

Advances in Group Theory and Applications
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17 (2023), pp. 129–141 ISSN: 2499-1287
DOI: 10.32037/agta-2023-019

Central Endomorphisms of Groups and Radical Rings *

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(*Received Mar. 31, 2023; Accepted Sept. 1, 2023 — Communicated by F. de Giovanni*)

Abstract

An endomorphism γ of a group G is called a *central endomorphism* if $x^{-1}x^{\gamma}$ lies into the centre $Z(G)$ of G for each element x of G. It is easy to show that all non-zero central endomorphisms of G are automorphisms if and only if the ring $R = \mathrm{Hom}(G, Z(G))$ of all homomorphisms of G into $Z(G)$ is a (Jacobson) *radical ring*, i.e. R is a group via the *circle* operation $f \circ g = f + g + fg$ for all $f, g \in R$. In this article it is proved that R is an artinian radical ring if and only if G has no non-trivial abelian direct factors and the additive group of R satisfies the minimal condition on subgroups. Moreover, it is pointed out how central automorphisms can be used to produce solutions of the Yang-Baxter equation.

Mathematics Subject Classification (2020): 20E36, 16Nxx, 16P20, 16T25

Keywords: central endomorphism; radical ring; Yang-Baxter equation

1 Introduction

An endomorphism γ of a group G is said to be a *central endomorphism* if $x^{-1}x^{\gamma}$ belongs to the centre $Z(G)$ of G for each element x of G. Clearly every central endomorphism of G is *normal*, i.e. it commutes with all inner automorphisms of G. Conversely, it is easy to show that

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every normal endomorphism which is surjective is central. Therefore, an automorphism is central if and only if it is normal. It follows that the set $\mathrm{Aut}_c(G)$ of all central automorphisms of G is a normal subgroup of the full automorphism group $\mathrm{Aut}(G)$ of G. A relevant theorem by Cooper [7] shows that the group $\mathrm{Aut}_c(G)$ contains all *power automorphisms* of G, i.e. the automorphisms which fix every subgroup of G. We point out that this result was extended to *cyclic automorphisms* in [11]; recall here that an automorphism γ of a group G is said to be a cyclic automorphism if the subgroup $\langle x, x^\gamma \rangle$ is cyclic for any element $x \in G$ — see also [3]. Central automorphisms have been investigated by several authors (see, for instance, [4],[8],[12],[15],[18],[19]), and play an important role in many problems concerning nilpotent groups.

Let γ be a central endomorphism of a group G. Clearly the map

$$f_\gamma : x \in G \mapsto x^{-1}x^\gamma \in Z(G)$$

is a homomorphism from G into $Z(G)$; in particular, γ acts trivially over the commutator subgroup G' of G. Conversely, any homomorphism

$$h : G \longrightarrow Z(G)$$

induces a central endomorphism γ_h of G, defined by setting $x^{\gamma_h} = xx^h$ for every element $x \in G$. Let

$$R = \mathrm{Hom}(G, Z(G))$$

be the ring consisting of all homomorphisms from G into $Z(G)$, with the natural operations defined by the rules

$$x^{f+g} = x^f + x^g \quad \text{and} \quad x^{fg} = (x^f)^g$$

for all homomorphisms $f, g \in \mathrm{Hom}(G, Z(G))$ and for every $x \in G$. Consider the *adjoint monoid* R^{ad} of R, i.e. the monoid constructed on R with *circle operation* $\circ$ defined by the rule

$$f \circ g = f + g + fg,$$

for all $f, g \in R$. Clearly, the zero homomorphism of R is the identity element with respect to $\circ$. It is easy to show that the map

$$f : \gamma \in \mathrm{End}_c(G) \mapsto f_\gamma \in R$$

is an isomorphism from the monoid $\mathrm{End}_c(G)$ of central endomorphisms of G onto R^{ad}. As a consequence, we have that the group $\mathrm{Aut}_c(G)$ is isomorphic to the *adjoint group* R^0 of R, i.e. the group of units of R^{ad}.

