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Power Graphs Defined on Non-Commutative Non-Associative Moufang Loops

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Abstract

For a non-abelian group G of order n and for an odd prime p we present the non-commutative non-associative Moufang loop $M = M_{pn}(G, p)$ of order pn. We name this loop the *generalized Chein loop* in celebrating the Chein loop defined in 1974 by Orin Chein. The loop M is indeed an extension of G by the cyclic group Z_p. The *power graphs* and the *connectivity degrees* of M and of G have been studied and compared in this paper for every prime p. As concrete examples we present infinite classes of such loops which are non-isomorphic and are of the same order. This adds to the small class of the finite non-commutative non-associative classes of Moufang loops. This study gives us a construction method of Eulerian graphs of finite order, as well.

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Keywords: Moufang loop; power graph

1 Introduction

We begin with some preliminary definitions concerning non-associative algebraic structures. A non-empty set X with a binary operation is called a quasi-group if the equation $xy = z$ has a unique solution whenever two of tree elements $x, y, z \in X$ are specified. A quasi-group with the neutral element is called a loop. The loop L is called a Moufang loop if the identity $(ab)(ca) = [a(bc)]a$ holds for all $a, b, c \in L$.

Note that, the Moufang property is equivalent to any of the identities $a[b(cb)] = [(ab)c]b$ and $a[b(ac)] = [(ab)a]c$. For more and detailed information on the Moufang loops one may consult [6, 7].

Some preliminaries are needed in groups and graphs. For an algebraic structure S the *power graph* of S denoted by $P(S)$, is an undirected connected graph with the vertex set S and two distinct vertices x and y are adjacent if and only if $x = y^m$ or $y = x^m$, for some integer $m \geqslant 2$. The *proper power graph* of S is defined to be the graph $P^*(S) = P(S) - e$. A chronological report on these graphs which mainly studied for groups, may be the article [10] as forerun source and the references [2, 3, 4, 8]. Remind the definition of an *Eulerian graph* which is an undirected connected graph with each vertex of even degree. The *connectivity degree* (vertex connectivity degree) of an undirected graph Γ will be denoted by $\kappa(\Gamma)$. Note that the notation "$\sim$" will be used to denote the adjacency of two vertices in a graph.

Our notation in group theory is merely standard. The cyclic group of order m will be denoted by Z_m, the identity element of a group G will be denoted by e, and $o(g)$ denotes the order of an element g of a group. Finally, we'll use the well-known notation $\pi = \langle X \mid R \rangle$ for a presentation of a group or semigroup where X is called the generating set and R is a set of relations (or relators) for the considered group (or semigroup). For a useful and prolific information on the presentation theory one may consult [5, 9, 12]. As usual, the notation $gp(\pi)$ will be used to denote a group presented by the presentation π.

Section 2 of this paper is devoted to construct a non-commutative non-associative Moufang loop of order pn, for every odd prime p and a given non-abelian group G of order n. The power graphs and related topics such as the proper power graph and the connectivity degree of the considered graphs will be studied in Sections 3 and 4. Hereby, certain infinite classes of non-isomorphic Eulerian graphs have been constructed which are of same order.

2 The Moufang loop $M_{pn}(G, p)$

We follow [6] and review the Theorem 1. This theorem may be simplified and stated as: $G = \langle X \rangle$ *is a non-abelian group of order* n *and* X *is a minimal generating set.* $L = \langle u, X \rangle$ *is a non-associative Moufang loop where* u *is an element of order* 2 *and* $\{u, X\}$ *is a minimal generating set for* L.

Then every element of L *may be presented in the form* gu^i *where* $i \in \{0, 1\}$. *Moreover, by considering the set* $Gu = \{gu \mid g \in G\}$, *the set*

$$M_{2n}(G, 2) = G \cup Gu$$

possesses a binary operation $*$ *such that* $(M_{2n}(G, 2), *)$ *is a non-associative Moufang loop isomorphic to* L.

