A = \{1, \dots, d\}$, the vertices u_ω of T are labelled bijectively by elements ω of the free monoid A^* as follows: the root of T is labelled by the empty word $\varnothing$, and for each word $\omega \in A^*$ and letter $a \in A$ there is an edge connecting u_ω to $u_{\omega a}$. We say that u_ω precedes u_λ whenever ω is a prefix of λ. When convenient, we do not differentiate between A^* and vertices of T.

There is a natural length function on A^*. The words ω of length $|\omega| = n$, representing vertices u_ω that are at distance n from the root, are the nth-level vertices and form the *nth layer* of the tree. The elements of the boundary ∂T correspond naturally to infinite simple rooted paths, and they are in one-to-one correspondence with the d-adic integers.

Denote by T_u the full subtree of T that has its root at a vertex u and includes all vertices succeeding u. For any two vertices $u = u_\omega$ and $v = u_\lambda$, the map $u_{\omega\tau} \mapsto u_{\lambda\tau}$, induced by replacing the prefix ω by λ, yields an isomorphism between the subtrees T_u and T_v.

Every automorphism of T fixes the root, and the orbits of $\mathrm{Aut}(T)$ on the vertices of the tree T are precisely its layers. For $f \in \mathrm{Aut}(T)$, the image of a vertex u under f is denoted by u^f. The automorphism f induces a faithful action on the monoid A^* and for $a \in A$ we have $(\omega a)^f = \omega^f a'$ where $a' \in A$ is uniquely determined by ω and f.

From this we have a permutation $f(\omega)$ of A where

$$(\omega a)^f = \omega^f a^{f(\omega)}, \qquad \text{and hence} \quad (u_{\omega a})^f = u_{\omega^f a^{f(\omega)}},$$

and $f(\omega)$ is called the *label* of f at ω. The collection of all labels of f constitutes the *portrait* of f, and there is a one-to-one correspondence between automorphisms of T and portraits. The automorphism f is *rooted* if $f(\omega) = 1$ for $\omega \neq \varnothing$. It is *directed*, with directed path ℓ for some $\ell \in \partial T$, if the support $\{\omega \mid f(\omega) \neq 1\}$ of its labeling is infinite and marks only vertices at distance 1 from the set of vertices corresponding to the path ℓ. Additionally, the *section* of f at a vertex u is defined to be the unique automorphism f_u of $T \simeq T_u$ given by the condition $(uv)^f = u^f v^{f_u}$ for $v \in A^*$.

2.1 Subgroups of $\mathrm{Aut}(T)$

For $G \leqslant \mathrm{Aut}(T)$, the *vertex stabiliser* $\mathrm{st}_G(u)$ is the subgroup consisting of elements in G that fix the vertex u. For $n \in \mathbb{N}$, the *nth-level stabiliser* $\mathrm{St}_G(n) = \bigcap_{|\omega|=n} \mathrm{st}_G(u_\omega)$ is the subgroup consisting of automorphisms that fix all vertices at level n. Let $T_{[n]}$ be the finite subtree of T of vertices up to level n. Then $\mathrm{St}_G(n)$ is the kernel of the induced action of G on $T_{[n]}$, and we will denote by G_n the quotient $G/\mathrm{St}_G(n)$.

Each $g \in \mathrm{St}_{\mathrm{Aut}(T)}(n)$ can be described completely in terms of its restrictions to the subtrees rooted at vertices at level n. Indeed, the map

$$\psi_n \colon \mathrm{St}_{\mathrm{Aut}(T)}(n) \longrightarrow \prod_{|\omega|=n} \mathrm{Aut}(T_{u_\omega}) \simeq \underbrace{\mathrm{Aut}(T) \times \ldots \times \mathrm{Aut}(T)}_{d^n}$$

is a natural isomorphism mapping $g \in \mathrm{St}_{\mathrm{Aut}(T)}(n)$ to its nth-level sections. For ease of notation, we write $\psi = \psi_1$.

For $\omega \in A^*$, we further define:

$$\begin{aligned} \varphi_\omega : \mathrm{st}_{\mathrm{Aut}(T)}(u_\omega) &\longrightarrow \mathrm{Aut}(T_{u_\omega}) \simeq \mathrm{Aut}(T) \\ g &\longmapsto g_\omega \end{aligned}$$

A group $G \leqslant \mathrm{Aut}(T)$ is said to be *self-similar* if for every $g \in G$ and every vertex u, the section g_u, of g at u, is an element of G. For a self-similar group $G \leqslant \mathrm{Aut}(T)$, for $m \geqslant 2$ and $i \in A^{m-1}$, we will write

$$\psi_m^i = \psi \circ \varphi_i.$$

That is, for $g \in st_G(u_i)$ with $\varphi_i(g) \in St_G(1)$, we have $\psi_m^i(g)$ gives the components of $\varphi_i(g)$ corresponding to the children of the $(m-1)$th-level vertex i. In addition, we write

$$\phi_{n,m} : St_{G_n}(m) \longrightarrow \underbrace{G_{n-1} \times \ldots \times G_{n-1}}_{d^m}$$

for the corresponding ψ_m when working in the quotient. Likewise, we write $\phi_n = \phi_{n,1}$ for simplicity.

2.2 GGS-groups acting on the p^n-adic tree

Let T be the p^n-adic tree for a prime p and $n \in \mathbb{N}$. Given a non-zero vector $\mathbf{e} = (e_1, \ldots, e_{p^n-1}) \in (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1}$, the GGS-group $G = G_{\mathbf{e}}$ associated to the defining vector $\mathbf{e}$ is the group generated by the rooted automorphism a corresponding to the cycle $(1\,2\,\ldots\,p^n)$ and by a directed automorphism $b \in St_{Aut(T)}(1)$ defined recursively via

$$\psi(b) = (a^{e_1}, a^{e_2}, \ldots, a^{e_{p^n-1}}, b).$$

Unlike the GGS-groups acting on the p-adic tree, the GGS-groups acting on the p^n-adic tree for $n \geqslant 2$ are not all infinite. A necessary and sufficient condition for these groups to be infinite was given by Vovkivsky [23]. He proved that such a group is infinite if and only if there exists an $i \geqslant 0$ such that

$$R_0 \leqslant R_1 \leqslant \ldots \leqslant R_i = R_{i+1} = \ldots < n$$

where the sequence R_j is defined recursively as follows: R_0 is the largest integer such that $p^{R_0} \mid e_\ell$ for all $\ell \in \{1, \ldots, p^n - 1\}$; and then for $j \geqslant 0$ and while $R_j < n$, the number R_{j+1} is defined as the largest integer such that $p^{R_{j+1}}$ divides e_ℓ for all $\ell \in \{p^{R_j}, 2p^{R_j}, \ldots, p^n - p^{R_j}\}$.

Note that the order of a is p^n and the order of b is p^{n-R_0}. Further, from [6, Theorem 2.1] we get $G/G' \simeq C_{p^n} \times C_{p^{n-R_0}}$.

Vovkivsky further proved in [23] that a GGS-group acting on the p^n-adic tree is periodic if and only if for each $k \in \{0, \ldots, n-1\}$,

$$S[k] \equiv 0 \pmod{p^{k+1}}$$

where

$$S[k] = e_{p^k} + e_{2p^k} + \ldots + e_{p^n - p^k}. \tag{2.1}$$

We recall the prominent example of such a GGS-group, which is for the case $p = n = 2$:

Example 2.1 Let T be the 4-adic tree. The *second Grigorchuk group* $\Gamma \leqslant \mathrm{Aut}(T)$ is generated by two automorphisms a and b, where a is the rooted automorphism corresponding to the cycle $(1\,2\,3\,4)$, and $b \in \mathrm{St}_\Gamma(1)$ is recursively defined by $\psi(b) = (a, 1, a, b)$. The group Γ, which is periodic, was first defined in [15].

For more information on GGS-groups acting on the p^n-adic tree, see [23],[5],[6].

3 Properties of GGS-groups acting on the p^n-adic tree

For $G = G_{\mathbf{e}}$, a GGS-group acting on the p^n-adic tree, recall that R_0 is the largest integer such that $p^{R_0} \mid e_\ell$ for all $\ell \in \{1, \dots, p^n - 1\}$.

Suppose now that G is periodic. To each element $a^{ip^r}b^{jp^s}$ in G, where $0 \leqslant r < n$, $0 \leqslant s < n - R_0$ and $i, j \not\equiv 0 \pmod p$, we associate the following numbers:

(i) If $p^{n-s} \nmid S[r]$, let $t_0, t_1, \dots, t_{m_{r,s}} \in \mathbb{N}$ for some $1 \leqslant m_{r,s} < n$ be such that
- $t_k < n - s \leqslant t_{m_{r,s}}$ for all $k < m_{r,s}$,
- $p^{t_0} \mid S[r]$ and $p^{t_0+1} \nmid S[r]$,
- $p^{t_k} \mid S[t_{k-1} + s]$ and $p^{t_k+1} \nmid S[t_{k-1} + s]$ for $1 \leqslant k < m_{r,s}$,
- $p^{t_{m_{r,s}}} \mid S[t_{m_{r,s}-1} + s]$,

where the $S[\lambda]$ for $\lambda \in \{0, 1, \dots, n-1\}$ are as in (2.1).

(ii) If $p^{n-s} \mid S[r]$, then we set $m_{r,s} = 0$, and for convenience define $t_0 = n - s$.

If we are in case (i), note that t_0 does not depend on s.

Lemma 3.1 *Let $G = \langle a, b \rangle$ be a periodic GGS-group acting on the p^n-adic tree, let $0 \leqslant r < n$, $0 \leqslant s < n - R_0$ and $i, j \not\equiv 0 \pmod p$. Let $m := m_{r,s}$ and $t_0, t_1, \dots, t_m \in \mathbb{N}$ be as above. Then the order of $a^{ip^r}b^{jp^s}$ in both G and G_{m+3} is $p^{t(r,s)}$ where*

$$t(r,s) = (m+2)n - (t_0 + \dots + t_{m-1} + s(m+1) + r + R_0),$$

and if $p^{n-s} \mid S[r]$*, that is, when* $m = 0$*, we take* $t_0 + \ldots + t_{m-1} = 0$*.*

Proof — Clearly the order of $a^{ip^r}b^{jp^s}$ is a multiple of p^{n-r}. We have

$$\begin{aligned}\psi\Big((a^{ip^r}b^{jp^s})^{p^{n-r}}\Big) &= \psi\Big((b^{jp^s})^{a^{(p^{n-r}-1)ip^r}}\ldots(b^{jp^s})^{a^{2ip^r}}(b^{jp^s})^{a^{ip^r}}b^{jp^s}\Big)\\ &= \Big(\underbrace{(a^{jp^{s+R_0}})^*,\ldots,(a^{jp^{s+R_0}})^*}_{p^r-1}, a^{jp^sS[r]}(b^{jp^s})^{a^{jp^s(e_{ip^r}+e_{2ip^r}+\ldots+e_{p^r})}},\\ &\quad \underbrace{(a^{jp^{s+R_0}})^*,\ldots,(a^{jp^{s+R_0}})^*}_{p^r-1}, a^{jp^sS[r]}(b^{jp^s})^{a^{jp^s(e_{ip^r}+e_{2ip^r}+\ldots+e_{2p^r})}},\\ &\qquad\qquad \ldots,\ldots,\\ &\quad \underbrace{(a^{jp^{s+R_0}})^*,\ldots,(a^{jp^{s+R_0}})^*}_{p^r-1}, a^{jp^sS[r]}(b^{jp^s})^{a^{jp^s(e_{ip^r}+e_{2ip^r}+\ldots+e_{(p^{n-r}-1)p^r})}},\\ &\qquad \underbrace{(a^{jp^{s+R_0}})^*,\ldots,(a^{jp^{s+R_0}})^*}_{p^r-1}, a^{jp^sS[r]}b^{jp^s}\Big)\end{aligned} \tag{3.1}$$

where the $*$ denotes an unspecified exponent, and the last equality follows from the fact that the set $\{ip^r, 2ip^r, \ldots, (p^{n-r}-1)ip^r, p^n\}$ coincides, modulo p^n, with $\{p^r, 2p^r, \ldots, (p^{n-r}-1)p^r, p^n\}$. So the components of $\psi\big((a^{ip^r}b^{jp^s})^{p^{n-r}}\big)$ in positions a multiple of p^r are conjugates of $a^{jp^sS[r]}b^{jp^s}$ and the other components are powers of $a^{jp^{s+R_0}}$ whose order is at most p^{n-s-R_0}.

