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Rational Cohomology Fixed Points

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Abstract

For $n \in \mathbb{N}$, we introduce the notion of n-rational cohomology fixed points and we prove, under a certain assumption, that if X is a rational space, then X admits an $(n-1)$-rational cohomology fixed point, where $n = max\{i : k_n \neq 0\}$. Here k_n is the Postnikov n-invariant of X.

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1 Introduction

In this paper, a rational space is a simply connected CW-complex X of finite type, i.e., $\dim H^n(X;\mathbb{Z}) < \infty$ for all n, such that $H^*(X;\mathbb{Z})$ and $\pi_*(X)$ are vector spaces over $\mathbb{Q}$.

Let $\mathcal{E}_\#(X)$ denote the group of the homotopy self-equivalences of a space X inducing the identity on the homotopy groups (see [7, 4]).

Rational homotopy theory (see [8] for a general introduction) asserts that every simply connected space of finite type has a corresponding Sullivan algebra $(\Lambda V, \delta)$ called the Sullivan model of X, unique up to isomorphism, that encodes the rational homotopy of X. Moreover, there is a concept of "homotopy" between morphisms of Sullivan algebras, analogous in many respects to the topological notion of homotopy (see [8], p.321). Thus, for a given Sullivan

algebra $(\Lambda V, \delta)$, let $\mathcal{E}_{\#}(\Lambda V)$ denote its group of the homotopy self-equivalences inducing the identity on the graded vector space of the indecomposables V (see [6],[1],[3]). By virtue of the properties the Sullivan model we obtain the identification $\mathcal{E}_{\#}(X) \simeq \mathcal{E}_{\#}(\Lambda V)$.

A space X is said to have a m-*rational cohomology fixed point* if there exist $[\alpha] \in \mathcal{E}_{\#}(X^{[m]})$ and a non-zero cohomology class $x \in H^{m+2}(X^{[m]}; \mathbb{Z})$ such that $\alpha \not\simeq id$ and $H^m(\alpha)(x) = x$, where $X^{[m]}$ is the m-Postnikov section of X.

The aim of this paper is to establish the existence of an m-rational cohomology fixed point for rational spaces under certain condition involving the Postnikov m-invariant of X, namely the cohomology classes $k_m \in H^{m+1}(X^{[m-1]}; \pi_m(X))$, where $m > 1$. More precisely, making use of the Sullivan model we construct a homomorphism of groups $\psi : H^m(X^{[m-2]}; \mathbb{Z}) \rightarrowtail \mathcal{E}_{\#}(X^{[m]})$ for every $m \geqslant 2$ and we prove the following result.

Theorem 1 *Let X be an rational space X and let $n = max\{i : k_i \neq 0\}$. If ψ is not surjective, then X has an $(n-1)$-rational cohomology fixed point.*

The paper is organized as follows. In Section 2, we recall the basic properties of the Sullivan model in rational homotopy theory as well as some results regarding the group $\mathcal{E}_{\#}(X)$. In Section 3, we formulate and prove the main theorem in an algebraic setting and then we give a mere transcription of the above result in the topological context.

2 Sullivan model in rational homotopy theory

We show our results using standard tools of rational homotopy theory by working algebraically on the model Sullivan of X. We refer to [8] for a general introduction to these techniques. Recall that every rational space of finite type has a corresponding Sullivan algebra $(\Lambda V, \delta)$ called the Sullivan model of X, unique up to isomorphism, that encodes the homotopy of X.

It worth mentioning that there is a concept of "homotopy" between morphisms of Sullivan algebras, analogous in many respects to the topological notion of homotopy (see [8], p. 321). Thus, for a given Sullivan algebra $(\Lambda V, \delta)$, let $\mathcal{E}(\Lambda V)$ denote its group of homotopy self-equivalences and let $\mathcal{E}_{\#}(\Lambda V)$ be the subgroup of $\mathcal{E}(\Lambda V)$ formed by the elements inducing the identity on the graded vector space of the

indecomposables V (see [6],[1],[3]). By virtue of the properties of the model of Sullivan we can recover the homotopy data via the following identifications

$$H^n(X;\mathbb{Z}) \simeq H^*(\Lambda V), \quad \pi_*(X) \simeq V^*, \quad \mathcal{E}_\#(X) \simeq \mathcal{E}_\#(\Lambda V) \tag{2.1}$$

Furthermore, for every n, the sub-algebra $\left(\Lambda V^{\leqslant n}, \delta\right)$ can be considered as the Sullivan model of the Postnikov section $X^{[n]}$ implying that

$$\mathcal{E}_\#\left(X^{[n]}\right) \simeq \mathcal{E}_\#\left(\Lambda V^{\leqslant n}\right), \quad H^*\left(X^{[n]};\mathbb{Z}\right) \simeq H^*\left(\Lambda V^{\leqslant n}\right) \tag{2.2}$$