The aim of this section is to generalize this theorem by letting $u \in L$ as an element of order p, an odd prime. Let $G = \langle X \rangle$, $L = \langle u, X \rangle$ and

$$Gu^i = \{gu^i \mid g \in G\}, (i = 1, 2, \ldots, p-1).$$

If $M_{pn}(G, p) = G \cup Gu \cup Gu^2 \cup \ldots \cup Gu^{p-1}$ then,

Theorem 2.1 *The non-empty set* $M_{pn}(G, p)$ *is a non-commutative non-associative Moufang loop of order* pn, *for every odd prime* p *and any non-abelian group* G.

Proof — As noted, any element a of $M_{pn}(G, p)$ belongs to G or is in the form $a = gu^\alpha$ where $g \in G$ and $\alpha \in \{1, 2, \ldots, p-1\}$. In the first case by using the notation $u^0 = e$ we let $a = au^0$ where, e is the identity element of the group G. So, it is sufficient to define a binary operation in $M_{pn}(G, p)$ which satisfies the Moufang property. For any two elements xu^α and yu^β we define the binary operation $*$ as:

$$(xu^\alpha) * (yu^\beta) = \begin{cases} (xy^{(-1)^\alpha})u^{\alpha+\beta}, & \alpha, \beta \in \{0, 1, \ldots, p-1\} \text{ and } \beta \text{ is even}, \\ (y^{(-1)^\alpha}x)u^{\alpha+\beta}, & \alpha, \beta \in \{0, 1, \ldots, p-1\}, \text{ and } \beta \text{ is odd}, \end{cases}$$

where, $\alpha + \beta$ is reduced modulo p. Since G is non-abelian then contains at least two elements y and z such that $yz \neq zy$. Hence, for any element $x \in G$ we get

$$(xu) * (y * z) = (xz^{-1}y^{-1})u \neq (xy^{-1}z^{-1})u = ((xu) * y) * z.$$

So, $(M_{pn}(G, p), *)$ is non-associative. To complete the proof we have to verify the Moufang relation

$$(a * b) * (c * a) = [a * (b * c)] * a$$

for every values of $a = xu^\alpha$, $b = yu^\beta$ and $c = zu^\gamma$ where $x, y, z \in G$

and $\alpha, \beta, \gamma \in \{0, 1, \ldots, p-1\}$. Let

$$A = (a * b) * (c * a) \quad \text{and} \quad B = [a * (b * c)] * a.$$

We consider the following possible cases of the parameters α, β and γ to prove $A = B$.

Case 1: *All of the parameters are even.*

$$\begin{aligned} A &= ((xy)(zx))u^{2\alpha+\beta+\gamma} = (xyzx)u^{2\alpha+\beta+\gamma}, \\ B &= ((xyz)x)u^{2\alpha+\beta+\gamma} = (xyzx)u^{2\alpha+\beta+\gamma}. \end{aligned}$$

Case 2: *All of the parameters are odd.*

$$\begin{aligned} A &= (y^{-1}x)(x^{-1}z)u^{2\alpha+\beta+\gamma} = (y^{-1}z)u^{2\alpha+\beta+\gamma}, \\ B &= (x^{-1})(xy^{-1}z)u^{2\alpha+\beta+\gamma} = (y^{-1}z)u^{2\alpha+\beta+\gamma}. \end{aligned}$$

Case 3: *The parity of one parameter is different from that of the others.* There are six possibilities. An almost tedious hand calculation is needed to prove the Moufang property. For instances we give the proof in two cases for γ.

If γ is even then α and β are odd. So,

$$A = (xz)(y^{-1}x)u^{2\alpha+\beta+\gamma} = (xzy^{-1}x)u^{2\alpha+\beta+\gamma},$$

$$B = \left(x^{(-1)^{\alpha+\beta+\gamma}}\right)(zy^{-1}x)u^{2\alpha+\beta+\gamma} = (xzy^{-1}x)u^{2\alpha+\beta+\gamma}.$$