Suppose that $p^{n-s} \mid S[r]$. Then using (3.1), we have

$$\begin{aligned}&\psi\Big((a^{ip^r}b^{jp^s})^{p^{n-r}}\Big) =\\ &\quad\Big(\underbrace{(a^{jp^{s+R_0}})^*,\ldots,(a^{jp^{s+R_0}})^*}_{p^r-1}, (b^{jp^s})^{a^{jp^s(e_{ip^r}+e_{2ip^r}+\ldots+e_{p^r})}},\ldots,\\ &\qquad\qquad \underbrace{(a^{jp^{s+R_0}})^*,\ldots,(a^{jp^{s+R_0}})^*}_{p^r-1}, b^{jp^s}\Big).\end{aligned}$$

Note that the order of b^{jp^s} is p^{n-s-R_0}. Thus, the order of $a^{ip^r}b^{jp^s}$ in G is $p^{t(r,s)}$, for $t(r,s) = 2n-(r+s+R_0)$. Also since $(b^{jp^s})^{p^{n-s-R_0-1}}$ does not belong to $St_G(2)$, it follows that $(a^{ip^r}b^{jp^s})^{p^{t(r,s)-1}}$ does not belong to $St_G(3)$. Thus the element $a^{ip^r}b^{jp^s}$ is of order $p^{t(r,s)}$ in G_3

as well.

It remains to settle the case $p^{n-s} \nmid S[r]$, which we will now assume till the end of the proof. Since by hypothesis $p^{t_0+s+1} \nmid jp^sS[r]$, the order of $a^{jp^sS[r]}b^{jp^s}$ is a multiple of $p^{n-(t_0+s)}$. Applying ψ we get

$$\psi_2^{p^n}\Big((a^{ip^r}b^{jp^s})^{p^{n-r}p^{n-(t_0+s)}}\Big) = \psi\Big((a^{jp^sS[r]}b^{jp^s})^{p^{n-(t_0+s)}}\Big)$$

$$= \psi\Big((b^{jp^s})^{a^{(p^{n-(t_0+s)}-1)jp^sS[r]}} \dots (b^{jp^s})^{a^{2jp^sS[r]}}(b^{jp^s})^{a^{jp^sS[r]}}b^{jp^s}\Big).$$

As before, it follows that

$$\{jp^sS[r], 2jp^sS[r], \dots, (p^{n-(t_0+s)}-1)jp^sS[r], p^n\}$$

coincides, modulo p^n, with the set

$$\{p^{t_0+s}, 2p^{t_0+s}, \dots, p^n - p^{t_0+s}, p^n\}.$$

This means that the b's appear in all positions a multiple of p^{t_0+s} and only in these positions, and we can write each of the elements in these positions as

$$a^{jp^sS[t_0+s]}(b^{jp^s})^{a^{jp^s(e_{jp^sS[r]}+e_{2jp^sS[r]}+\dots+e_{\ell p^{t_0+s}})}}$$

for certain $\ell \in \{1, \dots, p^{n-(t_0+s)}-1\}$. Since we are interested in the order of these elements, up to conjugation the order of these elements coincides with the order of the element $a^{jp^sS[t_0+s]}b^{jp^s}$. In the other components of

$$\psi\big((a^{jp^sS[r]}b^{jp^s})^{p^{n-(t_0+s)}}\big),$$

there is a power of $a^{jp^{s+R_0}}$ whose order is at most p^{n-s-R_0}. By hypothesis $p^{t_1+s+1} \nmid p^sS[t_0+s]$ thus the order of $a^{jp^sS[t_0+s]}b^{jp^s}$ is at least $p^{n-(t_1+s)}$. Proceeding in this way, after $m-1$ further steps we find

$$\psi_{m+1}^{p^{mn}}\Big((a^{ip^r}b^{jp^s})^{p^{n-r}p^{n-(t_0+s)}p^{n-(t_1+s)}\dots p^{n-(t_{m-1}+s)}}\Big)$$

$$= \psi\Big((b^{jp^s})^{a^{(p^{n-(t_{m-1}+s)}-1)jp^sS[t_{m-2}+s]}} \dots$$

$$(b^{jp^s})^{a^{2jp^sS[t_{m-2}+s]}}(b^{jp^s})^{a^{jp^sS[t_{m-2}+s]}}(b^{jp^s})\Big)$$

$$= \Big(\underbrace{(a^{jp^{s+R_0}})^*, \dots, (a^{jp^{s+R_0}})^*}_{p^{t_{m-1}}-1}, a^{jp^s S[t_{m-1}+s]} (b^{jp^s})^{a^*},$$
$$\underbrace{(a^{jp^{s+R_0}})^*, \dots, (a^{jp^{s+R_0}})^*}_{p^{t_{m-1}}-1}, a^{jp^s S[t_{m-1}+s]} (b^{jp^s})^{a^*}, \dots,$$
$$\underbrace{(a^{jp^{s+R_0}})^*, \dots, (a^{jp^{s+R_0}})^*}_{p^{t_{m-1}}-1}, a^{jp^s S[t_{m-1}+s]} (b^{jp^s}) \Big)$$

$$= \Big(\underbrace{(a^{jp^{s+R_0}})^*, \dots, (a^{jp^{s+R_0}})^*}_{p^{t_{m-1}}-1}, (b^{jp^s})^{a^*}, \dots, \underbrace{(a^{jp^{s+R_0}})^*, \dots, (a^{jp^{s+R_0}})^*}_{p^{t_{m-1}}-1}, b^{jp^s} \Big)$$

where the last equality holds since $p^{t_m+s} \mid p^s S[t_{m-1}+s]$, thus

$$p^n \mid p^s S[t_{m-1}+s]$$

by the definition of t_m. Since by hypothesis $j \not\equiv 0 \pmod p$ it follows that the order of the elements in positions a multiple of $p^{t_{m-1}}$ is p^{n-s-R_0} and the orders of the other elements are at most p^{n-s-R_0}. This proves that the order of $a^{ip^r} b^{jp^s}$ is $p^{t(r,s)}$, where

$$t(r,s) = (m+2)n - (t_0 + \dots + t_{m-1} + s(m+1) + r + R_0).$$

Finally, let us prove that the element

$$(a^{ip^r} b^{jp^s})^{p^{t(r,s)-1}} \notin St_G(m+3),$$

which implies that $a^{ip^r} b^{jp^s}$ is of order $p^{t(r,s)}$ in G_{m+3}. We observe that

$$\psi_{m+1}^{p^{mn}} \left((a^{ip^r} b^{jp^s})^{p^{t(r,s)-1}} \right) =$$
$$\Big(\underbrace{(a^{jp^{n-1}})^*, \dots, (a^{jp^{n-1}})^*}_{p^{t_{m-1}}-1}, (b^{jp^{n-R_0-1}})^{a^*}, \dots$$
$$\underbrace{(a^{jp^{n-1}})^*, \dots, (a^{jp^{n-1}})^*}_{p^{t_{m-1}}-1}, b^{jp^{n-R_0-1}} \Big).$$

Since the components $(b^{jp^{n-R_0-1}})^{a^*}$ are non-trivial in G_2, the result follows. □

In the next three results, we identify a sort of branching struc-

ture within certain subgroups of GGS-groups. For convenience, we set $R_{-1} = 0$.

Lemma 3.2 *Let $G = \langle a, b \rangle$ be a GGS-group acting on the p^n-adic tree, for p a prime and $n \in \mathbb{N}$. Then for $j \in \{0, 1, \dots, n-1\}$, we have*

$$\varphi_v\big(\mathrm{st}_G(v)\big) \geqslant \langle a^{p^{R_{j-1}}}, b \rangle$$

for any vertex $v = u_1 \dots u_j \in A^j$ of length j, where $u_i \equiv 0 \pmod{p^{R_{i-2}}}$ for $i \in \{1, \dots, j\}$.

Proof — For a vertex v of length one, the result is clear from considering the sections of $b, b^a, \dots, b^{a^{p^n-1}}$, which are the generators of $\mathrm{st}_G(v)$. As the subgroup generated by the sections of b is $\langle a^{p^{R_0}}, b \rangle$, it follows that for a vertex w of length 2,

$$b, b^{a^{p^{R_0}}}, \dots, b^{a^{p^n - p^{R_0}}} \in \varphi_v\big(\mathrm{st}_G(w)\big),$$

where v is the prefix of w of length 1. It then follows that for vertices $w = u_1u_2$, with $u_1, u_2 \in A$ such that $u_2 \equiv 0 \pmod{p^{R_0}}$, we have $\varphi_w\big(\mathrm{st}_G(w)\big) \geqslant \langle a^{p^{R_1}}, b \rangle$. The result now follows recursively. □

Lemma 3.3 *Let $G = \langle a, b \rangle$ be a GGS-group acting on the p^n-adic tree, for p a prime and $n \in \mathbb{N}$. Let $\alpha \in \{0, \dots, n-1\}$ and let p^R be the highest power of p dividing e_{ip^α} for all $i \in \{1, \dots, p^{n-\alpha} - 1\}$. Suppose that*

(i) *the highest power of p dividing $e_{jp^{\alpha+1}}$ for all $j \in \{1, \dots, p^{n-\alpha-1} - 1\}$ is strictly greater than p^R, and*

(ii) *there exists $\ell \in \{1, \dots, p^{n-\alpha} - 1\}$ such that $p^{R+1} \mid e_{\ell p^\alpha}$.*

Then, writing $N_\alpha = \langle a^{p^\alpha}, b \rangle$ and $N_R = \langle a^{p^R}, b \rangle$, we have

$$\underbrace{1 \times \dots \times 1}_{p^\alpha - 1} \times \gamma_3(N_R) \times \dots \times \underbrace{1 \times \dots \times 1}_{p^\alpha - 1} \times \gamma_3(N_R) = \psi\big(\gamma_3\big(\mathrm{St}_{N_\alpha}(1)\big)\big).$$

Note that if $\alpha = n - 1$, condition (i) is vacuous.

