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Quasi-Projections and Factorizations of Monoids

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Abstract

The notions of left and right quasi-projections on a monoid are introduced. They are used to give a full classification of factorizations of monoids in terms of complementary pairs of quasi-projections. Connections with descent 1-cocycles and Rota–Baxter operators on groups are established.

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1 Introduction

The present article is a continuation of the previous work on the problem of describing factorizations of monoids and groups [3]. Although factorizations of wide classes of monoids and groups have been extensively studied, there has been no full classification of such factorizations. Here, we will describe how to fully classify all the monoid factorizations of a given monoid by introducing the notions of left and right quasi-projections on a monoid.

The paper is organized as follows. After fixing some basic notations and concepts in Section 2, we introduce in Section 3 the notions of left and right quasi-projections on monoids and give some basic properties of them. Also, a full classification of those factorizations

of a monoid whose one factor is a subgroup is obtained in terms of left quasi-projections. Section 4 contains results about complementary pairs of quasi-projections on monoids. In Section 5, we present our main result (see Theorem 5.1) asserting that there is a one-to-one bijection between the set of the monoid factorizations of an fixed monoid and the set of the complementary pairs of quasi-projections on the monoid. Some particular cases of the main result are also obtained. Finally, Section 6 is devoted to the study of monoid factorizations arising from semidirect product of monoids. Moreover, connections with non-abelain 1-cocycles, Rota–Baxter operators on groups and QR-factorization of complex matrices are established.

2 Preliminaries

This section fixes some conventions and recalls some basic definitions. Given a monoid A, we denote the unit of A as 1. When we deal with more than one monoid in context, we add subscripts. For two elements a and b of A, we usually write ab for their product (we sometimes write $a \cdot b$ if this helps to avoid confusion). The subgroup of the invertible elements of A is denoted by $U(A)$. The fact that X is a submonoid (resp. subgroup) of a monoid (resp. group) A is denoted by $X \leqslant A$. In this case we write i_X for the canonical embedding $X \subseteq A$. If B is another monoid, the map which sends each element of A to the unit element of B is denoted by $0_{A,B}$. For any homomorphism $f : A \to B$ of monoids, we put

$$\mathsf{Ker}(f) = \big\{a \in A : f(a) = 1\big\} \quad \text{and} \quad \mathsf{K}[f] = \big\{(a, a') \in A \times A : f(a) = f(a')\big\}.$$

If in addition B is a group and $t : A \to B$ is any map, $t^{-1} : A \to B$ is the map defined by $a \longmapsto (t(a))^{-1}$.

If X is a set (or a monoid), we shall denote by $\mathsf{End}(X)$ the monoid consisting of all endomaps (resp. endomorphisms) of the set (resp. monoid) X. The identity map of X is denoted by Id_X.

Definition 2.1 A monoid A is said to be *factorizable* if it contains two submonoids X and B such that the multiplication map

$$X \times B \to A, \ (x, b) \longmapsto xb$$

is bijective. When this is the case, A is said to be *factorized* by its submonoids X and B. X and B are called, respectively, the *first* and *second factor* of this factorization. If A is a group, then A is factorizable if in addition X and B are subgroups of A.

Thus, a monoid A is factorizable by its submonoids X and B if each element $a \in A$ may be written uniquely in the form $a = xb$ with $x \in X$ and $b \in B$. The relation $a = xb$ is called the (X, B)-*decomposition* of the element $a \in A$.

For a given monoid A, we write $\mathsf{FAC}(A)$ for the set of all factorizations of A:

$$\mathsf{FAC}(A) = \big\{(X, B) : X, B \leqslant A \text{ and } A \text{ factorizes by } X \text{ and } B\big\}.$$

Note that $\mathsf{FAC}(A)$ always contains at least two elements, namely $\big(\{1\}, A\big)$ and $\big(A, \{1\}\big)$. They are called the *trivial factorizations of* A. For a fixed submonoid X of A, let $\mathsf{FAC}(A/X)$ be denote the set of those submonoids $B \leqslant A$ for which the pair (X, B) is a monoid factorization of A.

2.1 Bicrossed product of monoids

Factorizations of monoids arise naturally from bicrossed product of monoids (in fact, these two notions are equivalent, in the sense explained below). Let us recall their definition.

Given two morphisms $k : X \to A$ and $l : B \to A$ of monoids, we write $[k, l]$ for the map

$$[k, l] : X \times B \to A, \quad (x, b) \longmapsto k(x)l(b).$$

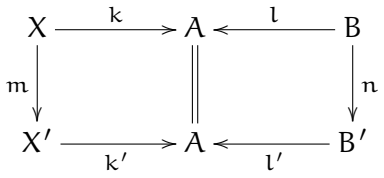
Definition 2.2 A diagram $X \xrightarrow{k} A \xleftarrow{l} B$ of monoids and their homomorphisms for which the map

$$[k, l] : X \times B \to A$$

is bijective is called a *bicrossed product diagram of monoids*. When this is the case, we say that A is a *bicrossed product of the monoids* X *and* B.

Two bicrossed diagrams $X \xrightarrow{k} A \xleftarrow{l} B$ and $X' \xrightarrow{k'} A \xleftarrow{l'} B'$ of monoids are *isomorphic* if there exist isomorphisms $m : X \to X'$

and $n : B \to B'$ of monoids making the diagram

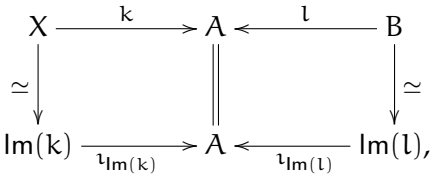


commute.

Proposition 2.3 *A monoid is factorizable if and only if it can be represented as a bicrossed product of monoids.*

Proof — For the sufficiency part, we note that particular instances of bicrossed product diagrams of monoids are given by factorizations of monoids: it follows at once from Definition 2.1 that given a factorization of a monoid A by its submonoids X and B, $X \xrightarrow{\iota_X} A \xleftarrow{\iota_B} B$ is a bicrossed product diagram of monoids.

As for the necessity part, it can be checked easily that for any bicrossed product diagram of monoids $(X \xrightarrow{k} A \xleftarrow{l} B)$, both k and l are injective. It then follows that the diagram



where the left and right horizontal maps are the canonical bijections, commutes. Using the commutativity of the diagram, it is clear that the bottom row is a bicrossed product diagram of monoids (and hence a factorization of A) provided that the top one is so. □

Remark 2.4 It follows from the proof of the preceding proposition that any bicrossed product diagram of monoids is isomorphic to one induced by a factorization of a monoid.

Recall that for two maps $f : X \to Y$ and $g : X \to Y'$, the map

$$\langle f, g\rangle : X \to Y \times Y'$$

is the unique map with $\pi_Y \circ \langle f, g\rangle = f$ and $\pi_{Y'} \circ \langle f, g\rangle = g$.

Let a monoid A factorizes by its submonoids X and B. Write $q_{X,B}$ and $p_{X,B}$ for the compositions

$$A \xrightarrow{[\iota_X,\iota_B]^{-1}} X \times B \xrightarrow{\pi_X} X$$

and

$$A \xrightarrow{[\iota_X,\iota_B]^{-1}} X \times B \xrightarrow{\pi_B} B,$$

respectively. Then, since

$$\begin{aligned}\langle q_{X,B}, p_{X,B}\rangle &= \langle \pi_X \circ [\iota_X,\iota_B]^{-1}, \pi_B \circ [\iota_X,\iota_B]^{-1}\rangle \\ &= \langle \pi_X, \pi_B\rangle \circ [\iota_X,\iota_B]^{-1} \\ &= \mathsf{Id}_{X\times B} \circ [\iota_X,\iota_B]^{-1} \\ &= [\iota_X,\iota_B]^{-1},\end{aligned}$$

it follows that

$$[\iota_X,\iota_B] \circ \langle q_{X,B}, p_{X,B}\rangle = \mathsf{Id}_A. \tag{2.1}$$

But for any element $a \in A$,

$$\begin{aligned}\big([\iota_X,\iota_B] \circ \langle q_{X,B}, p_{X,B}\rangle\big)(a) &= \big(m_A \circ (\iota_X \times \iota_B) \circ (q_{X,B} \times p_{X,B})\big)(a) \\ &= \big(m_A \circ \big((\iota_X q_{X,B}) \times (\iota_B p_{X,B})\big)\big)(a) \\ &= (\iota_X q_{X,B})(a) \cdot (\iota_B p_{X,B})(a)\end{aligned}$$

and hence

$$\big(\iota_X q_{X,B}\big)(a) \cdot (\iota_B p_{X,B})(a) = a. \tag{2.2}$$

Since $q_{X,B}(a) \in X$ and $p_{X,B}(a) \in B$, it follows from the uniqueness property of monoid factorizations that (2.2) is the (X, B)-decomposition of a.