And if γ is odd then α and β are even. So,

$$\begin{aligned} A = (xy)u^{\alpha+\beta} * (zx^{-1})u^{\gamma+\alpha} &= (zx^{-1})^{(-1)^{\alpha+\beta}}xyu^{2\alpha+\beta+\gamma} \\ &= (zx^{-1}xy)u^{2\alpha+\beta+\gamma} = (zy)u^{2\alpha+\beta+\gamma}, \end{aligned}$$

$$\begin{aligned} B &= [xu^{\alpha} * (zy)u^{\gamma+\beta}] * (xu^{\alpha}) \\ &= [(zyx)u^{\alpha+\beta+\gamma}] * (xu^{\alpha}) = (zyxx^{-1})u^{2\alpha+\beta+\gamma} = (zy)u^{2\alpha+\beta+\gamma}. \end{aligned}$$

This completes the proof showing that $(M_{pn}(G, p), *)$ is a non-commutative non-associative Moufang loop of order pn. □

3 Power graphs of $M_{pn}(G,p)$, $p=2$

The power graph and the proper power graph of the loop $M_{2n}(G,2)$ are studied in this section. Our main result is:

Theorem 3.1 *Let G be a finite non-abelian group of order n. If $M=M_{2n}(G,2)$, then:*

(i) $P(M)$ *is not an Eulerian graph.*

(ii) $\overline{P(M)}\setminus\{e\}$ *is an Eulerian graph if and only if the graph* $\overline{P(G)}\setminus\{e\}$ *is Eulerian.*

(iii) $P^*(M)=P^*(G)\cup\overline{K}_n$.

(iv) $\kappa\big(P(M)\big)=1$.

Proof — Evidently, $(gu)^2=e$, for every $g\in G$. Hence,

$$\deg(e)=(|G|-1)+|G|=2\times|G|-1$$

is an odd integer. This proves (i).

By the definition of $\overline{P(M)}$, the vertex e is a singleton and the vertices gu and h are adjacent in this graph, for every $g,h\in G$. Also, the vertices $\{gu\mid g\in G\}$ constitute the complete subgraph K_n. So, every vertex of $\overline{P(M)}\setminus\{e\}$, in the form gu, is of the even degree

$$(n-1)+(n-1)=2n-2.$$

Consequently, all of the vertices of the graph $\overline{P(M)}\setminus\{e\}$ are of even degree if and only if $\overline{P(G)}\setminus\{e\}$ is an Eulerian graph.

The proof of (iii) is easy because of the relations $(gu)^2=e$, for each g in $G\setminus\{e\}$. The part (iv) is a result of (iii). □

Corollary 3.2 *There exist two non-isomorphic Chein loops M_1 and M_2 of order 2^{m+1}, for every integer $m\geqslant 3$. Moreover, $\overline{P(M_i)}\setminus\{e\}$, $(i=1,2)$ are Eulerian graphs.*

Proof — We present two non-associative non-commutative loops of the same order 2^{m+1}, for every positive integer $m\geqslant 3$. Let

$$M_i=M_{2n}(G_i,2),\quad (i=1,2)$$

where $n = 2^m$, $G_1 = Q_8 \times Z_2^{m-3}$ and $G_2 = D_{2\times 4} \times Z_2^{m-3}$ (Q_8 is the quaternion group of order 8 and $D_{2\times 4}$ is the dihedral group of order 8). These groups are non-commutative and non-isomorphic, then the loops M_i, $(i = 1, 2)$ are too. To complete the proof we use the Theorem 3.1. Then, it is sufficient to prove that $\overline{P(G_i)} \setminus \{e\}$ are Eulerian graphs for $i = 1, 2$. Each of these groups may be generated by $m-1$ generators $a, b, c_1, c_2, \ldots, c_{m-3}$ and presented as,

$$G_1 = \langle a, b, c_1, \ldots, c_{m-3} \mid a^4 = e, b^2 = 1, ba = a^3b, c_i^2 = e,$$
$$ac_i = c_i a, bc_i = c_i b,\ i = 1, 2, \ldots, m-3\rangle,$$

$$G_2 = \langle a, b, c_1, \ldots, c_{m-3} \mid a^4 = e, b^2 = a^2, ba = a^3b, c_i^2 = e,$$
$$ac_i = c_i a, bc_i = c_i b,\ i = 1, 2, \ldots, m-3\rangle.$$