A well known problem concerning central endomorphisms is to investigate the conditions under which every non-zero central endomorphism of a group G is an automorphism. For the above considerations this is equivalent to saying that the ring $\mathrm{Hom}(G, Z(G))$ is *radical* (in the sense of Jacobson [14]). In 1965, Adney and Yen [1] proved that for a *finite* group G the ring $\mathrm{Hom}(G, Z(G))$ is radical if and only if G is *purely non-abelian*, i.e. G has no abelian non-trivial direct factors. This result was extended by Franciosi *et al.* [9] to a wider class containing all periodic groups with no infinite abelian sections of prime exponent (see also [18]).

The aim of this paper is to give some further contribution to this topic. In particular, it will be proved that for a purely non-abelian group G the ring $\mathrm{Hom}(G, Z(G))$ is radical whenever its additive group satisfies the minimal condition on subgroups.

In 2007, Rump [17] found a surprising relationship between radical rings and the *Yang-Baxter equation*, an important equation in theoretical physics and in statistical mechanics that has become the key tool in many areas of research, such as quantum groups, Hopf algebras, knot theory and tensor categories. In fact, Rump proved that a radical ring (or more general, a *brace*) produces an *involutive non-degenerate* solution of the Yang-Baxter equation (see the survey [5] for an interesting introduction to this topic). In the last part of the article we will point out how the central automorphisms of a group can be used to obtain set theoretical solutions of the Yang-Baxter equation.

Most of our notation is standard and can be found in [16]. Moreover, for some results concerning the theory of radical rings, we refer to [13, Chapter IX].

2 Results

Let G be a group having a direct decomposition $G = H \times K$, where H is a non-trivial abelian subgroup. Then the rules $x^h = x^{-1}$ if $x \in H$ and $x^h = 1$ if $x \in K$, define a homomorphism h of G into its centre $Z(G)$ such that the induced central endomorphism γ_h of G is not injective provided that $K \neq \{1\}$. It follows that a non-abelian group

in which every non-zero central endomorphism is an automorphism must be purely non-abelian. The converse is true for finite groups by the quoted result of Adney and Yen [1]. On the other hand, it is possible to construct infinite purely non-abelian groups having non-trivial central endomorphisms which are not automorphisms.

In fact, let $A = \langle x \rangle$ and $B = \langle y \rangle$ be a cyclic group of order 3 and an infinite cyclic group, respectively. Consider the semidirect product

$$G = B \ltimes A,$$

where $x^y = x^2$ and $[x, B^2] = \{1\}$. Clearly, $Z(G) = B^2$, $G' = A$ and G is the unique non-split extension of $\mathbb{Z}$ by S_3, the symmetric group of order 3, so G is purely non-abelian. Moreover, $\mathrm{Hom}(G/Z(G), Z(G)) = \{1\}$, and hence $|\mathrm{Aut}_c(G)| = 2$. But

$$\mathrm{Hom}(G, Z(G)) \simeq \mathrm{Hom}(G/G', Z(G)) \simeq \mathrm{Hom}(\mathbb{Z}, Z(G)) \simeq \mathbb{Z},$$

and hence G has infinitely many central endomorphisms.

On the other hand, we recall that if G is a purely non-abelian group such that $Z(G)$ or G/G' is periodic with no infinite sections of prime exponent, then every central non-zero endomorphism of G is an automorphism (see [9, Theorem 2.15] and [18, Corollary 2.8]).

Let R be a radical ring, and consider an *idempotent* element e in R, i.e. an element such that $e^2 = e$. As R coincides with its adjoint group R^0, there exists an element $r \in R$ such that $-e + r - re = 0$. Therefore $e = r - re$, and hence

$$e = e^2 = re - re^2 = re - re = 0.$$

Our next result shows that the absence of non-zero idempotents in the ring $\mathrm{Hom}(G, Z(G))$ is equivalent to the group G being purely non-abelian.

Lemma 2.1 *A group G is purely non-abelian if and only if the zero homomorphism is the only idempotent element of the ring $\mathrm{Hom}(G, Z(G))$.*

Proof — First suppose that G has a direct decomposition $G = H \times K$, where H is a non-trivial abelian subgroup. Clearly, the map

$$f : G \longrightarrow Z(G)$$

defined by $x^h = x^{-1}$ if $x \in H$ and $x^h = 1$ if $x \in K$, is a non-zero

idempotent element of the ring $\mathrm{Hom}(G, Z(G))$.