2.2 Descent cohomology sets

Let $\iota : X \to A$ be an (injective) homomorphism of monoids. We follow [3] (see also [7] and [1]) in calling a map $q : A \to X$ a (*left*) 1-*descent cocycle of the monoid* A *with coefficients in the monoid* X if the following conditions are satisfied:

(DL1) $q\iota = \mathsf{Id}_X$,

(DL2) $q\big(\iota(x) \cdot a\big) = x \cdot q(a) \quad (\forall a \in A, \forall x \in X)$, and

(DL3) $q(a \cdot a') = q\big(a \cdot (\iota q)(a')\big) \quad (\forall a, a' \in A)$.

The set of the left 1-descent cocycles of A with coefficients in X is denoted by $\mathcal{Z}^1_l(A/X, \iota)$. Two 1-cocyles $q_1, q_2 \in \mathcal{Z}^l(A/X, \iota)$ are called *cohomologous* if there exists an element $x_0 \in U(X)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{q_1} & B \\ {\scriptstyle -\cdot\iota(x_0)}\downarrow & & \downarrow{\scriptstyle -\cdot x_0} \\ A & \xrightarrow[q_2]{} & B \end{array}$$

commutes; i.e., if for any $a \in A$,

$$q_1(a) \cdot x_0 = q_2\big(a \cdot \iota(x_0)\big). \tag{2.3}$$

It is shown in [3] that the cohomology relation is an equivalence relation on $\mathcal{Z}^l(A/B, \iota)$. The set of equivalence class of cohomologous cocycles is denoted by $\mathsf{Desc}^1_l(A/B, \iota)$.

Symmetrically, a map $p : A \to X$ is called a (*right*) 1-*descent cocycle of* A *with coefficients in the monoid* C if the following conditions are satisfied:

(DR1) $p\,\iota = \mathsf{Id}_X$,

(DR2) $p\big(a \cdot \iota(x)\big) = p(a) \cdot x \quad (\forall a \in A, \forall x \in X)$, and

(DR3) $p(a \cdot a') = p\big((\iota p)(a) \cdot a'\big) \quad (\forall a, a' \in A)$.

The set of (equivalence class of cohomologous of) right 1-descent cocycles of A with coefficients in the monoid X is denoted by $\mathcal{Z}^1_r(A/X, \iota)$ (resp. $\mathsf{Desc}^1_r(A/X, \iota)$).

Typical examples of 1-descent cocycles arise from monoid factorizations. Direct inspection proves the following:

Proposition 2.5 *Let a monoid A factorizes by its submonoids X and B. Then $q_{XB} \in \mathcal{Z}^1_l(A/X, \iota_X)$ and $p_{XB} \in \mathcal{Z}^1_r(A/B, \iota_B)$. Moreover, $\mathsf{Ker}(q_{XB}) = B$ and $\mathsf{Ker}(p_{XB}) = X$.*

Remark 2.6 By preceding proposition, if a monoid A factorizes through its submonoids X and B, $\mathcal{Z}^1_l(A/X, \iota_X)$ and $\mathcal{Z}^1_r(A/B, \iota_B)$ can be regarded as pointed sets, whose base-points are q_{XB} and p_{XB}, respectively.

Next we recall from [3] the notion of (non-abelian) cohomology pointed sets of monoids.

2.3 Non-abelian cohomology pointed sets

Let X, B be monoids. A *left action* of B on X is a homomorphism

$$\Phi : B \to \mathsf{End}(X)$$

of monoids. In this situation, one also say that B *acts on* X *from the left*, and for all $b \in B$ and $x \in X$, one writes

$$b \bullet x = \Phi(b)(x).$$

A map $\chi : B \to X$ is called a (*non-abelian*) 1-*cocycle*, if $\chi(1) = 1$ and for arbitrary $b_1, b_2 \in B$:

$$\chi(b_1 \cdot b_2) = \chi(b_1) \cdot \big(b_1 \bullet \chi(b_2)\big).$$

We denote the set of all 1-cocycles $B \to X$ with $\mathcal{Z}^1(B, X)$. Then $\mathcal{Z}^1(B, X)$ is turned into a pointed set by defining the distinguished element to be the map $0_{B,X}$. Two 1-cocyles $\chi, \chi' : B \to X$ are called *equivalent* if there exists an invertible element $x_0 \in U(X)$ such that

$$\chi(b)(b \bullet x_0) = x_0 \chi'(b)$$

for all $b \in B$. It is shown in [3] that this provides an equivalence relation on $\mathcal{Z}^1(B, X)$. The set of equivalence classes of 1-cocycles is called the *first non-abelian* 1-*cohomology set* of B with coefficients in X and is denoted by $H^1(B, X)$. It is a pointed set with base-point the equivalence class of the map $0_{B,X}$.

Note that in the case when both X and B are groups, the pointed set $H^1(B, X)$ coincide with the ones introduced by Serre in [8].

2.4 Semidirect product of monoids

Let

$$X \xrightarrow{\iota_X} A \xleftarrow{\iota_B} B$$

be a factorization of a monoid A. Any left action $\Phi : B \to \mathsf{End}(X)$ of a monoid B on a monoid X gives rise to the *semiderect product* $X \rtimes_\Phi B$, where $X \rtimes_\Phi B$ is a monoid whose underlying set is the product $X \times B$ equipped with the structure of a monoid given by

$$(x_1, b_1) \cdot (x_2, b_2) = \big(x_1 \cdot (b_1 \bullet x_2), b_1 b_2\big), \quad \text{and} \quad 1_{X \rtimes_\Phi B} = (1_X, 1_B).$$

We write $\jmath_X$ and $\jmath_B$ for the maps

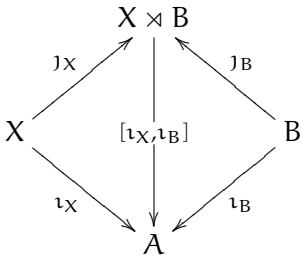
$$\jmath_X : X \to X \rtimes B, \qquad \iota_X(x) = (x, 1_B)$$

and

$$\jmath_B : B \to X \rtimes B, \qquad \iota_B(b) = (1_X, b).$$

Note that both $\jmath_X$ and $\jmath_B$ are monoid homomorphisms. Since for any pair $(x, b) \in X \times B$, one has that $(x, b) = (x, 1_B) \cdot (1_X, b) = \jmath_X(x) \cdot \jmath_B(b)$ in $X \rtimes B$, it follows that the map $[\jmath_X, \jmath_B] : X \times B \to A$ is bijective, or equivalently, that each element of $X \rtimes B$ can be written uniquely as the product $\jmath_X(x) \cdot \jmath_B(b)$. Therefore, the diagram $X \xrightarrow{\jmath_X} X \rtimes B \xleftarrow{\jmath_B} B$ is a bicrossed product diagram.

Remark 2.7 (i) The map $[\jmath_X, \jmath_B]$ is in fact a monoid isomorphism from the semidirect product $B \rtimes X$ into the monoid A that makes the diagram



commute.

(ii) Since the map $X \rtimes B \xrightarrow{\pi_B} B$ is easily seen to be a homomorphism of monoids,

$$p_{XB} : A \xrightarrow{[\jmath_X, \jmath_B]^{-1}} X \times B \xrightarrow{\pi_B} B$$

is also a homomorphism of monoids.