Proof — By assumption (i), there exists $k \in \{1, \dots, p^{n-\alpha} - 1\}$ with $k \not\equiv 0 \pmod p$ such that $e_{kp^\alpha} = \kappa p^R$ for some $\kappa \not\equiv 0 \pmod p$. Up to replacing b with a suitable power of itself, we may assume

without loss of generality that $\kappa = 1$. It suffices to show, since we can conjugate by a^{p^α}, that

$$\underbrace{1 \times \ldots \times 1}_{kp^\alpha - 1} \times \gamma_3(N_R) \times 1 \times \ldots \times 1 \leqslant \psi\big(\gamma_3\big(St_{N_\alpha}(1)\big)\big).$$

Case 1: Suppose $e_{(p^{n-\alpha}-k)p^\alpha} = 0$. Then, as

$$\gamma_3(N_R) = \big\langle \big[a^{p^R}, b, a^{p^R}\big], \big[a^{p^R}, b, b\big] \big\rangle^{N_R},$$

the result follows from

$$\psi\Big(\big[b, b^{a^{kp^\alpha}}, b\big]\Big) = \big(\underbrace{1, \ldots, 1}_{kp^\alpha - 1}, \big[a^{p^R}, b, a^{p^R}\big], 1, \ldots, 1\big)$$

and

$$\psi\Big(\big[b, b^{a^{kp^\alpha}}, b^{a^{kp^\alpha}}\big]\Big) = \big(\underbrace{1, \ldots, 1}_{kp^\alpha - 1}, \big[a^{p^R}, b, b\big], 1, \ldots, 1\big)$$

and using Lemma 3.2.

Case 2: Suppose $e_{(p^{n-\alpha}-k)p^\alpha} \neq 0$ and let $\chi \in \{1, \ldots, p^{n-R} - 1\}$ be such that $e_{(p^{n-\alpha}-k)p^\alpha} = \chi p^R$.

If $p \mid \chi$, consider

$$\psi\Big(\big[b, b^{a^{kp^\alpha}}, b\big]\Big) = \big(\underbrace{1, \ldots, 1}_{kp^\alpha - 1}, \big[a^{p^R}, b, a^{p^R}\big], 1, \ldots, 1, \big[b, a^{\chi p^R}, b\big]\big),$$

$$\psi\Big(\big[b, b^{a^{kp^\alpha}}, b^{a^{kp^\alpha}}\big]\Big) = \big(\underbrace{1, \ldots, 1}_{kp^\alpha - 1}, \big[a^{p^R}, b, b\big], 1, \ldots, 1, \big[b, a^{\chi p^R}, a^{\chi p^R}\big]\big).$$

Let us refer to those commutators in the final component as the *error terms*. We proceed to cancel those error terms by respectively multiplying with the following:

$$\psi\Big(\big(\big[b^{a^{kp^\alpha}}, b^\chi, b^{a^{kp^\alpha}}\big]^{-1}\big)^{a^{-kp^\alpha}}\Big) = \big(1, \ldots, 1, \big[a^{\chi p^R}, b^\chi, a^{\chi p^R}\big]^{-1}, \underbrace{1, \ldots, 1}_{kp^\alpha - 1}, \big[b, a^{\chi p^R}, b\big]^{-1}\big),$$

$$\psi\Big(\big([b^{a^{kp^\alpha}}, b^\chi, b^\chi\big]^{-1}\big)^{a^{-kp^\alpha}}\Big) = \big(1,\dots,1,\big[a^{\chi p^R}, b^\chi, b^\chi\big]^{-1}, \underbrace{1,\dots,1}_{kp^\alpha-1}, \big[b, a^{\chi p^R}, a^{\chi p^R}\big]^{-1}\big).$$

This introduces a new set of error terms, this time in the $(p^n - kp^\alpha)$th component. We note that the new set of error terms involve a higher p-power of a or b. Hence, after repeating this process a finite number of times, we will eventually have a set of error terms consisting of commutators with at least one component being the trivial element $a^{p^n} = b^{p^{n-R}0}$. In other words, we have completely canceled any error terms, showing that

$$\big(\underbrace{1,\dots,1}_{kp^\alpha-1}, \big[a^{p^R}, b, a^{p^R}\big], 1,\dots,1\big) \in \psi\big(\gamma_3\big(St_{N_\alpha}(1)\big)\big)$$

and

$$\big(\underbrace{1,\dots,1}_{kp^\alpha-1}, \big[a^{p^R}, b, b\big], 1,\dots,1\big) \in \psi\big(\gamma_3\big(St_{N_\alpha}(1)\big)\big).$$

Now suppose that $p \nmid \chi$. By hypothesis, there is $\ell \in \{1,\dots,p^{n-\alpha}-1\}$ such that $e_{\ell p^\alpha} = \lambda p^R$ with $p \mid \lambda$. We claim that we may choose ℓ such that $e_{(\ell-k)p^\alpha} = \xi p^R$ with $p \nmid \xi$. Indeed, suppose on the contrary that for all choices of ℓ satisfying condition (ii), we have $p^{R+1} \mid e_{(\ell-k)p^\alpha}$. Since $k \not\equiv 0 \pmod p$, repeatedly replacing ℓ with $\ell - k$, this means that $e_{jp^\alpha} \equiv 0 \pmod{p^{R+1}}$ for all $j \in \{1,\dots,p^{n-\alpha}-1\}$, a contradiction. Hence the claim. For convenience, we write $e_{(k-\ell)p^\alpha} = \zeta p^R$, and let $\nu \in \{1,\dots,p^{n-R}-1\}$ be such that $\nu\xi \equiv \chi \pmod{p^{n-R}}$. Then we have

$$\psi\Big(\big[b, (b^\nu)^{a^{(k-\ell)p^\alpha}}, b\big]^{-1}\Big) = \Big(\underbrace{1,\dots,1}_{(k-\ell)p^\alpha-1}, \big[a^{\zeta p^R}, b^\nu, a^{\zeta p^R}\big]^{-1}, 1,\dots,1, \big[b, a^{\chi p^R}, b\big]^{-1}\Big),$$

$$\psi\Big(\big[b, b^{a^{kp^\alpha}}, (b^\nu)^{a^{(k-\ell)p^\alpha}}\big]^{-1}\Big) = \big(\underbrace{1,\dots,1}_{kp^\alpha-1}, \big[a^{p^R}, b, a^{\nu\lambda p^R}\big]^{-1}, 1,\dots,1, \big[b, a^{\chi p^R}, a^{\chi p^R}\big]^{-1}\big).$$

If $p \mid \zeta$, we proceed to cancel the error terms as in the above argument. If $p \nmid \zeta$, let $\theta \in \{1, \dots, p^{n-R}-1\}$ be such that $\theta\chi \equiv \zeta \pmod{p^{n-R}}$. Then, since $p \mid \lambda$, we can use the following element to proceed:

$$\psi\Big(\big[(b^{\theta})^{a^{kp^{\alpha}}}, b^{\nu}, (b^{\theta\nu})^{a^{(k-\ell)p^{\alpha}}}\big]\Big) =$$

$$\big(\underbrace{1, \dots, 1}_{kp^{\alpha}-1}, \big[b^{\theta}, a^{\nu p^{R}}, a^{\theta\nu\lambda p^{R}}\big], 1, \dots, 1, \big[a^{\zeta p^{R}}, b^{\nu}, a^{\zeta p^{R}}\big]\big).$$

The statement is proved. □

Lemma 3.4 *Let $G = \langle a, b\rangle$ be a GGS-group acting on the p^n-adic tree, for p an odd prime and $n \in \mathbb{N}$. Let $R \leqslant n-1$ with $e_{kp^R} \equiv 0 \pmod{p^R}$ for all $k \in \{1, \dots, p^{n-R}-1\}$ and suppose that $e_{kp^R} \not\equiv 0 \pmod{p^{R+1}}$ for all $k \in \{1, \dots, p^{n-R}-1\}$. Suppose further that $S[R] \equiv 0 \pmod{p^{R+1}}$. Then, writing $N = \langle a^{p^R}, b\rangle$, we have*

$$\underbrace{1 \times \dots \times 1}_{p^R-1} \times \gamma_3(N) \times \dots \times \underbrace{1 \times \dots \times 1}_{p^R-1} \times \gamma_3(N) = \psi\big(\gamma_3\big(St_N(1)\big)\big).$$

Proof — Let

$$e_{kp^R} = i_k p^R$$

for $k, i_k \in \{1, \dots, p^{n-R}-1\}$ with $i_k \not\equiv 0 \pmod{p}$. Further, by replacing b with an appropriate power, we may assume that $i_1 = 1$. As reasoned in the previous proof, it suffices to show that

$$\underbrace{1 \times \dots \times 1}_{p^R-1} \times \gamma_3(N) \times 1 \times \dots \times 1 \leqslant \psi\big(\gamma_3\big(St_N(1)\big)\big).$$

We proceed as in [12, Lemma 3.2] by considering the following two cases:

(a) There is $\ell \in \{2, \dots, p^{n-R}-2\}$ such that $i_\ell^2 - i_{\ell-1}i_{\ell+1} \not\equiv 0 \pmod{p}$.

(b) For all $\ell \in \{2, \dots, p^{n-R}-2\}$, we have $i_\ell^2 - i_{\ell-1}i_{\ell+1} \equiv 0 \pmod{p}$.

Note that if $p = 3$ and $R = n-1$, then Case (b) trivially holds.