Every $g \in G_i$ is in the form $g = a^\alpha b^\beta \gamma$ where, $\gamma = \prod_{i=1}^{m-3} c_i^{\epsilon_i}$. Moreover, $\alpha \in \{0, 1, 2, 3\}$, $\beta \in \{0, 1\}$. Evidently, $\gamma = e$ if and only if

$$\epsilon_1 = \epsilon_2 = \ldots = \epsilon_{m-3} = 0.$$

By using these comments we are able to classify the elements of the groups and calculate the degrees of vertices of the associated power graphs of them. Let $\delta = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_{m-3}$.

Let $g \in G_1$, then we get the key relations hold in G_1 as follows:

(1) $(\gamma^2 = (a^2\gamma)^2 = e, \delta \neq 0)$, $(b\gamma)^2 = e$, $(a^2b\gamma)^2$,

(2) $(a\gamma)^4 = (a^3\gamma)^4 = e$, $(ab\gamma)^2 = (a^3b\gamma)^2 = e$,
$(a\gamma)^3 = a^3\gamma$, $(a^3\gamma)^3 = a\gamma$,

(3) $(a\gamma)^2 = (a^3\gamma)^2 = a^2$, $(a^2)^2 = e$.

The relations of (1) yield $\deg(g) = 1$ for $g \in \{\gamma, b\gamma, a^2\gamma, a^2b\gamma\}$. By the relations of (2) we conclude that

$$\deg(g) = 3, \quad \text{when} \quad g \in \{a\gamma,\ a^3\gamma,\ ab\gamma,\ a^3b\gamma\}.$$

Finally, the relations of (3) give us $\deg(a^2) = 1 + 2^{m-2}$ because of existing $2 \times 2^{m-3}$ possible values for ϵ_i,s and the adjacency of the vertices e and a^2. Since G_1 is of order 2^m then $\deg(e) = 2^m - 1$. Hence, the degree of any vertex of the graph $P(G_1)$ belongs to $\{1, 3, 1 + 2^{m-2}, 2^m - 1\}$. Consequently, the vertices of $\overline{P(G_i)} \setminus \{e\}$ are of even degrees which yields that $\overline{P(M_1)} \setminus \{e\}$ is an Eulerian graph.

In a similar method we may check that the degree of any vertex of $P(G_2)$ belongs to $\{1, 3, 1+3\times 2^{m-2}, 2^m-1\}$. This in turn yields that $\overline{P(M_2)}\setminus\{e\}$ is an Eulerian graph. □

4 Power graphs of $M_{pn}(G,p)$, $(p>2)$

In the case when $p>2$, the method of investigation is almost different from that of the last section and needs to discover certain preliminary results. Consider $(M_{2n}(G,p),*)$ where G is a finite non-abelian group of order n and $p>2$ is a prime. Evidently, the powers of any element $gu^\alpha \in M_{2n}(G,p)$ is defined inductively as

$$(gu^\alpha)^2 = (gu^\alpha)*(gu^\alpha),$$

and

$$(gu^\alpha)^k = (gu^\alpha)*(gu^\alpha)^{k-1},$$

for every positive integer $k\geqslant 2$ and every $\alpha\in\{1,2,\dots,p-1\}$.

Throughout this section we use the notation $\deg(gu^\alpha)$ to denote the degree of the vertex $v\in P\big(M_{pn}(G,p)\big)$ which is corresponded to the element $gu^\alpha\in M_{pn}(G,p)$, for simplicity.