3 Quasi-projections on monoids

In this section, we introduce the notions of left and right quasi-projections on monoids and give some basic properties of them.

Definition 3.1 A *left quasi-projection on a monoid* A is a set map $l : A \to A$ such that for all $a, a' \in A$:

(L1) $l(1) = 1$.

(L2) $l(a) \cdot l(a') = l\big(l(a) \cdot a'\big)$.

(L3) $l(a \cdot a') = l\big(a \cdot l(a')\big)$.

Remark 3.2 Putting $a = 1$ in (L3) gives $ll = l$, i.e. that l is idempotent.

When the monoid A contains no nontrivial idempotents (i.e., when there exists no element $a \neq 1$ with $a^2 = a$, which is always the case if, for example, A is a group), the definition of left quasi-projection on A can be simplified, as in the following lemma.

Lemma 3.3 *Let A be a monoid containing no nontrivial idempotents and $l : A \to A$ be a set map satisfying* (L2) *and* (L3)*. Then it also satisfies* (L1)*.*

Proof — Since l satisfies (L3), it follows from Remark 3.2 that $ll = l$. Then putting $a = a' = 1$ in (L2), we get that $l(1)l(1) = l(l(1)) = l(1)$. Thus, $l(1)$ is an idempotent and hence, by our assumption on A, must equal to 1, proving that (L1) also holds. □

Symmetrically, a *right quasi-projection on a monoid* A is a set map $r : A \to A$ satisfying the following conditions:

(R1) $r(1) = 1$.

(R2) $r(a) \cdot r(a') = r\big(a \cdot r(a')\big)$ $(\forall a, a' \in A)$.

(R3) $r(a \cdot a') = r\big(r(a) \cdot a'\big)$ $(\forall a, a' \in A)$.

Condition (R3) implies that r is idempotent.

A straightforward calculation proves the following result.

Proposition 3.4 *For a monoid A, an idempotent endomap $u : A \to A$ of sets is at the same time left and right quasi-projection on A if and only if u is an idempotent endomorphism of the monoid A.*

Note that for any monoid A, there are always the two trivial left (as well as right) quasi-projections on A, namely Id_A and $0_{A,A}$.

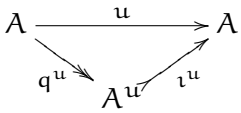
Let A be a monoid. To each map $u : A \to A$ of sets, one can associate two subsets of A:

$$A^u = \{a \in A : u(a) = a\}$$

and

$$A_u = \mathsf{Ker}(u) = \{a \in A : u(a) = 1\}.$$

We write $\iota^u : A^u \rightarrowtail A$ and $\iota_u : A_u \rightarrowtail A$ for the canonical inclusions. If, in addition, u is an idempotent map, then it factorizes through A^u in the sense that there is a (necessarily unique) surjection $q^u : A \twoheadrightarrow A^u$ such that the diagram (called the *splitting diagram of the idempotent* u)



commutes.

Proposition 3.5 *If l is a left quasi-projection on a monoid A, then A^l and A_l are both submonoids of A. If, in addition, A is a group, then A^l and A_l are subgroups of A.*

Proof — Suppose l is a left quasi-projection on A. Then, by (L1), 1 belongs to both subsets A_l and A^l. Next, if $a, a' \in A_l$ (resp. $a, a' \in A^l$) are two arbitrary elements, then

$$l(a) = l(a') = 1$$

(resp. $l(a) = a$ and $l(a') = a'$), and we calculate:

$$l(aa') \overset{(L3)}{=} l\big(al(a')\big) = l(a1) = l(a) = 1$$

(resp.

$$l(aa') = l\big(l(a)\, a'\big) \overset{(L2)}{=} l(a)\, l(a') = aa').$$

Thus, $aa' \in A_l$ (resp. $aa' \in A^l$), and hence the first part of the proposition follows.

Suppose now that A is a group. If $a \in A_l$, then $l(a) = 1$ and since

$$l(a^{-1}a) = l(1) = 1,$$

it follows that

$$1 = l(a^{-1}a) \overset{(L3)}{=} l\big(a^{-1}l(a)\big) = l(a^{-1}1) = l(a^{-1}),$$

and thus $a^{-1} \in A_l$. Next, if $a \in A^l$, then $l(a) = a$ and we have

$$l(a)\,l(a^{-1}) \overset{(L2)}{=} l\big(l(a)a^{-1}\big) = l(aa^{-1}) = l(1) \overset{(L1)}{=} 1,$$

implying that $l(a)^{-1} = l(a^{-1})$. But $l(a)^{-1} = a^{-1}$, since $l(a) = a$. Thus, $l(a^{-1}) = a^{-1}$, proving that $a^{-1} \in A^l$. This completes the proof. □

Symmetrically, we have the following result.

Proposition 3.6 *Let r be a right quasi-projection on a monoid A. Then A^r and A_r are both submonoids of A. If, in addition, A is a group, then A^r and A_r are subgroups of A.*

Proposition 3.7 *Let A be a monoid. A map $u : A \to A$ is a left* (*resp. right*) *quasi-projection on A if and only if the inclusion $\iota^u : A^u \to A$ makes A^u a submonoid of A and there is* (*necessarily*) *a unique element*

$$q^u \in \mathcal{Z}^1_l(A/A^u, \iota^u)$$

(*resp.* $p^u \in \mathcal{Z}^1_r(A/A^u, \iota^u)$) *such that* $u = \iota^u q^u$ (*resp.* $u = \iota^u p^u$).

Proof — Suppose first that u is a left quasi-projection on A and consider its splitting $u = \iota^u q^u$ through A^u. Then, by Proposition 3.5, A^u a submonoid of A and hence the canonical inclusion $\iota^u : A^u \to A$ makes A^u a submonoid of A. We are now going to show that q^u satisfies (DL1), (DL2) and (DL3) (and hence lies in $\mathcal{Z}^1_l(A/A^u, \iota^u)$). Since $1 = u(1) = \iota^u(q^u(1))$ and since $\iota^u : A^u \to A$ is a monomorphism of monoids (and hence, in particular, $\iota^u(1) = 1$), it follows that $q^u(1) = 1$. Thus, q^u satisfies (DL1). Next, since for all $a, b \in A$,

$$\begin{aligned}
(\iota^u q^u)\big((\iota^u q^u)(a) \cdot b\big) &= u(u(a) \cdot b) && [\text{since } u = \iota^u q^u]\\
&= u(a) \cdot u(b) && [\text{by (L2)}]\\
&= (\iota^u q^u)(a) \cdot (\iota^u q^u)(b) && [\text{since } u = \iota^u q^u]\\
&= \iota^u\big(q^u(a) \cdot q^u(b)\big) && [\text{as } \iota^u \text{ is a morphism of monoids}]
\end{aligned}$$

and

$$\begin{aligned}(\iota^u q^u)(ab) &= u(ab) && [\text{since } u = \iota^u q^u]\\ &= u\big(a \cdot u(b)\big) && [\text{by (L3)}]\\ &= (\iota^u q^u)\big(a \cdot (\iota^u q^u)(b)\big) && [\text{since } u = \iota^u q^u]\\ &= \iota^u\big(q^u\big(a \cdot (\iota^u q^u)(b)\big)\big)\end{aligned}$$

and since ι^u is injective, it follows that q^u satisfies (DL2) and (DL3).

Conversely, suppose that $u : A \to A$ is a map such that the canonical inclusion $\iota^u : A^u \to A$ makes A^u a submonoid of A and there is a unique element $q^u \in \mathcal{Z}^1_l(A/A^u, \iota^u)$ such that $u = \iota^u q^u$. Then since ι^u is a morphism of monoids, it follows that $\iota^u(1) = 1$ and we have:

$$u(1) = \iota^u\big(q^u(1)\big) \overset{(DL1)}{=} \iota^u(1) = 1.$$

Thus, u satisfies (L1). The following calculations

$$\begin{aligned}u(a) \cdot u(b) &= (\iota^u q^u)(a) \cdot (\iota^u q^u)(b)\\ &= \iota^u\big(q^u(a) \cdot q^u(b)\big) && [\text{since } \iota^u \text{ is a morphism of monoids}]\\ &= \iota^u\big(q^u\big((\iota^u q^u)(a) \cdot b\big)\big) && [\text{by (DL2)}]\\ &= u\big(u(a) \cdot b\big)\end{aligned}$$

and

$$u(ab) = \iota^u\big(q^u(ab)\big) \overset{(DL3)}{=} \iota^u\big(q^u\big(a \cdot (\iota^u q^u)(b)\big)\big) = u\big(a \cdot u(b)\big)$$

show that u also satisfies (L2) and (L3). Hence u is a left quasi-projection on A. This completes the proof since the case of right quasi-projections can be proved similarly. □

For a given monoid A, we write $\mathsf{LQ}(A)$ (resp. $\mathsf{RQ}(A)$) for the set of all left (resp. right) quasi-projections on A.