Suppose we are in Case (a). Let us define g_ℓ as follows:

$$g_\ell = \big((b^{i_\ell})^{a^{p^R}} b^{-i_{\ell-1}}\big)^{a^{-\ell p^R}}$$

Writing

$$\mu = i_\ell^2 - i_{\ell-1}i_{\ell+1},$$

we have

$$\psi(g_\ell) = \big(\underbrace{*,\dots,*}_{p^R-1}, a^{\mu p^R}, *, \dots, *, 1\big)$$

where $*$ represents unspecified elements. By assumption, we have $\mu \not\equiv 0 \pmod p$. Hence there is a power g of g_ℓ satisfying

$$\psi(g) = \big(\underbrace{*,\dots,*}_{p^R-1}, a^{p^R}, *, \dots, *, 1\big).$$

Furthermore, for simplicity let $\lambda := i_{p^{n-R-1}} \not\equiv 0 \pmod p$. Then

$$\psi\Big(b^{a^{p^R}}\big(b^{a^{-p^R}}\big)^{-\lambda}\Big) = \big(\underbrace{*,\dots,*}_{p^R-1}, ba^{-\lambda e_{2p^R}}, *, \dots, *, 1\big),$$

and multiplying by an appropriate power of g we obtain an element $h \in \mathrm{St}_N(1)$ with

$$\psi(h) = \big(\underbrace{*,\dots,*}_{p^R-1}, b, *, \dots, *, 1\big).$$

Then the result follows from:

$$\psi\Big(\big[b, b^{a^{p^R}}, g\big]\Big) = \big(\underbrace{1,\dots,1}_{p^R-1}, \big[a^{p^R}, b, a^{p^R}\big], 1, \dots, 1\big)$$

and

$$\psi\Big(\big[b, b^{a^{p^R}}, h\big]\Big) = \big(\underbrace{1,\dots,1}_{p^R-1}, \big[a^{p^R}, b, b\big], 1, \dots, 1\big).$$

Now, suppose we are in Case (b). Then, if $j \in \{2, \dots, p^{n-R}-1\}$, we have $i_j = \lambda^{j-1}$ for some $\lambda \not\equiv 0 \pmod p$. But, as $S[R] \equiv 0 \pmod{p^{R+1}}$, we have $\lambda \not\equiv 1 \pmod p$. Further we have

$$\lambda^{p^{n-R-1}}p^R \equiv p^R \pmod{p^{R+1}}.$$

Note that

$$\psi\Big(b\big(b^{a^{p^R}}\big)^{-\lambda}\Big) = \big(\underbrace{*,\dots,*}_{p^R-1}, a^{p^R}b^{-\lambda}, \underbrace{*,\dots,*}_{p^R-1}, 1,\dots,$$

$$\underbrace{*,\dots,*}_{p^R-1}, 1, \underbrace{*,\dots,*}_{p^R-1}, ba^{-p^R+\nu p^{R+1}}\big)$$

for some ν, and $*$ represents some power of a. Let χ be such that $\lambda\chi \equiv 1 \pmod{p^{n-R_0}}$; recall that p^{n-R_0} is the order of b. Then

$$\psi\Big(b^{\chi}\big(b^{-\chi\lambda}\big)^{a^{p^R}}\Big) = \psi\Big(b^{\chi}\big(b^{-1}\big)^{a^{p^R}}\Big)$$

$$= \big(\underbrace{*,\dots,*}_{p^R-1}, a^{\chi p^R}b^{-1}, \underbrace{*,\dots,*}_{p^R-1}, 1,\dots,\underbrace{*,\dots,*}_{p^R-1}, 1, \underbrace{*,\dots,*}_{p^R-1}, b^{\chi}*\big).$$

Thus, since $\chi \not\equiv 1 \pmod{p}$ and hence

$$N = \big\langle a^{\chi p^R}b^{-1}, ba^{-p^R+\nu p^{R+1}}\big\rangle = \big\langle a^{(\chi-1)p^R+\nu p^{R+1}}, ba^{-p^R+\nu p^{R+1}}\big\rangle,$$

the result follows from the observations below:

$$\psi\Big(\big[b^{(1-\nu p)}, b^{a^{p^R}}, b^{a^{p^R}}(b^{a^{2p^R}})^{-\lambda}\big]\Big) =$$

$$\big(\underbrace{1,\dots,1}_{p^R-1}, \big[a^{p^R-\nu p^{R+1}}, b, ba^{-p^R+\nu p^{R+1}}\big], 1,\dots,1\big),$$

$$\psi\Big(\big[(b^{a^{2p^R}})^{\lambda}, b^{a^{p^R}}, b^{\chi}\big(b^{-\chi\lambda}\big)^{a^{p^R}}\big]\Big) =$$

$$\big(\underbrace{1,\dots,1}_{p^R-1}, \big[a^{p^R-\nu p^{R+1}}, b, a^{\chi p^R}b^{-1}\big], 1,\dots,1\big).$$

The statement is proved. □

Next, we record some properties of certain finite GGS-groups acting on the 2^n-adic tree, which will be needed in Section 5.

Lemma 3.5 *Let G be a periodic GGS-group acting on the 2^n-adic tree for $n \geqslant 2$, and assume $R_0 = n-1$. Then $\mathrm{St}_G(4)$ is trivial. Moreover, $\mathrm{St}_G(3)$ is non-trivial if and only if $e_i = e_{2^{n-1}+i}$ for all $i \in \{1,\dots,2^{n-1}-1\}$.*

PROOF — Note that as $R_0 = n-1$, we have that 2^{n-1} divides each e_i. Thus the e_i's are 0 or 2^{n-1}. Since G is periodic, it follows that $e_{2^{n-1}} = 0$ and hence $\langle b, b^{a^{2^{n-1}}} \rangle \simeq C_2 \times C_2$.

Let $g \in St_G(4)$. Certainly $\varphi_\omega(g) \in St_G(1)$ for all $\omega \in A$. Since

$$\psi\big(St_G(1)\big) \cap St_G(1) \times \ldots \times St_G(1) \subseteq \underbrace{\langle b, b^{a^{2^{n-1}}} \rangle \times \ldots \times \langle b, b^{a^{2^{n-1}}} \rangle}_{2^n},$$

it follows that

$$\varphi_\omega(g) \in \big\{1, b, b^{a^{2^{n-1}}}, bb^{a^{2^{n-1}}}\big\}.$$

As $\langle b, b^{a^{2^{n-1}}} \rangle \cap St_G(3) = 1$, the first claim follows.

For the second statement, let g be a non-trivial element in $St_G(3)$, in particular $\varphi_\omega(g) \in St_G(2)$ for all $\omega \in A$. From the previous paragraph, it follows that $\varphi_\omega(g) \in \big\{1, bb^{a^{2^{n-1}}}\big\}$. Thus there exists $\omega \in A$ such that $\varphi_\omega(g) = bb^{a^{2^{n-1}}}$. We observe that

$$\psi\Big(bb^{a^{2^{n-1}}}\Big) = \big(a^{e_1+e_{2^{n-1}+1}}, \ldots, a^{e_{2^{n-1}-1}+e_{2^n-1}}, b, \\ a^{e_1+e_{2^{n-1}+1}}, \ldots, a^{e_{2^{n-1}-1}+e_{2^n-1}}, b\big),$$

so $\varphi_\omega(g) \in St_G(2)$ if and only if $e_i = e_{2^{n-1}+i}$ for all $i \in \{1, \ldots, 2^{n-1}-1\}$, as required. □

Lemma 3.6 *Let G be a periodic GGS-group acting on the 2^n-adic tree for $n \geqslant 2$. Suppose $R_0 = n-1$ and $S[0] \equiv 0 \pmod{p^n}$. Then*

$$\gamma_{2^n+1}(G) \leqslant St_G(1)'.$$

Moreover, $[b, \underbrace{a, \ldots, a}_{2^{n-1}}]^2 = 1$ and $[b, \underbrace{a, \ldots, a}_{2^{n-1}+i}]^2 \in \gamma_{2^n+i+2}(G)$ for all $i \in \mathbb{N}$.

PROOF — As $St_G(1)/St_G(1)'$ is elementary abelian, we have that $(G')^2 \leqslant St_G(1)'$. Therefore, from [19, Proposition 1.1.32(ii)], we have

$$1 = \big[a^{2^n}, b\big] \equiv [b, \underbrace{a, \ldots, a}_{2^n}] \pmod{St_G(1)'},$$

which gives $\gamma_{2^n+1}(G) \leqslant St_G(1)'$, as required.

Also from [19, Proposition 1.1.32(ii)], we obtain

$$\left[a^{2^{n-1}}, b\right] \equiv [a, b, \underbrace{a, \ldots, a}_{2^{n-1}-1}] \pmod{\mathrm{St}_G(1)'}$$

and so

$$[b, \underbrace{a, \ldots, a}_{2^{n-1}}]^2 = 1,$$

as the group $\mathrm{St}_G(1)'$ is elementary abelian and $[\mathrm{St}_G(1), \mathrm{St}_G(1)'] = 1$.

For the last statement, we proceed by induction on i. For $i = 1$, the statement follows from

$$1 = \left[[b, \underbrace{a, \ldots, a}_{2^{n-1}}]^2, a\right] = [b, \underbrace{a, \ldots, a}_{2^{n-1}+1}]^2 \left[b, \underbrace{a, \ldots, a}_{2^{n-1}+1}, [b, \underbrace{a, \ldots, a}_{2^{n-1}}]\right],$$

and similarly for the inductive step, as

$$\left[[b, \underbrace{a, \ldots, a}_{2^{n-1}+i}]^2, a\right] = [b, \underbrace{a, \ldots, a}_{2^{n-1}+i+1}]^2 \left[b, \underbrace{a, \ldots, a}_{2^{n-1}+i+1}, [b, \underbrace{a, \ldots, a}_{2^{n-1}+i}]\right].$$

The statement is proved. □

Lemma 3.7 *Let* G *be a GGS-group satisfying the conditions of* Lemma 3.6. *Then* $\gamma_i(G)/\gamma_{i+1}(G)$ *is isomorphic to a subgroup of* $C_2 \times C_2$, *for* $i \geqslant 2$.

Furthermore $|\gamma_i(G) : \gamma_{i+1}(G)| \leqslant 2$ *for* $i > 2^n$.

Proof — We note that

$$\psi\left(\mathrm{St}_G(1)'\right) \subseteq S,$$

where

$$S := \left\{ \left(\left[b, a^{2^{n-1}}\right]^{\epsilon_1}, \ldots, \left[b, a^{2^{n-1}}\right]^{\epsilon_{2^n}} \right) \mid \epsilon_1, \ldots, \epsilon_{2^n} \in \{0, 1\} \right\}.$$

Since $|S| = 2^{2^n}$, it follows that $\log_2 |\mathrm{St}_G(1)'| \leqslant 2^n$.

First we consider the case when the defining vector $\mathbf{e}$ is non-symmetric, that is, there exists $j \in \{1, \ldots, 2^{n-1} - 1\}$ with $e_j \neq e_{2^n - j}$. Then, there exists some $k \in \{1, \ldots, 2^n - 1\}$ such that

$$\psi\left(\left[b, b^{a^k}\right]\right) = \left(1, \ldots, 1, \left[b, a^{2^{n-1}}\right]\right) \in \psi\left(\mathrm{St}_G(1)'\right).$$

Hence for **e** non-symmetric, we have $\log_2 |\mathrm{St}_G(1)'| = 2^n$, and

$$\begin{aligned}\mathrm{St}_G(1)' &= \left\langle \left[b, b^{a^k}\right], \left[b, b^{a^k}\right]^a, \left[b, b^{a^k}\right]^{a^2}, \ldots, \left[b, b^{a^k}\right]^{a^{2^n-1}} \right\rangle \\ &= \left\langle \left[b, b^{a^k}\right], \left[b, b^{a^k}, a\right], \left[b, b^{a^k}, a^2\right], \ldots, \left[b, b^{a^k}, a^{2^n-1}\right] \right\rangle \\ &= \left\langle \left[b, b^{a^k}\right], \left[b, b^{a^k}, a\right], \left[b, b^{a^k}, a, a\right], \ldots, \left[b, b^{a^k}, \underbrace{a, \ldots, a}_{2^n-1}\right] \right\rangle.\end{aligned}$$