Lemma 4.1 *Let p be an odd prime and G is a finite non-abelian group of order n. For every integer $k\geqslant 2$ and any element $gu^\alpha\in M_{pn}(G,p)$ where $\alpha\in\{1,2,\dots,p-1\}$,*

$$\big(gu^\alpha\big)^k = \begin{cases} g^k u^{\alpha k}, & \text{if } \alpha \text{ is even},\\ g^{\frac{1-(-1)^k}{2}} u^{\alpha k}, & \text{if } \alpha \text{ is odd}. \end{cases}$$

Proof — We use an induction method on k. For $k=2$ we have:

$$\big(gu^\alpha\big)^2 = (gu^\alpha)*(gu^\alpha) = \begin{cases} \big(gg^{(-1)^\alpha}\big)u^{2\alpha} = g^2u^{2\alpha}, & \text{if } \alpha \text{ is even},\\ \big(g^{(-1)^\alpha}g\big)u^{2\alpha} = u^{2\alpha}, & \text{if } \alpha \text{ is odd}. \end{cases}$$

The rest of the proof is straightforward by considering the induction

hypothesis. We consider two cases for α. For even values of α we get:

$$\begin{aligned}(gu^{\alpha})^{k+1} &= (gu^{\alpha}) * (gu^{\alpha})^{k},\\ &= (gu^{\alpha}) * (g^{k}u^{k\alpha}),\\ &= g(g^{k})^{(-1)^{\alpha}}u^{(k+1)\alpha}, \qquad (\alpha \text{ is even integer}),\\ &= g^{k+1}u^{(k+1)\alpha}.\end{aligned}$$

And for odd values of α,

$$\begin{aligned}(gu^{\alpha})^{k+1} &= (gu^{\alpha}) * (gu^{\alpha})^{k},\\ &= (gu^{\alpha}) * \big(g^{\frac{1-(-1)^{k}}{2}}u^{k\alpha}\big),\\ &= g^{(-1)^{\alpha}\times\frac{1-(-1)^{k}}{2}}gu^{(k+1)\alpha},\\ &= g^{\frac{-1+(-1)^{k}}{2}+1}u^{(k+1)\alpha}, \qquad (\alpha \text{ is odd integer}),\\ &= g^{\frac{1+(-1)^{k}}{2}}u^{(k+1)\alpha},\\ &= g^{\frac{1-(-1)^{k+1}}{2}}u^{(k+1)\alpha}.\end{aligned}$$

This completes the proof. □

Lemma 4.2 *Let* p *be an odd prime and* G *is a finite non-abelian group of order* n. *Then,* $\deg(u^{\alpha}) = p + n - 2$, *for every* $\alpha \in \{1, 2, \ldots, p-1\}$

Proof — Let $v = u^{\alpha}$. Since $u^{p} = e$ then, v is adjacent with $p-1$ vertices

$$\{u^{j} \mid 1 \leqslant j \leqslant p,\ \ j \neq \alpha\},$$

for every $\alpha \in \{1, 2, \ldots, p-2\}$. To detect other vertices adjacent to v, we consider two cases.

Case 1: $\alpha \in \{1, 3, 5, \ldots, p-2\}$. For every $g \in G \setminus \{e\}$ and $k = p+1$ the Lemma 4.1 gives us:

$$(gu^{\alpha})^{k} = g^{\frac{1-(-1)^{k}}{2}}u^{k\alpha} = g^{0}u^{(p+1)\alpha} = u^{\alpha}.$$

Hence, v is also adjacent with $n-1$ vertices corresponding to gu^{α}, $(g \in G \setminus \{e\})$. Moreover, there is no vertex adjacent with v other than the the above identified vertices. In contrary, if for some $h \in G \setminus \{e\}$

and integers k and β,

$$u^{\alpha} = (hu^{\beta})^{k} = \begin{cases} h^{k}u^{\beta k}, & \text{if } \beta \text{ is even,} \\ h^{\frac{1-(-1)^{k}}{2}}u^{\beta k}, & \text{if } \beta \text{ is odd,} \end{cases}$$

then, β should be an odd integer and k should be even, which yields the contradiction $\alpha = k\beta$. Consequently,

$$\deg(v) = p - 1 + n - 1 = p + n - 2,$$

for every odd values of α.