Proposition 3.8 *For any submonoid $X \subseteq A$ of a monoid A, the assignment $q \longmapsto \iota_X q$ yields an injective map*

$$\kappa_X : \mathcal{Z}^1_l(A/X, \iota_X) \to \mathsf{LQ}(A).$$

Proof — By Proposition 3.7, $\kappa_X(q) = \iota_X q$ is a left quasi-projection

on A for any $q \in \mathcal{Z}^1_l(A/X, \iota_X)$. The injectivity of κ_X follows from the injectivity of ι_X. □

Theorem 3.9 *For any monoid A, the map*

$$\bigsqcup_{X \leqslant A} \kappa_X : \bigsqcup_{X \leqslant A} \mathcal{Z}^1_l(A/X, \iota_X) \to \mathsf{LQ}(A)$$

is bijective. Its inverse takes $l \in \mathsf{LQ}(A)$ to q^l, where q^l is the first factor in the factorization $l = \iota^l q^l$ of the idempotent map l.

Proof — Since for any two submonoids $X, Y \leqslant A$, $X \neq Y$ clearly implies that

$$\mathcal{Z}^1_l(A/X, \iota_X) \cap \mathcal{Z}^1_l(A/Y, \iota_Y) = \varnothing,$$

it follows from Proposition 3.8 that the map $\bigsqcup_{X \leqslant A} \kappa_X$ is injective. Hence it remains only to show surjectivity of $\bigsqcup_{X \leqslant A} \kappa_X$. To this end, fix an arbitrary $l \in \mathsf{LQ}(A)$ and consider its factorization

$$A \xrightarrow{q^l} A^l \xrightarrow{\iota^l} A.$$

Since

- A^l is a submonoid of A by Proposition 3.5, and
- $q^l : A \to A^l$ is an element of $\mathcal{Z}^1_l(A/A^l, \iota^l)$ by Proposition 3.7,

it follows that $\kappa_{A^l}(q^l) = l$. Therefore, the map $\bigsqcup_{X \leqslant A} \kappa_X$ is surjective and hence bijective. Moreover, it is clear that $\left(\bigsqcup_{X \leqslant A} \kappa_X\right)^{-1}(l) = q^l$. This completes the proof. □

It follows from the preceding theorem that the set $\mathsf{LC}(A)$ is a disjoint union of its subsets $\mathsf{Im}(\kappa_X), X \leqslant A$. We describe these subsets next.

We say that two elements $l, l' \in \mathsf{LQ}(A)$ are *equivalent*, denoted by $l \sim l'$, if $ll' = l'$ and $l'l = l$. It is easy to check that $\sim$ is an equivalence relation on the set $\mathsf{LQ}(A)$. We write $[l]$ for the equivalence class of $l \in \mathsf{LQ}(A)$.

Lemma 3.10 *Let A be a monoid. If $l, l' \in \mathsf{LQ}(A)$ are equivalent, then we have that $A^l = A^{l'}$.*

PROOF — Suppose that $l, l' \in \mathsf{LQ}(A)$ are equivalent. Then $ll'=l'$ and $l'l = l$ and hence, by symmetry, it is enough to show the inclusion $A^l \subseteq A^{l'}$. In order to do this, take an arbitrary element $a \in A^l$. Then $l(a) = a$ and we have

$$l'(a) = l'\big(l(a)\big) = l(a) = a$$

implying $a \in A^{l'}$. Thus, $A^l \subseteq A^{l'}$. □

For any fixed left quasi-projection $l : A \to A$ on a monoid A, the map $q^l : A \to A^l$ lies in $\mathcal{Z}^1_l(A/A^l, \iota^l)$ by Proposition 3.7 and hence taking it as a base-point, $\mathcal{Z}^1_l(A/A^l, \iota^l)$ becomes a pointed set. Then clearly $\mathsf{Desc}^1_l(A/A^l, \iota^l)$ is also a pointed set with base-point the class of q^l. Moreover, the set $[l]$ can be obviously regarded as a pointed set, whose base-point is l.

Proposition 3.11 *For any left quasi-projection $l : A \to A$ on a monoid A, the map κ_{A^l} corestricts to an isomorphism of pointed sets*

$$\ell_l : \mathcal{Z}^1_l(A/A^l, \iota^l) \to [l].$$

Its inverse ℓ_l^{-1} is given by $\ell_l^{-1}(l') = q^{l'}$, where $q^{l'}$ is the first factor in the factorization $A \xrightarrow{q^{l'}} A^{l'} \xrightarrow{\iota^{l'}} A$ of the idempotent map l'.

PROOF — In the light of Lemma 3.10, we get from Proposition 3.7 that the splitting of any $l' \in [l]$ has the form

$$A \xrightarrow{q^{l'}} A^{l'} = A^l \xrightarrow{\iota^l} A$$

for some $q^{l'} \in \mathcal{Z}^1_l(A/A^{l'}, \iota^{l'}) = \mathcal{Z}^1_l(A/A^l, \iota^l)$. It follows that κ_{A^l} corestricts to a surjective map

$$\ell_l : \mathcal{Z}^1_l(A/A^l, \iota^l) \to [l],$$

which is also injective, since κ_{A^l} is. Hence, ℓ_l is a bijective map. Finally, since $\ell_l(q^l) = \iota^l q^l = l$, it follows that ℓ_l is an isomorphism of pointed sets. □

Fix a left quasi-projection l on a monoid A and define a relation $\cong_l$ on the set $[l]$ by $l_1 \cong_l l_2$ if and only if there exists an element $a_0 \in A$

with $l(a_0) \in U(A)$ satisfying the following property:

$$l_1(a)l(a_0) = l_2\big(al(a_0)\big) \text{ for all } a \in A. \tag{3.1}$$

When we want to indicate the dependence of the relation $l_1 \cong_l l_2$ on the element a_0, we write it as $l_1 \cong_{l,a_0} l_2$.

Lemma 3.12 *The definition of the relation $\cong_l$ does not depend on the choice of representatives for the equivalence class of l.*

Proof — Suppose that l' is another representative of the class $[l]$. Then $ll' = l'$ and $l'l = l$. Now, if $l_1 \cong_{l,a_0} l_2$, then $l(a_0)$ is invertible and hence $l'\big(l(a_0)\big) = l(a_0)$ is also invertible. Moreover, since

$$l_1(a)l'\big(l(a_0)\big) = l_1(a)l(a_0) \overset{(3.1)}{=} l_2\big(al(a_0)\big) = l_2\big(al'\big(l(a_0)\big)\big)$$

for all $a \in A$, it follows that $l_1 \cong_{l',\,l(a_0)} l_2$. Therefore, $\cong_l$ does not depend on the choice of representatives for the equivalence class $[l]$. □

Proposition 3.13 *The relation $\cong_l$ is obtained by pulling back of the equivalence relation on the set $\mathcal{Z}^1_l(A/A^l, \iota^l)$ of being cohomologous (see* Section 2.2*) along the map ℓ_l^{-1}. In particular, $\cong_l$ is an equivalence relation on the set $[l]$.*

Proof — Two elements $q_1, q_2 \in \mathcal{Z}^1_l(A/A^l, \iota^l)$ are cohomologous (or, equivalently, $[q_1] = [q_2]$ in $\mathrm{Desc}^1_l(A/A^l, \iota^l)$) if there is an element $x_0 \in U(A^l)$ such that

$$q_1(a) \cdot x_0 = q_2\big(a \cdot \iota^l(x_0)\big)$$

for all $a \in A$. Since ι^l is a homomorphism of monoids, it follows that

$$(\iota^l q_1)(a) \cdot \iota^l(x_0) = (\iota^l q_2)\big(a \cdot \iota^l(x_0)\big) \tag{3.2}$$

and since $x_0 \in U(A^l)$, and hence $\iota^l(x_0) = l\big(\iota^l(x_0)\big) \in U(A)$, to say that (3.2) holds for all $a \in A$ is just to say that $\kappa_l(q_1) \cong_{l,\iota^l(x_0)} \kappa_l(q_2)$.