Indeed, the last equality follows from the more general fact that

$$\langle h, [h, a], [h, a^2], \ldots, [h, a^\ell] \rangle = \langle h, [h, a], [h, a, a], \ldots, [h, \underbrace{a, \ldots, a}_{\ell}] \rangle$$

for $h \in \mathrm{St}_G(1)'$ and $0 \leqslant \ell \leqslant 2^n - 1$. This is proved by induction on ℓ, using the identity

$$[h, a^{\ell+1}] = [h, a][h, a^\ell][h, a^\ell, a].$$

From this, we deduce that for each $i \geqslant 2$, either

$$\mathrm{St}_G(1)' \cap \gamma_i(G) = \mathrm{St}_G(1)' \cap \gamma_{i+1}(G)$$

or

$$\big(\mathrm{St}_G(1)' \cap \gamma_i(G)\big)/\big(\mathrm{St}_G(1)' \cap \gamma_{i+1}(G)\big) \simeq C_2.$$

As

$$\gamma_i(G) = \langle [b, \underbrace{a, \ldots, a}_{i-1}] \rangle \big(\mathrm{St}_G(1)' \cap \gamma_i(G)\big)\gamma_{i+1}(G),$$

the result follows, with the final statement coming from the fact that

$$[b, \underbrace{a, \ldots, a}_{2^n}] \in \mathrm{St}_G(1)'.$$

Now we suppose that **e** is symmetric, so if $j \in \{1, \ldots, 2^{n-1} - 1\}$, then $e_j = e_{2^n - j}$. Thus, there exists a minimal $k \in \{1, \ldots, 2^{n-1} - 2\}$ such that

$$\big(1, \ldots, 1, \left[b, a^{2^{n-1}}\right], \underbrace{1, \ldots, 1}_{k-1}, \left[b, a^{2^{n-1}}\right], 1, \ldots, 1\big) \in \psi\big(\mathrm{St}_G(1)'\big).$$

Let $g \in \mathrm{St}_G(1)'$ be such an element, chosen such that $g \in \gamma_\ell(G)$ with ℓ minimal. We observe that $k = 2^\lambda$ for some $\lambda \in \{0, 1, \ldots, n-2\}$, else k will not be minimal. Also,

$$\big(1, \ldots, 1, \big[b, a^{2^{n-1}}\big]\big) \notin \psi\big(\mathrm{St}_G(1)'\big).$$

Looking at the $2^n \times 2^n$ circulant matrix C defined by the vector

$$(1, \underbrace{0, \ldots, 0}_{2^\lambda - 1}, 1, 0, \ldots, 0) \in (\mathbb{F}_2)^{2^n},$$

a direct generalisation of [12, Lemma 2.7(i)] yields that the rank of C is $2^n - 2^\lambda$, as the polynomial associated to the circulant matrix is $1 + x^{2^\lambda} = (1+x)^{2^\lambda}$. Therefore we obtain $\log_2 |\mathrm{St}_G(1)'| = 2^n - 2^\lambda$, and

$$\begin{aligned}\mathrm{St}_G(1)' &= \big\langle g, g^a, g^{a^2}, \ldots, g^{a^{2^n - 2^\lambda - 1}} \big\rangle \\ &= \big\langle g, [g, a], [g, a, a], \ldots, [g, \underbrace{a, \ldots, a}_{2^n - 2^\lambda - 1}] \big\rangle.\end{aligned}$$

The result now follows as in the non-symmetric case. □

Lemma 3.8 *Let* G *be a GGS-group satisfying the conditions of* Lemma 3.6. *Then*

$$\langle [b, \underbrace{a, \ldots, a}_{2^n - 1}, b] \rangle^G = \big\{ \big[(ab)^{2^n}, g\big] \mid g \in G \big\}.$$

Proof — If $h = (ab)^{2^n}$, we know from [19, Proposition 1.1.32(i)] that

$$h = [b, \underbrace{a, \ldots, a}_{2^{n-1} - 1}]^2 [b, \underbrace{a, \ldots, a}_{2^n - 1}] c$$

for some $c \in \gamma_{2^n}(G) \cap \mathrm{St}_G(1)'$. Using Lemma 3.6, we observe that for all $1 \leqslant i \leqslant 2^n - 1$, we have

$$[h, a^i] \in \gamma_{2^n+1}(G) \leqslant \mathrm{St}_G(1)'.$$

Furthermore, one can check that

$$[h, a] = [h, b] = [b, \underbrace{a, \ldots, a}_{2^n - 1}, b].$$

Let $j > 2^n$ be maximal such that

$$[b, \underbrace{a, \dots, a}_{2^n-1}, b] \in \gamma_j(G).$$

Thus, since $\log_2 |St_G(1)'| \leqslant 2^n$ as seen in the proof of Lemma 3.7, we have

$$\gamma_j(G) = \langle [h, b], [h, b, a], \dots, [h, b, \underbrace{a, \dots, a}_{2^n-1}] \rangle.$$

Our goal is now to show that

$$\{[h, g] \mid g \in G\} = \langle [h, b], [h, b, a], \dots, [h, b, \underbrace{a, \dots, a}_{2^n-1}] \rangle.$$

To this purpose, as seen in the previous proof it suffices to prove

$$\left\{[h, g] \mid g \in G\right\} = \left\langle [h, b], [h, b]^a, \dots, [h, b]^{a^{2^n-1}} \right\rangle. \tag{3.2}$$

For any $1 \leqslant i \leqslant 2^n - 1$, we have that

$$[h, b]^{a^i} = \left[h^{a^i}, b^{a^i}\right] = \left[h\left[h, a^i\right], b^{a^i}\right] = [h, b^{a^i}], \tag{3.3}$$

where the last equality follows from the fact that

$$[h, a^i] \in \gamma_{2^n+1}(G) \leqslant St_G(1)'.$$

Next we note that

$$[h, a^k] = [h, b][h, b]^a \dots [h, b]^{a^k-1} \tag{3.4}$$

for all $1 \leqslant k \leqslant 2^n - 1$. Now, every element $g \in G$ is of the form

$$g = a^\lambda b^{a^{\ell_1}} \dots b^{a^{\ell_d}}$$

for some $d \geqslant 0$ and $\lambda, \ell_1, \dots, \ell_d \in \{0, 1, \dots, 2^n - 1\}$. Then taking into account (3.3) and (3.4), we get

$$[h, g] = [h, b]^{a^{\ell_d}} \dots [h, b]^{a^{\ell_1}} [h, a^\lambda],$$

from which we deduce that (3.2) holds. □

4 Beauville structures for quotients of infinite GGS-groups acting on the p^n-adic tree

We are now ready to prove that certain quotients of G are Beauville groups, for G an infinite periodic GGS-group acting on the p^n-adic tree where p is a prime and $n \geqslant 2$. Recall that for a 2-generator finite group H, the sets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ form a Beauville structure for H, if and only if

$$\langle x \rangle \cap \langle y^g \rangle = 1 \tag{4.1}$$

for all $x \in X = \{x_1, x_2, x_1x_2\}$, $y \in Y = \{y_1, y_2, y_1y_2\}$ and $g \in H$. We also remark that we will only consider $G/\mathrm{St}_G(k)$ for k large enough such that the order of each element in X (or in Y) is the same in G and in $G/\mathrm{St}_G(k)$. The reason for this will become apparent when we prove Theorem 1.1 later in this section.

Recall the meaning of $m_{r,s}$ from the beginning of Section 3. For G an infinite GGS-group, recall that there is some $1 \leqslant q \leqslant n-1$ such that $R_q = R_{q+1} < n$. We write p^d for the order of $\left[a^{p^{R_q}}, b, a^{p^{R_q}}\right]b$ in G, and we denote by λ' the minimal level where all sections of the element $\left(\left[a^{p^{R_q}}, b, a^{p^{R_q}}\right]b\right)^{p^{d-1}}$ that involve b are of the form $(b^*)^{a^*}$ for some unspecified exponents $*$. For the rest of this section, we set

$$m = \max\{m_{0,0} + \lambda',\ m_{1,0}\}.$$

Theorem 4.1 *Let G be an infinite periodic GGS-group acting on the p^n-adic tree, for p a prime and $n \geqslant 2$. Then the quotient $G/\mathrm{St}_G(m+3)$ is a Beauville group.*

Proof — Case 1: Suppose that $e_{kp^{n-1}} \neq 0$ for some $k \in \{1, \dots, p-1\}$. Note that by assumption the prime p must be odd. Let $1 \leqslant q \leqslant n-1$ be such that

$$R := R_q = R_{q+1} < n.$$

Suppose first that $e_{\ell p^{n-1}} = 0$ for some $\ell \in \{1, \dots, p-1\}$. Then writing $\mu = m_{0,0}$ and using Lemmata 3.3 and 3.2 repeatedly we obtain

$$c = \psi_{\mu+1}^{-1}\big(\big(\underbrace{1, \dots, 1}_{p^{(\mu+1)n-1}}, \left[a^{p^R}, b, a^{p^R}\right]\big)\big) \in \gamma_3(G).$$

By abuse of notation, we still write a, b for their images in G_{m+3}.

We set

$$X = \{a^{p-1}, ab, a^p b\} \quad \text{and} \quad Y = \{ac, b, acb\}.$$

Assume first that $x = a^{p-1}$, and consider $y = ac$. We write p^τ for the order of ac in G_{m+3}, which is strictly greater than p^n. In order to prove our claim for this choice of x and y, it suffices to show that

$$\left\langle a^{p^{n-1}} \right\rangle \neq \left\langle (ac)^{p^{\tau-1}} \right\rangle^g,$$

for all $g \in G_{m+3}$. Note that the element $(ac)^{p^{\tau-1}}$ is in $\mathrm{St}_{G_{m+3}}(1)$ and so is any conjugate of $(ac)^{p^{\tau-1}}$. However $a^{p^{n-1}} \notin \mathrm{St}_{G_{m+3}}(1)$, hence (4.1) holds here. Similarly for $y \in \{b, acb\}$.

Suppose next that $x = ab$, and consider $y = b$. By Lemma 3.1, the element ab has order $p^{t(0,0)}$. It suffices to consider the intersection

$$\left\langle (ab)^{p^{t(0,0)-1}} \right\rangle \cap \left\langle b^{p^{n-R_0-1}} \right\rangle^g.$$

We observe that $\psi\left(\left(b^{p^{n-R_0-1}}\right)^g\right)$ has exactly one component consisting of purely a non-rooted automorphism in $\mathrm{St}_G(1)$, whereas $\psi\left((ab)^{p^{t(0,0)-1}}\right)$ has p^n such components. Therefore (4.1) holds. For $y = ac$, we observe that $\psi_{\mu+1}\left((ac)^{p^{\tau-1}}\right)$ has p^n components consisting of non-rooted automorphisms in $\gamma_3(G)$, whereas all components of $\psi_{\mu+1}\left((ab)^{p^{t(0,0)-1}}\right)$ consisting of non-rooted automorphisms are in $\mathrm{St}_G(1)\backslash\gamma_3(G)$. Hence the result holds for this pair.

Let $x = a^p b$. Since $(a^p b)^{p^{n-1}}$ has p^{n-1} components consisting of non-rooted automorphisms at the first level, whereas both $(ac)^{p^n}$ and $(acb)^{p^n}$ have p^n such components and $b^{p^{n-R_0-1}}$ has one such component, we have that (4.1) holds here.