Thus, if $[\alpha] \in \mathcal{E}_\#(X)$, then $[\alpha]$ induces an element $[\beta] \in \mathcal{E}_\#(\Lambda V)$ such that the following diagram commutes

$$\begin{array}{ccc} H^n(X;\mathbb{Z}) & \xrightarrow{H^*(\alpha)} & H^n(X;\mathbb{Z}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^*(\Lambda V) & \xrightarrow{H^*(\beta)} & H^*(\Lambda V) \end{array} \tag{2.3}$$

2.1 A certain short exact sequence

See [5] for more details. Let $\left(\Lambda(V^q \oplus V^{\leqslant k}), \delta\right)$ be a Sullivan algebra, where $q > k$ and let $b^q : V^q \to H^{q+1}\left(\Lambda V^{\leqslant k}\right)$ be the linear map defined by setting

$$b^q(v) = [\delta(v)], \quad v \in V^q \tag{2.4}$$

Here $[\delta(v)]$ denotes the homology class of $\delta(v) \in \Lambda^{q+1}\left(V^{\leqslant k}\right)$.

Let $\mathcal{G}_k^q$ be the subgroup of $\mathcal{E}_\#\left(\Lambda V^{\leqslant k}\right)$ consisting of the elements $[\gamma]$ making the following diagram commutes

$$\begin{array}{ccc} V^q & & \\ \downarrow b^q & \searrow^{b^q} & \\ H^{q+1}\left(\Lambda V^{\leqslant k}\right) & \xrightarrow{H^{q+1}(\gamma)} & H^{q+1}\left(\Lambda V^{\leqslant k}\right) \end{array}$$

Theorem 2 ([5], Theorem 2.12) *Let* $\left(\Lambda(V^q \oplus V^{\leqslant k}), \delta\right)$ *be a Sullivan*

algebra, where $q > k$. *There is a short exact sequence of groups*

$$\mathrm{Hom}\big(V^q, H^q(\Lambda V^{\leqslant k})\big) \rightarrowtail \mathcal{E}_{\#}\big(\Lambda(V^q \oplus V^{\leqslant k}), \delta\big) \twoheadrightarrow \mathcal{G}^q_k \tag{2.5}$$

3 Main results

3.1 Rational cohomology fixed points

Definition 3 A space X is said to have an m-*rational cohomology fixed point* if there exist $[\alpha] \in \mathcal{E}_{\#}\big(X^{[m]}\big)$ and a non-zero cohomology class $x \in H^{m+2}\big(X^{[m]}; \mathbb{Z}\big)$ such that $\alpha \not\simeq \mathrm{id}$ and $H^m(\alpha)(x) = x$, where $X^{[m]}$ is the m-Postnikov section of X.

Remark 4 Let X be a rational space. It is clear from Definition 3 that if the group $\mathcal{E}_{\#}\big(X^{[m]}\big)$ is trivial, then X has not an m-rational cohomology fixed point for every $m \in \mathbb{N}$.

Let $(\Lambda V, \delta)$ be the Sullivan model of a rational space X. Let

$$n = \max\big\{i : b^i \neq 0\big\} \tag{3.1}$$

Lemma 5 *If* $(\Lambda V, \delta)$ *is the Sullivan model of a rational space* X, *then for every* n *we have the following short exact sequence*

$$\mathrm{Hom}\big(V^n, H^n\big(\Lambda V^{\leqslant n-1}\big)\big) \xrightarrow{\Phi} \mathcal{E}_{\#}\big(\Lambda V^{\leqslant n}\big) \twoheadrightarrow \mathcal{G}^n_{n-1} \tag{3.2}$$

Proof — It suffices to apply the short exact sequence (2.5) for $q = n$, $k = n-1$. □

Remark 6 Using the identifications (2.2), the homomorphism ϕ induces a homomorphism of groups

$$\psi : H^n\big(X^{[n-1]}; \mathbb{Z}\big) \rightarrowtail \mathcal{E}_{\#}\big(X^{[n]}\big). \tag{3.3}$$

Proposition 7 *Let* $(\Lambda V, \delta)$ *be the Sullivan model of a rational space* X. *If the linear map* $b^n = 0$ *for all* n, *then* X *has not an* n*-rational cohomology fixed point for all* n.