Conversely, if $\kappa_l(q_1) \cong_{l,y_0} \kappa_l(q_2)$ for some $y_0 \in A$ with $l(y_0) \in U(A)$, then

$$\kappa_l(q_1)(a) \cdot l(y_0) = \kappa_l(q_2)\big(a \cdot l(y_0)\big) \text{ for all } a \in A,$$

or, equivalently, since $\kappa_l(q_1) = \iota^l q_1$, $\kappa_l(q_2) = \iota^l q_2$ and $l = \iota^l q^l$,

$$(\iota^l q_1)(a) \cdot \iota^l\big(q^l(y_0)\big) = (\iota^l q_2)\big(a \cdot \iota^l\big(q^l(y_0)\big)\big) \text{ for all } a \in A,$$

whence — since ι^l is injective homomorphism of monoids —

$$q_1(a) \cdot q^l(y_0) = q_2\big(a \cdot \iota^l\big(q^l(y_0)\big)\big) \quad \text{for all } a \in A,$$

which — since $l(y_0) \in U(A)$ and $l(y_0) \in A^l$ and hence $q^l(y_0) \in U(A^l)$ — just means that q_1 and q_2 are cohomologous.

Summing up, we have proved that $q_1, q_2 \in \mathcal{Z}^1_l(A/A^l, \iota^l)$ are cohomologous if and only if $\kappa_l(q_1) \cong_l \kappa_l(q_2)$. This completes the proof of the first statement of the proposition. The second one then follows from the facts that the relation on the set $\mathcal{Z}^1_l(A/A^l, \iota^l)$ of being cohomologous is an equivalence relation and that the map ℓ_l^{-1} is bijective. □

An important consequence of this proposition is the following result.

Theorem 3.14 *For any left quasi-projection $l : A \to A$ on a monoid A, the isomorphism of pointed sets*

$$\kappa_l : \mathcal{Z}^1_l(A/A^l, \iota^l) \xrightarrow{\simeq} [l]$$

induces an isomorphism

$$\overline{\kappa}_l : \mathsf{Desc}^1_l(A/A^l, \iota^l) \xrightarrow{\simeq} [l]/_{\cong}$$

of pointed sets.

For any submonoid B of a monoid A and any $a \in U(A)$, we write B^a for aBa^{-1}. B^a is called a $U(A)$-*conjugate* of B. According to [2, Lemma 2.19], for every submonoid X of a monoid A, the group $U(A)$ acts by conjugation on the set $\mathsf{FAC}(A/X)$. In other words, if $(X, B) \in \mathsf{FAC}(A)$, then for any $U(A)$-conjugate B^a of B, A is factorized by the submonoids X and B^a. We denote the corresponding quotient set by $\mathsf{FAC}(A/X)/U(A)$

Theorem 3.15 *Let l be left quasi-projection on a monoid A such that A^l is a group. Then the passage*

$$[l]/_{\cong} \longmapsto \mathsf{FAC}(A/A^l)/U(A)$$

$$l' \longmapsto \mathsf{Ker}(l')$$

is bijective.

Proof — By [2, Proposition 2.17], the map

$$\mathcal{Z}^{l}_{l}(A/A^{l}, \iota^{l}) \xrightarrow{q \to \mathsf{Ker}(q)} \mathsf{FAC}(A/A^{l})$$

is bijective, which in view of [3, Proposition 4.6] yields a bijection

$$\mathsf{Desc}^{l}_{l}(A/A^{l}, \iota^{l}) \simeq \mathsf{FAC}(A/A^{l})/\mathcal{U}(A).$$

It then follows from Theorem 3.14 that the map

$$[l]/_{\cong} \xrightarrow{l' \to \mathsf{Ker}(q^{l'})} \mathsf{FAC}(A/A^{l})/\mathcal{U}(A)$$

is bijective. The proof of the theorem is now completed by noting that $\mathsf{Ker}(q^{l'}) = \mathsf{Ker}(l')$, since $l' = q^{l'}\iota^{l'}$ and since $\iota^{l'}$ is injective. □

4 Complementary pairs of quasi-projections

For any set X, let $\triangle_X$ be the diagonal in $X \times X$, i.e., $\triangle_X = \{(x, x) : x \in X\}$.

Definition 4.1 Let l and r be left and right quasi-projections, respectively, on a monoid A. We say that (l, r) is a *complementary pair of quasi-projections on* A provided the following conditions are satisfied:

1. $A^{l} = \mathsf{Ker}(r)$ and $A^{r} = \mathsf{Ker}(l)$.
2. $K[l] \cap K[r] = \triangle_A$.

Given a monoid A, we write $\mathsf{CP}(A)$ for the set of complementary pair of quasi-projections an A.

Proposition 4.2 *Let A be a monoid and l and r be respectively left and right quasi-projections on A. Then $(l, r) \in \mathsf{CP}(A)$ if and only if*

(i) $lr = rl = 0_{A,A}$ *and*

(ii) $l * r = \mathsf{Id}_A$,

where the map $l * r : A \to A$ is defined by $(l * r)(a) = l(a)r(a)$.

Proof — Suppose first that Conditions (i) and (ii) are satisfied. If $a \in A^l$, then $l(a) = a$ and hence

$$r(a) = r\big(l(a)\big) = 1$$

by (i). Thus $A^l \subseteq \mathsf{Ker}(r)$. Suppose now $a \in \mathsf{Ker}(r)$. Then $r(a) = 1$ and by (ii) we have $a = l(a)r(a) = l(a)$. Hence $\mathsf{Ker}(r) \subseteq A^l$. Consequently, $A^l = \mathsf{Ker}(r)$. In a similar manner, one shows that $A^r = \mathsf{Ker}(l)$. This proves (1) in Definition 4.1. For clause (2) in the definition, suppose that $a, a' \in A$ are such that $(a, a') \in \mathsf{K}[l] \cap \mathsf{K}[r]$. Then $l(a) = l(a')$ and $r(a) = r(a')$ and we have by (ii):

$$a = l(a)r(a) = l(a')r(a') = a'.$$

Therefore, $\mathsf{K}[l] \cap \mathsf{K}[r] = \triangle_A$. So that $(l, r) \in \mathsf{CP}(A)$.

Conversely, if $(l, r) \in \mathsf{CP}(A)$, then since $l(a) \in A^l$ for all $a \in A$, the equality $A^l = \mathsf{Ker}(r)$ implies that $rl = 0_{A,A}$. Symmetrically, the equality $A^r = \mathsf{Ker}(l)$ implies that $lr = 0_{A,A}$. Thus (i) holds. To show that (ii) also holds, consider an arbitrary pair $(x, y) \in A^l \times \mathsf{Ker}(l)$. Then $l(x) = x$, $l(y) = 1$ and we have:

$$l(xy) = l\big(l(x)y\big) \overset{(L2)}{=} l(x)l(y) = x1 = x.$$

Similarly, $x \in A^l = \mathsf{Ker}(r)$ implies that $r(x) = 1$ and we have:

$$r(xy) \overset{(R3)}{=} r\big(r(x)y\big) = r(1y) = r(y) = y,$$

where the last equation holds because $y \in A^r = \mathsf{Ker}(l)$. In particular, since $l(a) \in A^l$ and $r(a) \in A^r = \mathsf{Ker}(l)$ for any $a \in A$, we have

$$l\big(l(a)r(a)\big) = l(a) \quad \text{and} \quad r\big(l(a)r(a)\big) = r(a).$$

Hence

$$\big(l(a)r(a), a\big) \in \mathsf{K}[l] \cap \mathsf{K}[r].$$

Then $l(a)r(a) = a$, since $\mathsf{K}[l] \cap \mathsf{K}[r] = \triangle_A$ by (2). Thus, (ii) also holds. This completes the proof of the proposition. □

Proposition 4.3 *Let A be a monoid and $(l, r) \in \mathsf{CP}(A)$. Then A factorizes through A^l and A_l; i.e., the map*

$$[\iota^l, \iota_l] : A^l \times A_l \to A$$

is bijective.