Hence, it remains to establish (4.1) for $x = ab$ with $y = acb$. Recall from the proof of Lemma 3.1 that

$$\psi_{\mu+1}^{p^{\mu n}}\left((ab)^{p^{t(0,0)-n+R_0}}\right) = \Big(\underbrace{\left(a^{p^{R_0}}\right)^*, \ldots, \left(a^{p^{R_0}}\right)^*}_{p^{t_{\mu-1}}-1}, b^{a^*}, \ldots, \underbrace{\left(a^{p^{R_0}}\right)^*, \ldots, \left(a^{p^{R_0}}\right)^*}_{p^{t_{\mu-1}}-1}, b\Big).$$

However,

$$\psi_{\mu+1}^{p^{\mu n}}\big((acb)^{p^{t(0,0)-n+R_0}}\big) = \Big(\underbrace{\big(a^{p^{R_0}}\big)^*,\dots,\big(a^{p^{R_0}}\big)^*}_{p^{t_{\mu-1}}-1},\big([a^{p^R},b,a^{p^R}]b\big)^{a^*},\dots,\underbrace{\big(a^{p^{R_0}}\big)^*,\dots,\big(a^{p^{R_0}}\big)^*}_{p^{t_{\mu-1}}-1},[a^{p^R},b,a^{p^R}]b\Big).$$

As $\psi\big([a^{p^R},b,a^{p^R}]b\big)$ has more than one section with a non-rooted automorphism, whereas $\psi(b)$ has only one, the result follows. Indeed,

$$\psi\big([a^{p^R},b,a^{p^R}]b\big) = \Big(\underbrace{a^*,\dots,a^*}_{p^R-1},(ba^{-e_{p^n-p^R}}b)^{a^{e_{p^R}}},\underbrace{a^*,\dots,a^*}_{p^R-1},(a^{e_{p^R}}b^{-1}a^{e_{p^R}})^{a^{e_{2p^R}}},a^*,\dots,a^*,b^{-1}a^{2e_{p^n-p^R}-e_{p^n-2p^R}}b\Big),$$

and so if $e_{p^R}=e_{p^n-p^R}=0$ then

$$\psi\big([a^{p^R},b,a^{p^R}]b\big) = \Big(\underbrace{a^*,\dots,a^*}_{p^R-1},b^2,\underbrace{a^*,\dots,a^*}_{p^R-1},(b^{-1})^{e_{2p^R}},a^*,\dots,a^*,(a^{-e_{p^n-2p^R}})^b\Big)$$

and the result is clear. If either e_{p^R} or $e_{p^n-p^R}$ is non-zero, we see from the proof of Lemma 3.1 that, writing p^χ for the order of acb in G, the minimal level λ of the tree, for which all λ-level sections of $(acb)^{p^{\chi-1}}$ that involve b are of the form $(b^*)^{a^*}$ for some unspecified exponents $*$, is strictly greater than $\mu+1$. In particular, all these cases imply that the number of sections of $(acb)^{p^{\chi-1}}$ at level λ consisting of purely non-rooted automorphisms is strictly greater than $p^{n-t_{\mu-1}}$, which is the corresponding number for $(ab)^{p^{t(0,0)-1}}$ at level λ.

Suppose now that $e_{\ell p^{n-1}}\neq 0$ for all $\ell\in\{1,\dots,p-1\}$. The result follows similarly using Lemma 3.4.

Case 2: Suppose that $e_{kp^{n-1}}=0$ for all $k\in\{1,\dots,p-1\}$. Since G is infinite, for some $1\leqslant q\leqslant n-1$ we have $R:=R_q=R_{q+1}<n-1$. Then we proceed as in the previous case, but note that for $p=2$ we

replace c with

$$\psi_{\mu+1}^{-1}\big(\big(\underbrace{1,\dots,1}_{2^{(\mu+1)n}-1}, \big[a^{2^R}, b, a^{2^{R+1}}\big]\big)\big),$$

since then, recalling that $R < n-1$,

$$\begin{aligned}\psi\big(\big[a^{2^R}, b, a^{2^{R+1}}\big]b\big) = \\ \Big(\underbrace{a^*,\dots,a^*}_{2^R-1}, \big(ba^{-e_{2^n-2^{R+1}}+e_{2^n-2^R}}\big)^{a^{e_{2^R}}}, \\ \underbrace{a^*,\dots,a^*}_{2^R-1}, \big(a^{e_{2^R}-e_{2^n-2^R}}b\big)^{a^{e_{2^{R+1}}}}, \\ \underbrace{a^*,\dots,a^*}_{2^R-1}, \big(a^{e_{2^{R+1}}}b^{-1}a^{e_{2^R}}\big)^{a^{e_{3\cdot 2^R}}}, \\ a^*,\dots,a^*, \big(a^{e_{2^n-2^R}-e_{2^n-3\cdot 2^R}+e_{2^n-2^{R+1}}}\big)^b\Big).\end{aligned}$$

The statement is proved. □

Proof of Theorem 1.1 — Let G be an infinite periodic GGS-group acting on the p^n-adic tree with p a prime and $n \geqslant 2$. Let t_0 be defined according to $r = s = 0$, and set $m_G = m + 3$. By Theorem 4.1, the quotient $G/\mathrm{St}_G(m_G)$ is a Beauville group with triples

$$X = \{a^{p-1}, ab, a^p b\} \quad \text{and} \quad Y = \{ac, b, acb\},$$

for c defined as in the proof of Theorem 4.1.

As the orders of each of the elements in X are the same in $G/\mathrm{St}_G(k)$ and $G/\mathrm{St}_G(m_G)$ for all $k \geqslant m_G$, it follows from [14, Lemma 4.2] that $G/\mathrm{St}_G(k)$ is a Beauville group for every $k \geqslant m_G$. □

Remark 4.2 We recall that a *mixed Beauville structure* for a finite group H is a triple (H, M, X) with $M \leqslant H$ a subgroup of index 2 and $X = \{x, y\}$ satisfying the following properties:

- $M = \langle x, y\rangle$,
- there exists $h_0 \in H\backslash M$ such that

$$h_0^{-1}\big(\Sigma(x,y)\big)h_0 \cap \Sigma(x,y) = \{1\},$$

- for all $h \in H \backslash M$, we have $h^2 \notin \Sigma(x, y)$.

Since M is of index 2, for finite p-groups, only 2-groups can admit mixed Beauville structures. Furthermore, if H admits a mixed Beauville structure (H, M, X), then M must be 2-generated. However, none of the finite 2-groups that we consider are mixed Beauville groups. This is because for G an infinite periodic GGS-group acting on the 2^n-adic tree for $n \geqslant 2$, no maximal subgroup of $G/St_G(k)$ is 2-generated. Indeed, consider G_k for a fixed $k \geqslant m_G$. Since $\Phi(G_k) = G_k^2$, there are three maximal subgroups of G_k, which are

$$M_1 := \langle a \rangle G_k^2, \quad M_2 := \langle b \rangle G_k^2, \quad M_3 := \langle ab \rangle G_k^2.$$

The Schreier index formula shows that M_1, M_2 and M_3 can be generated by at most 3 elements. Using the Reidemeister–Schreier method, we obtain

$$M_1 = \left\langle a, (ab)^2, b^2 \right\rangle, \quad M_2 = \left\langle a^2, b, (ab)^2 \right\rangle, \quad M_3 = \left\langle a^2, b^2, ab \right\rangle.$$

Clearly,

$$M_3/G_k' = \left\langle a^2, b^2, ab \right\rangle / G_k' \simeq \left\langle a^2 \right\rangle \times \left\langle b^2 \right\rangle \times \langle ab \rangle,$$

and thus $d(M_3) = 3$. Suppose for a contradiction that $d(M_1) = 2$. Since

$$M_1/G_k' = \left\langle a, b^2 \right\rangle G_k'/G_k' \simeq \langle a \rangle \times \left\langle b^2 \right\rangle,$$

and $[a, b] \notin \left\langle a, b^2 \right\rangle$, it follows that $d(M_1) = 3$. A similar argument holds for M_2.

Remark 4.3 In the above theorems, we only require that $S[i] \equiv 0 \pmod{p^{i+1}}$ for $i \in I$, for some subset $I \subseteq \{0, 1, \ldots, n-1\}$. Therefore there are non-periodic GGS-groups acting on the p^n-adic tree, for $n \geqslant 2$, with quotients being Beauville groups. For instance, if

- $t_0 > 2$ for the pairs $r = s = 0$ and $r = 1, s = 0$, and
- $R > 2$, and
- $S[2] \not\equiv 0 \pmod{p^3}$, but $S[i] \equiv 0 \pmod{p^{i+1}}$ for all other i,

the proofs above still hold.

Remark 4.4 In some cases, the proofs above can be extended to finite GGS-groups G. Since G is finite, there exists $d \in \{0, 1, \dots, n-1\}$ such that $R_{d+1} = n$. Writing $\mu = m_{0,0}$, we observe that $d \geqslant \mu - 1$. If $d > \mu$, then by repeated application of Lemmata 3.3 and 3.2, we have an element

$$c := \psi_{\mu+1}^{-1}\big((\underbrace{1, \dots, 1}_{p^{(\mu+1)n-1}}, \big[a^{p^{R_\mu}}, b, a^{p^{R_\mu}}\big])\big) \in \gamma_3(G).$$

Setting $X = \{a^{p-1}, ab, a^p b\}$ and $Y = \{ac, b, acb\}$, we are done as in Case 1 of the proof of Theorem 4.1.

If $d = \mu$, then using the same c and the sets X and Y as before, since $e_{\ell p^{R_\mu}} = 0$ for any ℓ, we observe that

$$\psi\big(\big[a^{p^{R_\mu}}, b, a^{p^{R_\mu}}\big]b\big) = \big(\underbrace{a^*, \dots, a^*}_{p^{R_\mu}-1}, b^2, \underbrace{a^*, \dots, a^*}_{p^{R_\mu}-1}, b^{-1}, a^*, \dots, a^*, 1\big),$$

and hence if p is odd, the same argument follows. For $p = 2$, the same argument can clearly be used upon replacing c with

$$\psi_{\mu+1}^{-1}\big((\underbrace{1, \dots, 1}_{2^{(\mu+1)n-1}}, \big[a^{2^{R_\mu}}, b, a^{2^{R_\mu+1}}\big])\big),$$

provided $R_\mu < n-1$, since then

$$\psi\big(\big[a^{2^{R_\mu}}, b, a^{2^{R_\mu+1}}\big]b\big) = \big(\underbrace{a^*, \dots, a^*}_{2^{R_\mu}-1}, b, \underbrace{a^*, \dots, a^*}_{2^{R_\mu}-1}, b,$$
$$\underbrace{a^*, \dots, a^*}_{2^{R_\mu}-1}, b^{-1}, a^*, \dots, a^*, 1\big).$$

However, for each of these finite groups G, one would only obtain finitely many (Beauville) quotients $G/St_G(k)$, for $k \geqslant m_G$.

In the next section, we identify a class of finite GGS-groups where no quotient by a level stabiliser is Beauville.