Conversely, let h be a non-zero homomorphism of G into $Z(G)$ such that $h^2 = h$. If x is any element of G, then $(x^h)^h = x^{h^2} = x^h$. Therefore

$$(x^{-1}x^h)^h = (x^{-1})^h(x^h)^h = 1,$$

and hence $x \in \mathrm{Im}(h)\,\mathrm{Ker}(h)$. Thus $G = \mathrm{Im}(h)\,\mathrm{Ker}(h)$. Furthermore, let y be any element of $\mathrm{Im}(h) \cap \mathrm{Ker}(h)$. Then $y^h = 1$ and there exists an element $x \in G$ such that $y = x^h$. It follows that

$$y = x^h = x^{h^2} = y^h = 1,$$

and hence $\mathrm{Im}(h) \cap \mathrm{Ker}(h) = \{1\}$. We have proved that G has the direct decomposition $G = \mathrm{Im}(h) \times \mathrm{Ker}(h)$, where $\mathrm{Im}(h)$ is a non-trivial abelian direct factor of G. □

Lemma 2.2 *Let R be a ring, and let D be a divisible subgroup of the additive group of R. If x is an element of finite additive order of R, then $dx = xd = 0$ for every element $d \in D$.*

Proof — Let m be the order of x. Since D is divisible, there exists an element $y \in D$ such that $d = my$. It follows that

$$dx = (my)x = (\underbrace{y + \ldots + y}_{m})x = \underbrace{yx + \ldots + yx}_{m}$$

$$= y(\underbrace{x + \ldots + x}_{m}) = y(mx) = 0.$$

Similarly one has $xd = 0$. □

We are now in a position to prove our main result.

Theorem 2.3 *Let G be a group. Then $\mathrm{Hom}(G, Z(G))$ is a radical artinian ring if and only if G is purely non-abelian and the additive group of $\mathrm{Hom}(G, Z(G))$ satisfies the minimal condition on subgroups.*

Proof — Put $R = \mathrm{Hom}(G, Z(G))$. Suppose first that R is a radical artinian ring. We have already pointed out that G is purely non-abelian. Moreover, R is *nilpotent* by [13, Chapter IX, Proposition 2.13], i.e. there exists a positive integer n such that every product of n elements of R is zero. Consider in R the series of ideals:

$$R \geqslant R^2 \geqslant \ldots \geqslant R^n = \{0\}.$$

Clearly, every subgroup of the additive group of the ring R^m/R^{m+1} is an ideal for all positive integers $m < n$. It follows that $(R, +)$ satisfies

the minimal condition on subgroups since this property is extension-closed.

Conversely, assume that G is purely non-abelian and $(R,+)$ satisfies the minimal condition on subgroups. Then $(R,+)$ is the direct sum of a finite subgroup F and a periodic divisible subgroup D (see [16, 4.2.11]). Assume for a contradiction that R contains a non-nilpotent element $f+d$, with $f \in F$ and $d \in D$. By Lemma 2.2, f is non-nilpotent too. Let n be the order of F. Clearly, $f^2 = g + d_1$ for some elements $g \in F$ and $d_1 \in D$. Then $nf^2 = nd_1$ and so $nf^3 = 0$ by Lemma 2.2. Thus, f^t belongs to the subgroup $R[n]=\{r\in R \mid nr=0\}$ of $(R,+)$ for all $t \geqslant 3$. Moreover, as $(R,+)$ satisfies the minimal condition on subgroups, we have that $R[n]$ is finite (see [16, 4.3.5]). It follows that there exist positive integers i and j such that $f^{i+j} = f^i$, and hence $f^{i+jk} = f^i$ for every $k \in \mathbb{N}$. In particular if $jk > i$, we have

$$(f^{jk})^2 = f^{jk-i}f^{jk+i} = f^{jk-i}f^i = f^{jk},$$

which is impossible since $f^{jk} \neq 0$ and R has no non-zero idempotent by Lemma 2.1.