Proof — Note first that by Proposition 3.7, both A^l and A_l are submonoids of A. We next claim that the map

$$\langle l, r\rangle : A \to A^l \times A_l$$

that takes $a \in A$ to the pair $\big(l(a), r(b)\big)$ is the inverse of the map $[\iota^l, \iota_l]$. Indeed, we have already established in the proof of the sufficient part of Proposition 4.2 that

$$\langle l, r\rangle[\iota^l, \iota_l] = \mathsf{Id}_{A^l \times A_l},$$

while the following calculation shows that $[\iota^l, \iota_l]\langle l, r\rangle = \mathsf{Id}_A$:

$$\big([\iota^l, \iota_l]\langle l, r\rangle\big)(a) = [\iota^l, \iota_l]\big(l(a), r(a)\big) = l(a)r(a) \overset{4.1(2)}{=} a$$

for all $a \in A$. So $[\iota^l, \iota_l]$ is inverse of the map $[\iota^l, \iota_l]$, as claimed. □

Proposition 4.4 *In any complementary pair of quasi-projections on a monoid A, the components define each other uniquely. In other words, if $(l, r), (l, r'), (l', r) \in \mathsf{CP}(A)$, then $l = l'$ and $r = r'$.*

Proof — If $(l, r), (l, r') \in \mathsf{CP}(A)$, then

$$A^r = A_l = A^{r'}$$

by Definition 4.1. It follows that $r(a) \in A^{r'}$ (i.e., $r'\big(r(a)\big) = r(a)$) for every $a \in A$. Moreover, since $(l, r) \in \mathsf{CP}(A)$, $a = l(a)r(a)$ by Proposition 4.2 and we have:

$$r'(a) = r'\big(l(a)r(a)\big) \overset{(R2)}{=} r'\big(l(a)\big)r(a) \overset{4.2(i)}{=} 1r(a) = r(a).$$

Therefore, $r = r'$. The equality $l = l'$ can be proved similarly. □

For future reference we state and prove the following result.

Lemma 4.5 *Let A be a monoid and $(l, r) \in \mathsf{CP}(A)$. Then for all $a, b \in A$, we have:*

(i) $l\big(a \cdot r(b)\big) = l(a)$.

(ii) $r\big(l(a) \cdot b\big) = r(b)$.

PROOF — By symmetry, it suffices to prove (i). Since $(l, r) \in \mathsf{CP}(A)$, $a = l(a) \cdot r(a)$, we compute:

$$\begin{aligned}
l\big(a \cdot r(b)\big) &= l\big(l(a) \cdot r(a) \cdot r(b)\big) \\
&= l(a) \cdot l\big(r(a) \cdot r(b)\big) && \text{[by (L2)]} \\
&= l(a) \cdot l\big(r(a) \cdot l\big(r(b)\big)\big) && \text{[by (L3)]} \\
&= l(a) \cdot l\big(r(a) \cdot 1\big) && \text{[by Proposition 4.2(i)]} \\
&= l(a) \cdot l\big(r(a)\big) \\
&= l(a) \cdot 1 = l(a). && \text{[by Proposition 4.2(i)]}
\end{aligned}$$

The statement is proved. □

Let A be a monoid and $l : A \to A$ a set-theoretical map such that A^l is a subgroup of A. Then since $l(a) \in A^l$, it follows that $l(a)^{-1} \in A^l$. Hence it makes sense to define a map $l^\triangleright : A \to A$ by $l^\triangleright(a) = l(a)^{-1}\, a$.

Proposition 4.6 *In the situation described above, suppose that l is a left quasi-projection on A. Then the map $l^\triangleright : A \to A$ is a right quasi-projection on A.*

PROOF — For (R1), we have $l^\triangleright(1) = l(1)^{-1} \cdot 1 \overset{(L1)}{=} 1^{-1} \cdot 1 = 1$. Next, since

$$l\big(a l(b)^{-1} b\big) \overset{(L3)}{=} l\big(a l(b)^{-1} l(b)\big) = l(a),$$

it follows that

$$\begin{aligned}
l^\triangleright\big(a l^\triangleright(b)\big) = l^\triangleright\big(a l(b)^{-1} b\big) &= l\big(a l(b)^{-1} b\big)^{-1} a l(b)^{-1} b \\
&= l(a)^{-1} a l(b)^{-1} b = l^\triangleright(a) l^\triangleright(b),
\end{aligned}$$

proving (R2). Finally, to prove (R3), we make the following calculation:

$$\begin{aligned}
l^\triangleright\big(l^\triangleright(a) b\big) &= l^\triangleright\big(l(a)^{-1} a b\big) && \text{[by definition of } l^\triangleright\text{]} \\
&= \big(l\big(l(a)^{-1} a b\big)\big)^{-1} \big(l(a)^{-1} a b\big) && \text{[by definition of } l^\triangleright\text{]} \\
&= \big(l\big(l\big(l(a)^{-1}\big) a b\big)\big)^{-1} \big(l(a)^{-1} a b\big) && \text{[since } l(a)^{-1} \in A^l\text{]} \\
&= \big(l(a)^{-1} l(a b)\big)^{-1} \big(l(a)^{-1} a b\big) && \text{[by (L2)]} \\
&= l(a b)^{-1} l(a) \big(l(a)^{-1} a b\big) \\
&= l(a b)^{-1} (a b) = l^\triangleright(a b). && \text{[by definition of } l^\triangleright\text{]}
\end{aligned}$$

Therefore, $l^{\triangleright} : A \to A$ is a right quasi-projection on A. □

Dually, we have the following result.

Proposition 4.7 *Let A be a monoid and let r be a right quasi-projection on A such that A^r is a subgroup of A. Then the map $r^{\triangleleft} : A \to A$, $r^{\triangleleft}(a) = ar(a)^{-1}$, is a left quasi-projection on A.*

In the situation of the preceding two propositions, we call $l^{\triangleright}$ and $r^{\triangleright}$ *right and left quasi-projections induced by* l *and* r, respectively.

Proposition 4.8 *Let A be a monoid and l (resp. r) be a left (resp. right) quasi-projection on A such that A^l (resp. A^r) is a subgroup of A. Then the pair $(l, l^{\triangleright})$ (resp. $(r^{\triangleleft}, r)$) is complementary.*

Proof — By symmetry, it suffices to prove the result for left quasi-projections. So suppose that l is a left quasi-projection on a monoid A such that A^l is a subgroup of A. Then $l^{\triangleright}(a) = l(a)^{-1}a$ and for any $a \in A$, we have:

$$l\big(l^{\triangleright}(a)\big) = l\big(l(a)^{-1}a\big) \overset{(L_2)}{=} l(a)^{-1}l(a) = 1$$

and

$$l^{\triangleright}\big(l(a)\big) = \big(l\big(l(a)\big)\big)^{-1}l(a) = l(a)^{-1}l(a) = 1.$$

Thus, Condition (i) in Proposition 4.2 is satisfied. Next, since

$$l(a)l^{\triangleright}(a) = l(a)l(a)^{-1}a = a$$

for all $a \in A$, it follows that $l * l^{\triangleright} = \mathrm{Id}_A$. Hence Condition (ii) in Proposition 4.2 also holds. Therefore, by Proposition 4.2, the pair $(l, l^{\triangleright})$ is complementary. □

According to Propositions 3.5 and 3.6, for any left or right quasi-projection u on a group A, A^u is a subgroup of A, and hence one can define the corresponding induced quasi-projection. The following follows immediately from Propositions 4.4 and 4.8.