5 Finite GGS-groups acting on the p^n-adic tree

We define the subclass $\mathcal{E}$ of finite GGS-groups to consist of:

(i) the GGS-groups acting on the p^n-adic tree, for p any prime and $n \geqslant 2$, with $t_0 = R_0, \ldots, t_{m_{0,0}-1} = R_{m_{0,0}-1}$ and $R_{m_{0,0}} = n$; and

(ii) the GGS-groups acting on the 2^n-adic tree with $R_0 = n-1$, $S[0] \equiv 0 \pmod{p^n}$, and $e_{2^{n-1}} = 0$.

We show here that quotients of the GGS-groups $G \in \mathcal{E}$ by level stabilisers do not yield Beauville groups. We first consider the subfamily (i) of groups in $\mathcal{E}$.

Theorem 5.1 *Let $G \in \mathcal{E}$ be a GGS-group acting on the p^n-adic tree for $n \geqslant 2$, and assume that $t_0 = R_0, \ldots, t_{m_{0,0}-1} = R_{m_{0,0}-1}$ and $R_{m_{0,0}} = n$, where $t_0, t_1, \ldots, t_{m_{0,0}-1}$ are defined according to $r = s = 0$. Then the quotient $G/St_G(k)$ is not a Beauville group for all $k \in \mathbb{N}$.*

Proof — The result is clear for $k = 1$ since $G/St_G(1)$ is cyclic, so we assume $k \geqslant 2$. Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be two systems of generators for G. At least one of x_1, x_2, x_1x_2, call it z_1, must be in the coset $a^i b^j G'$ for $i, j \not\equiv 0 \pmod{p}$. Likewise for y_1, y_2, y_1y_2, and call it z_2. By Lemma 3.1 and its proof, the order of ab is $p^{t(0,0)}$ if $k \geqslant m_{0,0}+3$, and p^{2n-R_0} if $k = 2$. For $3 \leqslant k \leqslant m_{0,0}+1$, the order of ab is at least

$$p^n p^{n-R_0} p^{n-R_1} \ldots p^{n-R_{k-2}},$$

and for $k = m_{0,0}+2$, the order of ab is at least

$$p^n p^{n-R_0} p^{n-R_1} \ldots p^{n-R_{m_{0,0}-1}}.$$

Suppose first that $k = 2$. For $i, j \not\equiv 0 \pmod{p}$ and $w \in G_2'$, as

$$\phi_2\big((a^i b^j w)^{p^n}\big) = \big(a^{jS[0]}, \ldots, a^{jS[0]}\big)$$

it follows that $\langle z_1^{p^{2n-R_0-1}} \rangle = \langle z_2^{p^{2n-R_0-1}} \rangle$, which gives the result.

Now let $k \geqslant m_{0,0}+3$. Here, writing $t = t(0,0)$, we claim that

$$\langle (ab)^{p^{t-1}} \rangle = \langle (a^i b^j w)^{p^{t-1}} \rangle \simeq C_p$$

for all $i, j \not\equiv 0 \pmod{p}$ and $w \in G'$.

To prove the claim, we first note, writing

$$\psi(w) = (w_1, \dots, w_{p^n}),$$

that the product of all the components $w_1, \dots, w_{p^n}$, irrespective of the order, has zero total exponent in a, and in b, and if written as

$$\left(b^{\beta_1}\right)^{a^{\alpha_1}} \dots \left(b^{\beta_d}\right)^{a^{\alpha_d}}$$

for some $d \geqslant 2$ with $\beta_1, \dots, \beta_d \in \mathbb{Z}/p^{n-R_0}\mathbb{Z}$ and $\alpha_1, \dots, \alpha_d \in \mathbb{Z}/p^n\mathbb{Z}$, then we have $\alpha_i \equiv 0 \pmod{p^{R_0}}$ for all $i \in \{1, \dots, d\}$.

Hence, each component of $\psi\left((a^i b^j w)^{p^n}\right)$ is of the form

$$a^{jS[0]} b^j v_0,$$

for some $v_0 \in \langle a^{p^{R_0}}, b\rangle'$. We note that, for $\omega \in A$ with $\omega \not\equiv 0 \pmod{p^{R_0}}$,

$$\varphi_\omega\left(\left(a^{jS[0]} b^j v_0\right)^{p^{n-R_0}}\right) = \varphi_\omega\left(\left(a^{S[0]} b\right)^{jp^{n-R_0}}\right) = a^{j\gamma_\omega}$$

where

$$\gamma_\omega = \sum_{\ell=0}^{p^{n-R_0}-1} e_{\omega+\ell p^{R_0}},$$

and for $\omega \in A$ with $\omega \equiv 0 \pmod{p^{R_0}}$,

$$\varphi_\omega\left(\left(\left(a^{jS[0]} b^j v_0\right)^{p^{n-R_0}}\right)^{a^{fp^{R_0}}}\right) = a^{jS[R_0]} b^j v_1$$

for $v_1 \in \left\langle a^{p^{R_1}}, b\right\rangle'$ and $f \in \left\{0, 1, \dots, p^{n-R_0} - 1\right\}$.

Thus, continuing in this manner, we see that it suffices to compare

$$\left(a^{jS[R_{\mu-2}]} b^j v_{\mu-1}\right)^{p^{n-R_{\mu-1}}} \qquad \text{with} \qquad \left(a^{S[R_{\mu-2}]} b\right)^{p^{n-R_{\mu-1}}},$$

where $\mu = m_{0,0}$ and $v_{\mu-1} \in \left\langle a^{p^{R_{\mu-1}}}, b\right\rangle'$. Recalling also that $R_\mu = n$, equivalently the components of $\psi(b)$ in positions

$$p^{R_{\mu-1}}, 2p^{R_{\mu-1}}, \dots, p^n - p^{R_{\mu-1}}$$

are trivial, we have

$$\psi\big((a^{jS[R_{\mu-2}]}b^j\nu_{\mu-1})^{p^{n-R_{\mu-1}}}\big) = \\ \big(a^{j\delta_1},\dots,a^{j\delta_{p^{R_{\mu-1}}-1}},b^j,\ \dots\ ,a^{j\delta_1},\dots,a^{j\delta_{p^{R_{\mu-1}}-1}},b^j\big)$$

where, for $q \in \{1,\dots,p^{R_{\mu-1}}-1\}$,

$$\delta_q = \sum_{\ell=0}^{p^{n-R_{\mu-1}}-1} e_{q+\ell p^{R_{\mu-1}}}.$$

As

$$\psi\big((a^{S[R_{\mu-2}]}b)^{p^{n-R_{\mu-1}}}\big) = \\ \big(a^{\delta_1},\dots,a^{\delta_{p^{R_{\mu-1}}-1}},b,\ \dots\ ,a^{\delta_1},\dots,a^{\delta_{p^{R_{\mu-1}}-1}},b\big),$$

and, from the above computations, we deduce that all other non-trivial labels in the portraits of $(ab)^{jp^{t-1}}$ and $(a^ib^jw)^{p^{t-1}}$ are equal powers of a. Hence, the claim follows.

Thus, we have $\langle z_1^{p^{t-1}}\rangle = \langle z_2^{p^{t-1}}\rangle$.

For $3 \leqslant k \leqslant \mu+2$, writing $p^{\hat{t}}$ for the order of ab modulo $St_G(k)$, the analogous result $\langle z_1^{p^{\hat{t}-1}}\rangle = \langle z_2^{p^{\hat{t}-1}}\rangle$ follows similarly from the component-wise description of

$$\big(a^{jS[0]}b^j\nu_0\big)^{p^{n-R_0}},\ \big(a^{jS[R_0]}b^j\nu_1\big)^{p^{n-R_1}},\ \dots,\ \big(a^{jS[R_{k-4}]}b^j\nu_{k-3}\big)^{p^{n-R_{k-3}}},$$

where we recall that $R_{-1} = 0$. □

Theorem 5.2 *Let* G *be a periodic GGS-group acting on the* 2^n*-adic tree for* $n \geqslant 2$*. Suppose that* $R_0 = n-1$ *and* $S[0] \equiv 0 \pmod{p^n}$*. Then, the quotient* $G/St_G(k)$ *is not a Beauville group for all* $k \in \mathbb{N}$*.*

Proof — From Lemma 3.5, we only need to consider $k \leqslant 4$. For $k = 1$ the result is clear since Beauville groups are not cyclic. For the remaining values of k, let

$$X = \{x_1, x_2, x_1x_2\} \quad \text{and} \quad Y = \{y_1, y_2, y_1y_2\}$$

be two systems of generators for G_k.

Assume now $k = 2$. Then there exists some $z_1 \in X$ with $z_1 \equiv a^i \pmod{G_2'}$ for some odd i, and there exists $z_2 \in Y$ with $z_2 \equiv a^j \pmod{G_2'}$ for some odd j. Noting that the order of both z_1 and z_2 is 2^n, we now prove that, for $w, v \in G_2'$, the element $(a^i w)^{2^{n-1}}$ is conjugate to the element $(a^j v)^{2^{n-1}}$.

We observe that $\mathrm{St}_{G_2}(1)$, and hence G_2', is elementary abelian. Furthermore for each odd $i \in \{1, \dots, 2^n - 1\}$ we have

$$G_2' \leqslant \left\langle [a^i, b], [a^i, b, a^i], \dots, [a^i, b, \underbrace{a^i, \dots, a^i}_{2^n-2}] \right\rangle; \tag{5.1}$$

compare Lemma 3.6 and the proof of Lemma 3.7. It then follows from [19, Proposition 1.1.32(i)] that

$$\left(a^i w\right)^{2^{n-1}} = a^{i 2^{n-1}} \left[w, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right].$$

From (5.1) there exist $j_1, \dots, j_{2^n-1} \in \{0, 1\}$ such that w can be expressed as:

$$w = [a^i, b]^{j_1} [a^i, b, a^i]^{j_2} \dots [a^i, b, \underbrace{a^i, \dots, a^i}_{2^n-2}]^{j_{2^n-1}}$$

Now for $h \in \mathrm{St}_{G_2}(1)$, bearing in mind that $\mathrm{St}_{G_2}(1)$ is elementary abelian, we obtain from [19, Proposition 1.1.32(ii)] that

$$\left[a^{i 2^{n-1}}, h\right] = \left[a^i, h, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right]$$

in G_2.

Thus, setting

$$h = b^{j_1} [a^i, b]^{j_2} \dots [a^i, b, \underbrace{a^i, \dots, a^i}_{2^n-3}]^{j_{2^n-1}}$$

we deduce that

$$\left[a^{i 2^{n-1}}, h\right] = \left[w, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right].$$

Indeed,

$$\begin{aligned}\left[a^{i2^{n-1}}, h\right] &= \left[a^i, b^{j_1}, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right]\left[a^i, [a^i, b]^{j_2}, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right]\dots \\ &\qquad \left[a^i, \left[a^i, b, \underbrace{a^i, \dots, a^i}_{2^n-3}\right]^{j_{2^n-1}}, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right] \\ &= \left[\left[a^i, b\right]^{j_1}, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right]\left[\left[a^i, b, a^i\right]^{j_2}, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right]\dots \\ &\qquad \left[\left[a^i, b, \underbrace{a^i, \dots, a^i}_{2^n-2}\right]^{j_{2^n-1}}, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right] \\ &= \left[w, \underbrace{a^i, \dots, a^i}_{2^{n-1}-1}\right]\end{aligned}$$

where the second equality holds since $\mathrm{St}_{G_2}(1)'$ is trivial and $[a, x] = [x, a]$ for all $x \in \mathrm{St}_{G_2}(1)$.