Case 2: $\alpha \in \{2, 4, 6, \ldots, p-1\}$. In this case each α is in the form $2^{\gamma} \times t$, for some odd integer t and a positive integer γ. Let $\beta = \frac{\alpha}{2^{\gamma}}$ and $k{=}2^{\gamma}$. Since β is an odd integer the Lemma 4.1 gives us:

$$\left(hu^{\beta}\right)^{k} = h^{\frac{1-(-1)^{k}}{2}}u^{k\beta} = h^{\frac{1-(-1)^{2^{\gamma}}}{2}}u^{2^{\gamma}\times\beta} = u^{\alpha},$$

for every $h \in G \setminus \{e\}$. Hence, v is adjacent with hu^{β} for all $h \in G \setminus \{e\}$. This proves that $\deg(v) = p - 1 + n - 1 = p + n - 2$, in this case. □

Lemma 4.3 *Let* p *be an odd prime and* G *is a finite non-abelian group of order* n. *If* $v = gu^{\alpha}$ *where,* $g \neq e$ *then,*

$$\deg(v) = \begin{cases} 2p, & \text{if } \alpha = 1, 3, \ldots, p-2, \\ o(g) - 1, & \text{if } \alpha = 2, 4, \ldots, p-1 \text{ and } p \mid m, \\ p \times o(g) - 1, & \text{if } \alpha = 2, 4, \ldots, p-1 \text{ and } p \nmid m, \end{cases}$$

Moreover, for odd values of α, v *is adjacent with the vertex corresponding to* g *however, for even values of* α, v *is adjacent with the vertex corresponding to* g *if and only if* $p \nmid o(g)$.

Proof — As well as in the last lemma we use the key Lemma 4.1 and consider two cases.
Case 1: $\alpha \in \{1, 3, \ldots, p-2\}$. For every $g \in G \setminus \{e\}$ and every $k \geqslant 2$ we have:

$$(gu^{\alpha})^{k} = g^{\frac{1-(-1)^{k}}{2}}u^{\alpha k}.$$

Since $u^{\alpha(p+k)} = u^{\alpha k}$ then, letting $k = 2, 3, \ldots, p+1$ yields that the vertex v is adjacent with all of the vertices corresponding to the ele-

ments of the set $A_1 \cup A_2$ where,

$$A_1 = \{u^{2\alpha}, u^{3\alpha}, \dots, u^{(p-1)\alpha}, e, u^{(p+1)\alpha}\},$$
$$A_2 = \{gu^{2\alpha}, gu^{3\alpha}, \dots, gu^{(p-2)\alpha}, gu^{(p-1)\alpha}, g\}.$$

Note that, in the set A_2 the element $gu^{(p+1)\alpha}$ is absent because of the definition of the power graph. Evidently, A_1 and A_2 contain p and $p-1$ different elements, respectively. Moreover, reducing the powers of u modulo p yields:

$$A_1 = \{e, u, u^2, \dots, u^{p-1}\}, \quad A_2 = \{g, gu^2, \dots, gu^{p-1}\}.$$

There is one more vertex adjacent with v. Indeed, reviewing the proof of Lemma 4.2 (Case 1) we see that the vertex corresponding to u^α is adjacent with v. Hence, $\deg(v) = p + (p-1) + 1 = 2p$, for every α in $\{1, 3, \dots, p-2\}$.

As a result of this proof v is adjacent with the vertex corresponding to g, for every odd values of α.

Case 2: $\alpha \in \{2, 4, \dots, p-1\}$. The Lemma 4.1 gives us $(gu^\alpha)^k = g^k u^{\alpha k}$, for every integer $k \geqslant 2$. Suppose that $o(g) = m$. Since $g^m = e$ and $u^{\alpha p} = e$ then,

$$g^{rm+i} = g^i, \quad u^{\alpha((r+p)m+i)} = u^{\alpha(rm+i)},$$

for every integers $0 \leqslant r \leqslant p-1$ and $2 \leqslant i \leqslant m+1$. To prove the assertions we have to consider two cases: $p \mid m$ and $p \nmid m$. Obviously, v is not adjacent with itself.