Proposition 4.9 *A complementary pair of quasi-projections on a group A has the form $(l, l^{\triangleright})$ for some $l \in \mathsf{LQ}(A)$ or, equivalently, the form $(r^{\triangleleft}, r)$ for some $r \in \mathsf{RQ}(A)$.*

Combining Propositions 4.4 and 4.9, we have the following result.

Proposition 4.10 *For any group A, the maps*

$$\mathsf{LQ}(A) \xrightarrow{l \longmapsto l^{\triangleright}} \mathsf{RQ}(A) \text{ and } \mathsf{RQ}(A) \xrightarrow{r \longmapsto r^{\triangleleft}} \mathsf{LQ}(A)$$

define mutually inverse 1–1 correspondences between the left and right quasi-projections on A.

Remark 4.11 It follows from Proposition 4.9 that for any group A, the maps

$$\mathsf{CP}(A) \xrightarrow{(l,r)\to l} \mathsf{LQ}(A) \text{ and } \mathsf{CP}(A) \xrightarrow{(l,r)\to r} \mathsf{RQ}(A)$$

are bijective with inverse maps $l \to l^{\triangleright}$ and $r \to r^{\triangleleft}$, respectively.

5 Factorizations and complementary pairs

In this section, we present our main results. We start with the following result.

Theorem 5.1 *For a monoid A, the correspondence*

$$(l, r) \longmapsto (A^l, A^r)$$

yields a bijection

$$\mathsf{CP}(A) \simeq \mathsf{FAC}(A).$$

Proof — According to Proposition 4.3, the map

$$\mathsf{CP}(A) \xrightarrow{(l,r)\longmapsto(A^l,A^r)} \mathsf{FAC}(A),$$

denoted by Φ, is well defined. In order to show that this map is bijective, we construct its inverse Ψ. For any fixed $(X, B) \in \mathsf{FAC}(A)$, consider the composites

$$l_{X,B} : A \xrightarrow{q_{X,B}} X \xrightarrow{\iota_X} A$$

and

$$r_{X,B} : A \xrightarrow{p_{X,B}} B \xrightarrow{\iota_B} A.$$

Since $q_{X,B} \in \mathcal{Z}^1_l(A/X, \iota_X)$ and $p_{X,B} \in \mathcal{Z}^1_r(A/B, \iota_B)$ by Proposition 2.5, it follows from Proposition 3.7 that $l_{X,B}$ is a left quasi-projection and that $r_{X,B}$ a right quasi-projection on A. We next claim that $(l_{X,B}, r_{X,B})$ belongs to $\mathsf{CP}(A)$. Since $l_{X,B} * r_{X,B} = \mathrm{Id}_A$ by (2.2), it suffices to check that

$$l_{X,B} \circ r_{X,B} = r_{X,B} \circ l_{X,B} = 0_{A,A}.$$

As for an arbitrary $a \in A$, (2.2) is the (X, B)-factorization of a, it follows that $p_{X,B} \circ \iota_X = 0_{X,B}$ and $q_{X,B} \circ \iota_B = 0_{B,X}$. Then

$$r_{X,B} \circ \iota_X = \iota_B \circ p_{X,B} \circ \iota_X = \iota_B \circ 0_{X,B} = 0_{X,A}$$

and similarly $l_{X,B} \circ \iota_B = 0_{B,A}$, whence the desired equalities follow. Now define

$$\Psi : \mathsf{FAC}(A) \to \mathsf{CP}(A)$$

by $\Psi(X, B) = (l_{X,B}, r_{X,B})$. By Proposition 2.5, clearly $\Phi\Psi = \mathrm{Id}_{\mathsf{FAC}(A)}$. Since for any $(l, r) \in \mathsf{FAC}(A)$, $q_{A^l,A^r} = q^l$, $p_{A^l,A^r} = q^r$, $\iota^l = \iota_{A^l}$, $\iota^r = \iota_{A^r}$, $l = \iota^l q^l$ and $r = \iota^r q^r$, it follows that $\Psi\Phi = \mathrm{Id}_{\mathsf{CP}(A)}$. Thus, the map Φ is bijective, as desired. □

In the light of Remark 4.11, we get from Theorem 5.1 the following

Theorem 5.2 *For any group A, we have a bijection*

$$\mathsf{LQ}(A) \to \mathsf{FAC}(A)$$

given by

$$l \longmapsto (A^l, A_l).$$

The inverse map is given by

$$(X, B) \longmapsto \big(A \xrightarrow{[\iota_X, \iota_B]^{-1}} X \times B \xrightarrow{\pi_X} X \xrightarrow{\iota_X} A\big).$$

Symmetrically, the correspondence

$$r \longmapsto (A_r, A^r)$$

yields a bijection $\mathsf{RQ}(A) \to \mathsf{FAC}(A)$, whose inverse takes $(X, B) \in \mathsf{FAC}(A)$ to the composite $A \xrightarrow{[\iota_X, \iota_B]^{-1}} X \times B \xrightarrow{\pi_B} B \xrightarrow{\iota_B} A$.

Let A group. Recall from [5] (see also [6]) that a map $B : A \to A$ is

said to be a *Rota–Baxter operator on* A, if for all $a, b \in A$ one has

$$B(a) \cdot B(b) = B\big(a \cdot B(a) \cdot b \cdot B(a)^{-1}\big)$$

A group A together with the map B is also called a *Rota–Baxter group*.

Proposition 5.3 (see [5], Lemma 2.6) *For any group A and for every $l \in \mathsf{LQ}(A)$, $\big(A, (l^{\triangleright})^{-1}\big)$ is a Rota–Baxter group.*

Proof — Since $l \in \mathsf{LQ}(A)$, we have $r = l^{\triangleright} \in \mathsf{RQ}(A)$ and $(l, r) \in \mathsf{CP}(A)$. Writing B for r^{-1}, for all $a, b \in A$, we have

$$\begin{aligned}
B(a) \cdot B(b) &= r(a)^{-1} \cdot r(b)^{-1} && [\text{as } B = r^{-1}] \\
&= \big(r(b) \cdot r(a)\big)^{-1} \\
&= \big(r\big(b \cdot r(a)\big)\big)^{-1} && [\text{by (R2)}] \\
&= \big(r\big(l(a) \cdot b \cdot r(a)\big)\big)^{-1} && [\text{by Lemma 4.5}] \\
&= \big(r\big(l(a) \cdot r(a) \cdot r(a)^{-1} \cdot b \cdot r(a)\big)\big)^{-1} && [\text{as } r(a) \cdot r(a)^{-1} = 1] \\
&= \big(r\big(a \cdot r(a)^{-1} \cdot b \cdot r(a)\big)\big)^{-1} && [\text{as } l * r = \mathsf{Id}_A] \\
&= B\big(a \cdot B(a) \cdot b \cdot B(a)^{-1}\big), && [\text{as } B = r^{-1}]
\end{aligned}$$

proving that r^{-1} is a Rota–Baxter operator on A, and so that $(A, l^{\triangleright})^{-1}$ is a Rota–Baxter group. □

6 The case of semidirect product

In this section we study monoid factorizations arising from semidirect product of monoids. First we give a necessary and sufficient condition for a bicrossed product diagram of monoid to be (isomorphic) the semidirect product of monoids.

Let A be a monoid and $Y \subseteq A$ its subset. We say that a submonoid X of A is *left Y-normal* if $yX \subseteq Xy$ for all $y \in Y$. Obviously, if A is a group, then any normal subgroup of A is left Y-normal for every subset $Y \subseteq A$. Moreover, if a subgroup is left A-normal, then it is a normal subgroup of A.