Proof — By virtue of the properties of the Sullivan model $(\Lambda V, \partial)$ of X, it is worth noting that for all $i \geqslant 2$, the linear map

$$b^n \in \mathrm{Hom}\big(V^n, H^{n+1}(\Lambda(V^{\leqslant n-1}))\big) \simeq \mathrm{Hom}\big(\pi_n(X), H^{n+1}\big(X^{[n-1]}\big)\big)$$

is the dual of the k_n-invariant

$$k_n \in H^{n+1}\big(X^{[n-1]};\pi_n(X)\big) \simeq \mathrm{Hom}\big(H_{n+1}(X^{[n-1]}),\pi_n(X)\big). \quad (3.4)$$

Therefore if $b^n = 0$, then the Postnikov invariant k_n is nil implying that $X^{[n+1]}$ has the homotopy type of $X^{[n]} \times K(\pi_{n+1}(X), n+2)$. Here $K\big(\pi_{n+1}(X), n+2\big)$ denotes the Eilenberg–Mac Lane space. As a result, if all the maps b^n are nil, then

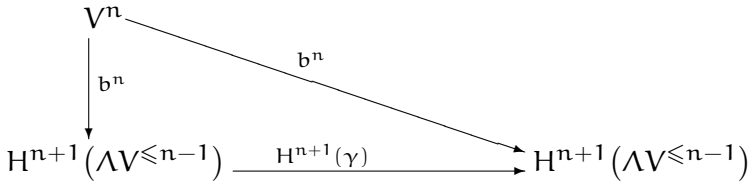
$$X \simeq \prod_n K\big(\pi_{n+1}(X), n+2\big).$$

It follows that the Sullivan model of X has the form $(\Lambda V, 0)$ and in this case the group $\mathcal{E}_\#\big(X^{[n]}\big)$ is trivial for all n. Consequently, has not an n-rational cohomology fixed point for all n according to Remark 4. □

Now we are ready to state the main theorem of this paper

Theorem 8 *Let X be an rational space X and $n = \max\{i : k_i \neq 0\}$. If the homomorphism ψ given (3.3) is not surjective, then X has an $(n-1)$-rational cohomology fixed point.*

Proof — Let $(\Lambda V, \delta)$ be the Sullivan model of X and let us consider the short exact sequence (3.2). As ψ is not surjective, it follows that ϕ is also not surjective. Thus, the two groups $\mathcal{E}_\#\big(\Lambda V^{\leqslant n}\big)$ and $\mathcal{G}^n_{n-1}$ are not trivial. Recall that $\mathcal{G}^n_{n-1}$ is the subgroup of $\mathcal{E}_\#\big(\Lambda V^{\leqslant n-1}\big)$ of self-homotopy classes $[\gamma]$ making the following diagram commutes



Now, if $v \in V^n$, then by virtue of the commutativity of the above diagram we obtain

$$H^{n+1}(\gamma) \circ b^n(v) = b^n(v)$$

Notice that as $n = \max\{i : k_n \neq 0\} \neq 0$, it follows, by (3.1) and (3.4), that $b^n(v) \neq 0$. Consequently, if we set $b^n(v) = x$, then there are $[\gamma]$ in $\mathcal{E}_\#\big(\Lambda V^{\leqslant n-1}\big)$ and a non-zero cohomology class $x \in H^{n+1}\big(\Lambda V^{\leqslant n-1}\big)$

such that

$$H^{n+1}(\gamma)(x) = x.$$

A mere transcription of the above result in the topological context by using the identifications (2.2) and the diagram (2.3), allows us to write that there exist $[\gamma] \in \mathcal{E}_{\#}\big(X^{[n-1]}\big)$ and a non-zero cohomology class $x \in H^{n+1}\big(X^{[n-1]}; \mathbb{Z}\big)$ such that

$$H^{n+1}(\gamma)(x) = x.$$

Thus, the space X has an $(n-1)$-rational cohomology fixed point. □

REFERENCES

[1] M. Benkhalifa: "Realizability of the group of self-homotopy equivalences and local homotopy theory", *J. Homotopy Relat. Struct.* 5 (2010), 361–372.

[2] M. Benkhalifa: "Postnikov decomposition and the group of self-equivalences of a rationalized spaces", *Homology, Homotopy Appl.* 19 (2017), 209–224.

[3] M. Benkhalifa: "On the group of self-homotopy equivalences of an elliptic space", *Proc. Amer. Math. Soc.* 148 (2020), 2695–2706.

[4] M. Benkhalifa: "The effect of cell-attachment on the group of self-equivalences of an elliptic space", *Michigan Math. J.* 71 (2021), 611–617.

[5] M. Benkhalifa: "On the Euler–Poincaré characteristics of a simply connected rationally elliptic CW-complex", *J. Homotopy Relat. Struct.* 17 (2022), 163–174.

[6] M. Benkhalifa: "On the group of self-homotopy equivalences of an almost formal space", *Quaest. Math.* (2022); doi:10.2989/16073606.2022.2044405.

[7] M. Benkhalifa: "The group of self-homotopy equivalences of a rational space cannot be a free abelian group", *J. Math. Soc. Japan*, to appear.

[8] Y. Félix – S. Halperin – J.-C. Thomas: "Rational Homotopy Theory", Springer, New York (2001).

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