If $p \mid m$ then v is adjacent with all of the different vertices corresponding to the elements

$$\{g^{kp} \mid k = 1, 2, \dots, \tfrac{m}{p} - 1\},$$
$$\{g^k u^{\alpha k} \mid k = 2, \dots, m-1, \ \ p \nmid k\},$$
$$\{g^m u^{\alpha m}\} = \{e\}.$$

On the other hand, according to the proof of Case 2 of Lemma 4.2, if $\alpha = 2^\gamma \times t$ then the vertex corresponding to u^α is adjacent with the corresponding vertex of $gu^{\frac{\alpha}{2^\gamma}}$. Since $\gamma > 0$ then $\frac{\alpha}{2^\gamma} \neq \alpha$. Hence, the vertex of v is not adjacent with the vertices $\{u^2, u^4, \dots, u^{p-1}\}$. A similar proof shows that the vertex of v is not adjacent with the ver-

tices $\{u, u^3, \ldots, u^{p-2}\}$ (by the proof of Case 2 of Lemma 4.2). So, $\deg(v) = \left(\frac{m}{p} - 1\right) + (m - 1 - \frac{m}{p} + 1) = m - 1 = o(g) - 1$.

If $p \nmid m$ again we consider the relations

$$g^{rm+i} = g^i \quad \text{and} \quad u^{\alpha((r+p)m+i)} = u^{\alpha(rm+i)}$$

with the prescribed conditions on i and r. For each possible value of the integer k the relation $(gu^\alpha)^k = g^k u^{\alpha k}$ gives us the element h where its corresponding vertex is adjacent with v. These elements are:

$$k = rm + 2 \Rightarrow h = g^2 u^{(rm+2)\alpha}, \quad (r = 0, 1, \ldots, p-1),$$

$$k = rm + 3 \Rightarrow h = g^3 u^{(rm+3)\alpha}, \quad (r = 0, 1, \ldots, p-1),$$

$$\ldots$$

$$k = rm + m \Rightarrow h = g^m u^{(rm+m)\alpha} = u^{(r+1)m\alpha}, \quad (r = 0, 1, \ldots, p-1),$$

$$k = rm + (m+1) \Rightarrow h = g^{m+1} u^{(rm+(m+1))\alpha} = gu^{((r+1)m+1)\alpha}, \quad (r = 0, 1, \ldots, p-2).$$

Finally, note that there is no other vertices adjacent with v except the above vertices (because of Lemma 4.1 and of Case 2 of Lemma 4.2). Hence,

$$\deg(v) = p(m-1) + (p-1) = pm - 1 = p \times o(g) - 1.$$

To complete the proof we show that v is adjacent with the vertex corresponding to g. Indeed, u is an element of order p, and m may be written in the form $m = pt + \gamma$, for some integers t and $0 < \gamma < p$. Then, $gu^{((r+1)m+1)\alpha} = gu^{((r+1)\gamma+1)\alpha}$. Since

$$1 \geqslant r + 1 \geqslant p - 1 \quad \text{and} \quad 1 \geqslant \gamma \geqslant p - 1,$$

then for every even values of α, the equivalence equation

$$(r+1)\gamma + 1) \equiv 0 \pmod{p}$$

has a solution for every γ, (for instance, $(\gamma, r) = (0, p-1)$ and $(\gamma, r) = (1, (p-1)/2)$). This proves that

$$u^{((r+1)m+1)\alpha} = e$$

for each solution of $(r+1)\gamma+1 \equiv 0 \pmod{p}$, i.e., v is adjacent with the vertex corresponding to g. □

Theorem 4.4 *Let p be an odd prime, G is a non-abelian group of a given order $n \geqslant 6$ and $M = M_{pn}(G,p)$. Then:*

(i) $\kappa\big(P(M)\big) \geqslant 2$.

(ii) *For every $p \equiv 1 \pmod{4}$ Eulerianity of the graph $P(G)$ is sufficient for the Eulerianity of the graph $P(M)$.*

Proof — As noted at the beginning of this section we use the notation $\deg(gu^\alpha)$ to denote the degree of the vertex $v \in P(M_{pn}(G,p))$ which is corresponded to the element $gu^\alpha \in M_{pn}(G,p)$.