Let γ be any central endomorphism of G so that there exists a homomorphism h of G into its centre $Z(G)$ such that $x^\gamma = xx^h$ for every $x \in G$. Let m be a positive integer such that $h^m = 0$, and consider the central endomorphism

$$\delta : x \in G \mapsto xx^{-h+h^2-h^3+\ldots+(-1)^{m-1}h^{m-1}} \in G.$$

For each element x of G, we have

$$\begin{aligned} x^{\gamma\delta} &= xx^h(xx^h)^{-h+h^2-h^3+\ldots+(-1)^{m-1}h^{m-1}} \\ &= xx^hx^{-h+h^2-h^3+\ldots+(-1)^{m-1}h^{m-1}}(x^h)^{(-h+h^2-h^3+\ldots+(-1)^{m-1}h^{m-1})} \\ &= xx^{h^2-h^3+\ldots+(-1)^{m-1}h^{m-1}}x^{-h^2+h^3+\ldots+(-1)^{m-1}h^m} = x. \end{aligned}$$

Thus $\gamma\delta = \iota_G$ and similarly it can be proved that $\delta\gamma = \iota_G$. Therefore γ is a central automorphism of G, whence R is a radical ring as required. Finally, it is clear that R is an artinian ring. □

We remark that the final part of the above proof can alternatively be obtained recalling that the nilpotent elements of the ring R belong to its *Jacobson radical*. For the convenience of the reader we have preferred to prove directly that a central endomorphism is an auto-

morphism under our assumption.

Let G be the central product of countably many copies of Q_8, the quaternion group of order 8. Then $Z(G) = G'$ has order 2 and $G/Z(G)$ is an infinite elementary abelian 2-group. Thus

$$\mathrm{Hom}(G, Z(G)) \simeq \mathrm{Hom}(G/G', Z(G)) \simeq \bigoplus_{n\in\mathbb{N}} C_n,$$

where each C_n has order 2. Clearly, $\mathrm{Hom}(G, Z(G))$ is a radical ring, but it is not artinian by Theorem 2.3. It is interesting to note that by a classical theorem of Fuchs and Szele [10], there exists an artinian ring whose additive group is isomorphic to $\big(\mathrm{Hom}(G, Z(G)), +\big)$.

Corollary 2.4 *Let G be a purely non-abelian group such that the additive group of the ring $\mathrm{Hom}(G, Z(G))$ satisfies the minimal condition on subgroups. Then every non-zero central endomorphism of G is an automorphism. Moreover, the group $\mathrm{Aut}_c(G)$ of all central automorphisms of G is nilpotent.*

Proof — Clearly, every non-zero central endomorphism of G is an automorphism since $R = \mathrm{Hom}(G, Z(G))$ is a radical ring by Theorem 2.3. Moreover, R is also an artinian ring, and hence it is nilpotent. It follows that there exists a positive integer n such that $R^n = \{0\}$ but $R^{n-1} \neq \{0\}$. It is well known (see for instance [2]) that the chain

$$R > R^2 > \ldots > R^{n-1} > R^n = \{0\}$$

is a central series of the adjoint group R^0 of R, so that R^0 (and hence also $\mathrm{Aut}_c(G)$) is a nilpotent group (of class at most $n-1$). □

Following Fuchs, a ring R is said to be a *zero-ring* if R has the zero multiplication, i.e. if $xy = 0$ for all $x, y \in G$. Clearly, if G is a group such that $\mathrm{Hom}(G, Z(G))$ is a radical zero-ring, then $\mathrm{Aut}_c(G)$ is isomorphic to the additive group of $\mathrm{Hom}(G, Z(G))$. Now we give some examples in order to illustrate Theorem 2.3.