Thus we have

$$\left(a^i w\right)^{2^{n-1}} = a^{i2^{n-1}}\left[a^{i2^{n-1}}, h\right] = \left(a^{i2^{n-1}}\right)^h = \left(a^{2^{n-1}}\right)^h$$

where the last equality holds since a has order 2^n and $i = 1 + 2d$ for some $d \in \mathbb{N}$. This proves that for all $i \in \{1, \dots, 2^n - 1\}$ and $w \in G_2'$ the element $(a^i w)^{2^{n-1}}$ is conjugate to $a^{2^{n-1}}$. Hence z_1 and z_2 are conjugate and G_2 is not a Beauville group.

Now assume $\mathrm{St}_G(3) \neq 1$. Then there is $z_1 \in X$ with $z_1 \equiv a^i b \pmod{G_3'}$ for some odd i, and there is $z_2 \in Y$ with $z_2 \equiv a^j b \pmod{G_3'}$ for some odd j. Observe that for $w \in G_3'$, we have

$$\phi_3\left((a^i b w)^{2^n}\right) = \left(b^{a^{\lambda_1 \cdot 2^{n-1}}}, \dots, b^{a^{\lambda_{2^n} \cdot 2^{n-1}}}\right)$$

where $\lambda_1, \dots, \lambda_{2^n} \in \{0, 1\}$. Moreover, since by Lemma 3.5 the elements b and $b^{a^{2^{n-1}}}$ coincide modulo $\mathrm{St}_G(2)$, we have

$$\phi_3\left((a^i b w)^{2^n}\right) = (b, \dots, b).$$

This proves that $z_1^{2^n}$ and $z_2^{2^n}$ are equal, and thus G_3 is not a Beauville group when $\mathrm{St}_G(3) \neq 1$.

We now consider the remaining cases, where here $G_k = G$. As before there exists some $z_1 \in X$ with $z_1 \equiv a^i b \pmod{G'}$ for some odd i, and there exists $z_2 \in Y$ with $z_2 \equiv a^j b \pmod{G'}$ for some odd j. Therefore it suffices to show, for $\delta \in \{0, 1, \dots, 2^{n-1} - 1\}$ and $w \in G'$, that $(a^{1+2\delta} bw)^{2^n}$ is conjugate to $(ab)^{2^n}$ in G.

Recall that $(G')^2 \leqslant St_G(1)'$ and that $St_G(1)'$ is elementary abelian. Hence, from [19, Proposition 1.1.32(i)] we have

$$(ab)^{2^n} = [b, \underbrace{a, \dots, a}_{2^{n-1}-1}]^2 [b, \underbrace{a, \dots, a}_{2^n-1}] c$$

for some $c \in \gamma_{2^n}(G) \cap St_G(1)'$. Let $i_1, i_2, i_3 \geqslant 2^n$ be such that

$$\big[[b, \underbrace{a, \dots, a}_{2^{n-1}-1}]^2, a\big] \in \gamma_{i_1}(G) \backslash \gamma_{i_1+1}(G),$$

$$[b, \underbrace{a, \dots, a}_{2^n}] \in \gamma_{i_2}(G) \backslash \gamma_{i_2+1}(G),$$

and

$$c \in \gamma_{i_3}(G) \backslash \gamma_{i_3+1}(G);$$

also set

$$\ell := \min\{i_1, i_2, i_3 + 1\}.$$

In other words, the integer ℓ is such that

$$\big[(ab)^{2^n}, a\big] \in \gamma_\ell(G) \backslash \gamma_{\ell+1}(G).$$

Now

$$\big(a^{1+2\delta} bw\big)^{2^n} = \big[bw, \underbrace{a^{1+2\delta}, \dots, a^{1+2\delta}}_{2^{n-1}-1}\big]^2 \big[bw, \underbrace{a^{1+2\delta}, \dots, a^{1+2\delta}}_{2^n-1}\big] cd,$$

where $d \in \gamma_{i_3+1}(G)$ is obtained as follows: starting with the commutator c, which is a product of commutators in b and a of weight at least 2^n, with weight exactly 2 in b; compare with [19, Proposition 1.1.32(i)], we have that $cd = \widetilde{c}$, where $\widetilde{c}$ is obtained from c by replacing all occurrences of b with bw. Here we have also used the standard identities

$$[xy, z] = [x, z]^y [y, z] \quad \text{and} \quad [x, yz] = [x, z][x, y]^z.$$

Next, using these standard identities together with Lemmata 3.6 and 3.7, we have

$$\begin{aligned}
&(a^{1+2\delta}bw)^{2^n} \\
&\equiv \big[bw, \underbrace{a^{1+2\delta}, \dots, a^{1+2\delta}}_{2^{n-1}-1}\big]^2 \big[bw, \underbrace{a^{1+2\delta}, \dots, a^{1+2\delta}}_{2^n-1}\big] c \pmod{\gamma_\ell(G)} \\
&\equiv \big[b, \underbrace{a, \dots, a}_{2^{n-1}-1}\big]^2 \big[b, \underbrace{a, \dots, a}_{2^n-1}\big] c \pmod{\gamma_\ell(G)}.
\end{aligned}$$

Therefore

$$\big(a^{1+2\delta}bw\big)^{2^n} \equiv (ab)^{2^n} \pmod{\gamma_\ell(G)}.$$

In other words, we have $(a^{1+2\delta}bw)^{2^n} = (ab)^{2^n}v$ for $v \in \gamma_\ell(G)$. By Lemma 3.8, we have that $v = [(ab)^{2^n}, g]$ for some $g \in G$, and hence we are done. □

Acknowledgements

We thank G.A. Fernández-Alcober for his valuable feedback and helpful suggestions. We also thank the referee for the helpful comments. The first author is supported by the Spanish Government, grant PID2020-117281GB-I00, partly with FEDER funds. The first and the second authors are supported by the Basque Government, grant IT974-16. The first author is also supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA-INdAM). The third author acknowledges support from EPSRC, grant EP/T005068/1, and from the Lincoln Institute of Advanced Studies. She also thanks the University of the Basque Country for its hospitality. The first and second authors would like to thank the University of Lincoln for its hospitality while this research was conducted.

REFERENCES

[1] N. Barker – N. Boston – B. Fairbairn: "A note on Beauville p-groups", *Exp. Math.* 21 (2012), 298–306.

[2] N. Barker – N. Boston – N. Peyerimhoff – A. Vdovina: "An infinite family of 2-groups with mixed Beauville structures", *Int. Math. Res. Not. IMRN* 11 (2015), 3598–3618.

[3] L. Bartholdi – R.I. Grigorchuk – Z. Šunić: "Branch Groups", *Elsevier*, Amsterdam (2003).

[4] F. Catanese: "Fibred surfaces, varieties isogenous to a product and related moduli spaces", *Amer. J. Math.* 122 (2000), 1–44.

[5] E. Di Domenico: "Some questions regarding groups of automorphisms of primary trees", *PhD thesis*, University of Trento and University of the Basque Country (2022).

[6] E. Di Domenico – G.A. Fernández-Alcober – N. Gavioli: "GGS-groups over primary trees: branch structures", *Monatsh. Math.* 200 (2023), 781–797.

[7] E. Di Domenico – Ş. Gül – A. Thillaisundaram: "Ramification structures for quotients of multi-EGS groups", *Int. J. Group Theory* 12 (2023), 237–252.

[8] B. Fairbairn: "Recent work on Beauville surfaces, structures and groups", in Groups St Andrews 2013, *Cambridge University Press*, Cambridge (2015), 225–241.

[9] B. Fairbairn: "Beauville p-groups: a survey", in Groups St Andrews 2017, *Cambridge University Press*, Cambridge (2019), 271–288.

[10] B. Fairbairn – E. Pierro: "New examples of mixed Beauville groups", *J. Group Theory* 18 (2015), 761–792.

[11] G.A. Fernández-Alcober – Ş. Gül: "Beauville structures in finite p-groups", *J. Algebra* 474 (2017), 1–23.

[12] G.A. Fernández-Alcober – A. Zugadi-Reizabal: "GGS-groups: order of congruence quotients and Hausdorff dimension", *Trans. Amer. Math. Soc.* 366 (2014), 1993–2017.

[13] Y. Fuertes – G. González-Diez –A. Jaikin-Zapirain: "On Beauville surfaces", *Groups Geom. Dyn.* 5 (2011), 107–119.

[14] Y. Fuertes – G.A. Jones: "Beauville surfaces and finite groups", *J. Algebra* 340 (2011), 13–27.

[15] R.I. Grigorchuk: "On Burnside's problem on periodic groups", *Funktsional. Anal. i Prilozhen* 14 (1980), 53–54.

[16] R.I. Grigorchuk: "Degrees of growth of finitely generated groups and the theory of invariant means", *Izv. Akad. Nauk SSSR Ser. Mat.* 48 (1984), 939–985.

[17] Ş. Gül – J. Uria-Albizuri: "Grigorchuk–Gupta–Sidki groups as a source for Beauville surfaces", *Groups Geom. Dyn.* 14 (2020), 689–704.

[18] G.A. Jones: "Beauville surfaces and groups: a survey", in Rigidity and Symmetry, *Springer*, New York (2014), 205–225.

[19] C.R. Leedham-Green – S. McKay: "The structure of groups of prime power order", *Oxford University Press*, Oxford (2002).

[20] M. Noce – A. Thillaisundaram: "Ramification structures for quotients of the Grigorchuk groups", *J. Algebra Appl.* 22 (2023), 2350047, 16pp.

[21] J.M. Petschick: "On conjugacy of GGS-groups", *J. Group Theory* 22 (2019), 347–358.

[22] J. Stix – A. Vdovina: "Series of p-groups with Beauville structure", *Monatsh. Math.* 181 (2016), 177–186.

[23] T. Vovkivsky: "Infinite torsion groups arising as generalizations of the second Grigorchuk group", in Algebra (Moscow, 1998), *de Gruyter*, Berlin (2000), 357–377.

[24] J.S. Wilson: "Groups with every proper quotient finite", *Proc. Cambridge Philos. Soc.* 69 (1971), 373–391.

Elena Di Domenico
Department of Mathematics
University of Trento
38123 Trento (Italy)

Department of Mathematics
University of the Basque Country UPV/EHU

48080 Bilbao (Spain)
e-mail: elena.didomenico@yahoo.it

Şükran Gül
Department of Mathematics
TED University
06420 Ankara (Turkey)
e-mail: sukran.gul@tedu.edu.tr

Anitha Thillaisundaram (*corresponding author*)
Centre for Mathematical Sciences
Lund University
223 62 Lund (Sweden)
e-mail: anitha.thillaisundaram@math.lu.se