To prove (i) consider the graph $P^*(M)$. By Lemma 4.2 any two vertices corresponding to u^α and u^β are adjacent and also, u^α is adjacent with gu^α. So, by removing the vertex e the remained graph is connected. This proves that $\kappa\big(P(M)\big) \geqslant 2$.

To prove (ii) suppose that $P(G)$ is an Eulerian graph. Evidently, n and the order of any element of G are odd integers. Hence,

$$\deg(e) = (n-1) + (p-1) + (n-1)(p-1) = pn - 1$$

is an even integer and by Lemma 4.2, $\deg(u^\alpha) = p + n - 2$ is also an even integer for each α. For identifying $\deg(gu^\alpha)$, where $g \in G \setminus \{e\}$, consider Lemma 4.3. Then, $\deg(gu^\alpha)$ is an even integer, for every α and every $g \in G \setminus \{e\}$. Finally, suppose that $\overline{\deg(g)}$ and $\deg(g)$ are the degrees of the vertex corresponding to g in $P(G)$ and $P\big(M_{pn}(G,p)\big)$, respectively. Then we get:

$$\deg(g) = \overline{\deg(g)} + \frac{p-1}{2}.$$

This is because of the proof of Lemma 4.3 where we showed that just for odd values of α ($\alpha = 1, 3, 5, \ldots, p-2$), gu^α is adjacent with g, for each $g \in G \setminus \{e\}$. Since $\overline{\deg(g)}$ and $\frac{p-1}{2}$ are even integers, every vertex of the graph $P\big(M_{pn}(G,p)\big)$ is of even degree and then $P\big(M_{pn}(G,p)\big)$ is an Eulerian graph. □

Corollary 4.5 *For every odd prime $p \equiv -1 \pmod{4}$ let G be a finite non-abelian p-group of order n. Then, the Eulerianity of the graph $P(G)$ is not sufficient for the Eulerianity of $P\big(M_{pn}(G,p)\big)$.*

Proof — We give two counter examples. These are the well-known extra special p-groups of order p^3, as follows:

$$G_1 = \langle a, b, c \mid a^p = b^p = c^p = 1, ac = ca, bc = cb, bac = ab \rangle,$$
$$G_2 = \langle a, b \mid a^{p^2} = b^p = 1, ab = ba^{p+1} \rangle$$

The first one is of exponent p and the second one is of exponent p^2. These groups studied recently for their power graphs in [11] and among all other things, the authors proved that $P(G_1)$ and $\overline{P(G_2)}$ are Eulerian graphs, for every odd prime $p \equiv \pm 1 \pmod 4$. So, $\overline{\deg(g)}$, the degree of any element $g \in G_i \setminus \{e\}$, $i = 1, 2$, is an even integer. However, in the case when $p \equiv -1 \pmod 4$ we see that

$$\deg(g) = \overline{\deg(g)} + \frac{p-1}{2}$$

is an odd integer. Showing that $P\big(M_{pn}(G, p)\big)$ is not Eulerian. □

5 Conclusion

For an odd prime p a p-group G is called extra special if the center and the commutator subgroups of G coincide. As a known fact such a group is of order $n = p^{2m+1}$, for some positive integer m. Let $M = M_{pn}(G,p)$. Then M is a non-commutative non-associative Moufang loop of order p^{2m+2}. The study of $P(M)$ for $m = 1$, as in the Corollary 4.5, motivates us to pose the following conjecture.

Conjecture 5.1 *For an odd prime* p *and the integer* $m \geqslant 1$ *let* G *be an extra special* p*-group of order* p^{2m+1} *and* $M = M_{p^{2m+2}}(G, p)$. *Then,* $P(G)$ *is an Eulerian graph. Moreover,* $P(M)$ *is an Eulerian graph of order* p^{2m+2} *if* $p \equiv 1 \pmod 4$.

Indeed, the Theorem 4.4 and the Corollary 4.5 verify the conjecture for $m = 1$. We suppose that by getting almost tedious presentation for G, the lengthy proofs of [11] could be formulated when $m \geqslant 2$.

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