Proposition 2.5 (a) *There exists a purely non-abelian group G such that the ring $\mathrm{Hom}(G, Z(G))$ is a zero-ring and its additive group is an infinite abelian group satisfying the minimal condition on subgroups.*

(b) *There exists a purely non-abelian group G of order 16 such that $\mathrm{Hom}(G, Z(G))$ is a zero-ring.*

(c) *If p is any odd prime, there exists a finite purely non-abelian group G of order p^7 such that $\mathrm{Hom}(G, Z(G))$ is not a zero-ring and $\mathrm{Aut}_c(G)$ is not abelian.*

Proof — (a) Let P and Q be groups of type 2^∞ with generators

$$1 = a_0, a_1, a_2, \dots \quad \text{and} \quad 1 = b_0, b_1, b_2 \dots,$$

respectively, where $a_{n+1}^2 = a_n$ and $b_{n+1}^2 = b_n$ for all n. Consider the automorphisms α and β of the direct product $X = P \times Q$ defined by the rules

$$a_n^\alpha = a_n^3,\ a_n^\beta = a_n b_n,\ b_n^\alpha = b_n^\beta = b_n,$$

for each non-negative integer n. Put $Y = \langle \alpha, \beta \rangle$ and

$$G = Y \ltimes X.$$

Clearly, $P = [P, \alpha]$ and $Q = [P, \beta]$, so that $Z(G) \leqslant X \leqslant G'$. It follows that G is purely non-abelian and every central automorphism of G acts trivially on $Z(G)$, whence the group $\mathrm{Aut}_c(G)$ is isomorphic to $\mathrm{Hom}(G, Z(G))$. Moreover, as the factor group $G/C_G(X/Q)$ is infinite cyclic, we have that $Y/Y' \simeq G/G'$ is a 2-generator infinite abelian group. On the other hand,

$$a_n^{\alpha\beta\alpha^{-1}\beta^{-1}} = (a_n^3 b_n^3)^{\alpha^{-1}\beta^{-1}} = (a_n b_n)^{\beta^{-1}} (b_n^2)^{\beta^{-1}} = a_n b_n^2 = a_n^{\beta^2},$$

and hence $\beta^2 = [\alpha^{-1}, \beta^{-1}]$. Thus $G/G' \simeq \mathbb{Z} \times \mathbb{Z}_2$, and so

$$\mathrm{Hom}(G, Z(G)) \simeq \mathrm{Hom}(G/G', Z(G)) \simeq \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}_2, \mathbb{Z}(2^\infty))$$
$$\simeq \mathbb{Z}_2 \times \mathbb{Z}(2^\infty).$$

Finally, let f be the element of order 2 of the above finite factor of the additive group of $\mathrm{Hom}(G, Z(G))$. Then $0 = f \circ f = f^2$, and hence $\mathrm{Hom}(G, Z(G))$ is a zero-ring by Lemma 2.2.

(b) Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be cyclic groups of order 4, and consider the semidirect product

$$G = B \ltimes A,$$

where $a^b = a^{-1}$ and $[a, b^2] = 1$. Clearly, $Z(G) = \langle a^2 \rangle \times \langle b^2 \rangle$ and $G' = \langle a^2 \rangle$. Also, G is purely non-abelian, so $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G, Z(G))|$. On the

other hand,

$$\mathrm{Hom}(G/G', Z(G)) \simeq \mathrm{Hom}(G/Z(G), Z(G)) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and hence the group $\mathrm{Aut}_c(G)$ is isomorphic to $\big(\mathrm{Hom}(G, Z(G)), +\big)$. Finally, it is easy to see that $\mathrm{Hom}(G, Z(G))$ is a zero-ring.

(c) Let $H = \langle x \rangle$ and $K = \langle y \rangle$ be cyclic groups of orders p^4 and p^3, respectively, and consider the semidirect product

$$G = K \ltimes H,$$

where $y^{-1}xy = x^{1+p^3}$. Clearly, $Z(G) = \langle x^p \rangle \times \langle y^p \rangle$ coincides with the Frattini subgroup of G, the factor group $G/Z(G)$ has order p^2 and $G' = \langle x^{p^3} \rangle$. Thus, G is purely non-abelian, and so $\mathrm{Hom}(G, Z(G))$ is a radical ring. Now, let

$$f : G \longrightarrow Z(G)$$

be the map defined by setting $(y^i x^j)^f = y^{ip}$, for all $i \in \{0, 1, \ldots, p^3 - 1\}$ and $j \in \{0, 1, \ldots, p^4 - 1\}$. It is easy to show that f is a homomorphism from G into $Z(G)$. Moreover, as $y^{f^2} = y^{p^2} \neq 1$, we have that $\mathrm{Hom}(G, Z(G))$ is not a zero-ring.

In order to prove that $\mathrm{Aut}_c(G)$ is not abelian, we first observe that

$$x = x^{y^{1+p}x^{-p}} \quad \text{and} \quad (y^{1+p}x^{-p})^{p^3} = 1.$$

It follows that the rules $x^\alpha = x$ and $y^\alpha = y^{1+p}x^{-p}$ define an endomorphism α of G. Moreover, since

$$(y^m x^n)^\alpha = (y^{1+p}x^{-p})^m x^n = y^m x^n y^{mp} x^{-mp}$$

and $y^{mp}x^{-mp} \in Z(G)$, we have that α is a central endomorphism. Thus α is a central automorphism because G is purely non-abelian. On the other hand, the relations

$$x^{y^{1+p}} = x \quad \text{and} \quad (y^{1+p})^{p^3} = 1$$

yield that the mapping β given by $x^\beta = x$ and $y^\beta = y^{1+p}$ is an endomorphism of G, which is even a central automorphism since

$$(y^m x^n)^\beta = y^m x^n y^{pm}$$

with $y^{p^m} \in Z(G)$. Clearly, α and β do not commute, since

$$y^{\beta\alpha} = (y^{1+p})^{\alpha} = (y^{1+p}x^{-p})^{1+p} = y^{(1+p)^2}x^{-p(1+p)}$$

and

$$y^{\alpha\beta} = (y^{1+p}x^{-p})^{\beta} = y^{(1+p)^2}x^{-p},$$

but $x^{-p^2} \neq 1$. □

Note that in the proof of part (a) of the above result we used a special case of a construction given in [8]. Moreover, we are grateful to professor R. Esteban-Romero (University of Valencia) for his useful comments in proving the non-commutativity of the group $Aut_c(G)$ in the part (c).

We finally show how central automorphisms can be used to obtain set-theoretic solutions of the Yang-Baxter equation.

Let X be a non-empty set. Consider a bijective map

$$r : X^2 \longrightarrow X^2$$

and write

$$r(x, y) = (\sigma_x(y), \gamma_y(x)),$$

where σ_x and γ_y are suitable maps $X \longrightarrow X$. Recall that the pair (X, r) is a *set-theoretic* (or *combinatorial*) solution of the *Yang-Baxter equation* if

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where

$$r_1 = r \times id_X : X^3 \longrightarrow X^3$$

and

$$r_2 = id_X \times r : X^3 \longrightarrow X^3.$$

Also, (X, r) is said to be *non-degenerate* and *involutive* if the maps σ_x and γ_y are bijective and $r^2 = id_{X^2}$.

Consider now a group G in which all non-zero central endomorphisms are automorphisms. Then

$$R = Hom(G, Z(G)) = \left\{ f_\gamma : x \in G \mapsto x^{-1}x^\gamma \in Z(G) \mid \gamma \in Aut_c(G) \right\}$$

is a radical ring. For every element $g \in R$, consider the map

$$\rho_g : f \in R \longrightarrow fg + f \in R.$$

Clearly ρ_g is bijective since R is a group via the circle operation

$$f \circ g = f + g + fg.$$

Then an easy application of [6, Lemma 2] (or [5, Lemma 8.5]) yields that the map

$$r : R^2 \longrightarrow R^2$$

defined by $r(f, g) = (\rho^{-1}_{\rho_g(f)}(g), \rho_g(f))$, for all $(f, g) \in R^2$, is a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.

The procedure just described can be implemented in GAP system to compute a solution of the Yang-Baxter equation considering for instance, the p-group G constructed in Proposition 2.5 (c) for small primes p. Note that

$$\begin{aligned} \mathrm{Hom}(G, Z(G)) &\simeq \mathrm{Hom}(\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3}, \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}) \\ &\simeq \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}, \end{aligned}$$

and hence in particular there exist 59049 central automorphisms of G for $p = 3$.

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