Proposition 6.1 *A bicrossed product diagram*

$$X \xrightarrow{\ k\ } A \xleftarrow{\ l\ } B$$

is (isomorphic to) a semidirect product diagram if and only if $\mathsf{Im}(k)$ *is left* $\mathsf{Im}(l)$*-normal in* A*.*

Proof — We identify X and B with their images in A under k and l, respectively. The "if" part follows from [2]. Conversely, if the diagram is a semidirect product, then p_{XB} is a morphism of monoids (see Remark 2.7) and then for any $x \in X$ and $b \in B$, we have:

$$p_{XB}(bx) = p_{XB}(b)p_{XB}(x) = p_{XB}(b) \overset{(DR1)}{=} b.$$

It then follows from Section 2.1 that

$$bx = q_{XB}(bx) \cdot p_{XB}(bx) = q_{XB}(bx) \cdot b,$$

implying — since $q_{XB}(bx) \in X$ — that $bX \subseteq Xb$ for all $b \in B$. Thus, X is left B-normal in A. □

Recall that any group A acts from the left on itself by conjugation, namely we have a left action

$$\bullet : A \times A \to A, \quad a \bullet b = aba^{-1}.$$

Thus one can define the set of all non-abelian 1-cocycles $\mathcal{Z}^1(A, A)$.

Proposition 6.2 *Let* A *be a group. The following are equivalent for an idempotent map* $l : A \to A$ *of sets:*

1. $l^{\triangleright}$ *is a group homomorphism.*
2. $l^{-1} \in \mathcal{Z}^1(A, A)$*, where* $l^{-1} : A \to A$ *is defined by* $l^{-1}(a) = \big(l(a)\big)^{-1}$*.*
3. l *is a left quasi-projection on* A *such that* A^l *is a normal subgroup of* A*.*

When these hold, A *is isomorphic to the semidirect product* $A^l \rtimes A^{l^{\triangleright}}$*.*

Proof — Since $l^{\triangleright}(a) = l^{-1}(a) \cdot a$, we have for all $a, b \in A$:

$$l^{\triangleright}(a \cdot b) = l^{-1}(a \cdot b) \cdot a \cdot b$$

and

$$l^{\triangleright}(a) \cdot l^{\triangleright}(b) = l^{-1}(a) \cdot a \cdot l^{-1}(b) \cdot b.$$

Comparing these equalities, we see that $l^{\triangleright}$ is a homomorphism of groups if and only if

$$l^{-1}(a \cdot b) \cdot a = l^{-1}(a) \cdot a \cdot l^{-1}(b),$$

or equivalently, if and only if

$$l^{-1}(a \cdot b) = l^{-1}(a) \cdot \big(a \cdot l^{-1}(b) \cdot a^{-1}\big). \tag{6.1}$$

Moreover, $l^{-1}(1) = 1^{-1} = 1$. Since the equality $l^{-1}(1) = 1$ together with (6.1) expresses the fact that l^{-1} is an element of the set $\mathcal{Z}^1(A, A)$, one concludes that the conditions (1) and (2) are equivalent.

(1) $\implies$ (3) It is readily seen that $(l^{\triangleright})^{\triangleleft} = l$ and since $l^{\triangleright}$, being a group homomorphism, is a right quasi-projection on A (see Proposition 3.4), it follows from Proposition 4.7 that l is a left quasi-projection on A. By Proposition 4.8, $(l, l^{\triangleright})$ is then a complementary pair of quasi-projections on A. Then $A^l = \mathsf{Ker}(l^{\triangleright})$; so that A^l is a normal subgroup of A.

(3) $\implies$ (1) As we have seen, to show that $l^{\triangleright}$ is a homomorphism of groups is to show that (6.1) holds. But since A^l is normal in A and since $l(b) \in A^l$, it follows that $a \cdot l(b) \cdot a^{-1} \in A^l$ and we have:

$$a \cdot l(b) \cdot a^{-1} \cdot l(a) \overset{(DL2)}{=} l\big(a \cdot l(b) \cdot a^{-1} \cdot a\big) = l\big(a \cdot l(b)\big) \overset{(DL3)}{=} l(a \cdot b).$$

Hence (6.1) holds, proving that $l^{\triangleright}$ is a homomorphism of groups. □

A corollary follows immediately.

Corollary 6.3 *For any group A, the passage*

$$\mathsf{LQ}(A) \longmapsto \mathcal{Z}^1_{\mathsf{Id}}(A, A), \quad l \to l^{-1}$$

is a bijection, with inverse

$$\mathcal{Z}^1_{\mathsf{Id}}(A, A) \longmapsto \mathsf{LQ}(A), \quad \chi \to \chi^{-1}.$$

Here, $\mathcal{Z}^1_{\mathsf{Id}}(A, A)$ is the subset of $\mathcal{Z}^1(A, A)$ whose elements are idempotent non-abelian 1-cocycles $A \to A$.

In light of Theorem 5.2, we get from Proposition 6.2 and Corollary 6.3 the following result.

Theorem 6.4 *For any group A, the assignment*

$$(X, B) \longmapsto (A \xrightarrow{[\iota_X, \iota_B]^{-1}} X \times B \xrightarrow{\pi_X} X \xrightarrow{\iota_X} A),$$

establishes a bijection

$$\mathsf{FAC}(A) \simeq \mathcal{Z}^1_{\mathsf{Id}}(A, A).$$

The following proposition connects 1-descent cocycles with Rota–Baxter operators.

Proposition 6.5 *Let A be a group. For any $\chi \in \mathcal{Z}^1_{\mathsf{Id}}(A, A)$, the map*

$$a \longmapsto \big(\chi(a) \cdot a\big)^{-1}$$

is a Rota–Baxter operator on A.

Proof — By combining Proposition 5.3 and Corollary 6.3, we see that for all $\chi \in \mathcal{Z}^1_{\mathsf{Id}}(A, A)$, the map

$$\big((\chi^{-1})^{\triangleright}\big)^{-1} : A \to A$$

is a Rota–Baxter operator on A. Since for any $a \in A$,

$$\big((\chi^{-1})^{\triangleright}\big)^{-1}(a) = \big((\chi^{-1})^{\triangleright}(a)\big)^{-1} = \big(\big(\chi^{-1}(a)\big)^{-1} \cdot a\big)^{-1} = \big(\chi(a) \cdot a\big)^{-1},$$

the result follows. □

Let X be a normal subgroup of a group A. Then $(a, x) \longmapsto axa^{-1}$ yields a left action of the group A on the group X and so we may define the pointed set $\mathcal{Z}^1(A, X)$. We write $\mathcal{Z}^1_\bullet(A, X)$ for the subset of $\mathcal{Z}^1(A, X)$ containing of the maps $\chi : A \to X$ such that $\chi \circ \iota_X = \mathsf{Id}_X^{-1}$ (equivalently, $\chi(x) = x^{-1}$ for all $x \in X$).

Proposition 6.6 *In the above situation, the assignment $q \longmapsto q^{-1}$ yields a bijection*

$$\mathcal{Z}^1_l(A/X, \iota_X) \simeq \mathcal{Z}^1_\bullet(A, X)$$

of sets. Its inverse is given by $\chi \longmapsto \chi^{-1}$.

Proof — Note first that since the group A acts on both groups X and A by conjugation and since X is normal in A, the canonical inclusion $\iota_X : X \to A$ is A-*equivariant*, i.e., $\iota_X(a \bullet x) = a \bullet \iota_X(x)$ for

all $a \in A$ and all $x \in X$. It follows — since ι_X is an injective homomorphism of groups — that $\chi \in \mathcal{Z}^1(A,X)$ if and only if $\iota_X \circ \chi \in \mathcal{Z}^1(A,A)$. Now, if $q \in \mathcal{Z}^1_l(A/X, \iota_X)$, then $l = \iota_X q$ is a left quasi-projection on A by Proposition 3.7; and, moreover, quite obviously, $A^l = X$. Then, by applying Proposition 6.2, we see that $l^{-1} = (\iota_X q)^{-1} = \iota_X q^{-1}$ lies in $\mathcal{Z}^1(A,A)$; or, equivalently, q^{-1} lies in $\mathcal{Z}^1(A,X)$. Since q belongs to $\mathcal{Z}^1_l(A/X, \iota_X)$, we have $q\iota_X = \mathrm{Id}_X$ by (DL1). Then

$$q^{-1}\iota_X = (q\iota_X)^{-1} = \mathrm{Id}_X^{-1}.$$

Thus, $q^{-1} \in \mathcal{Z}^1_\bullet(A,X)$ and so the map $\mathcal{Z}^1_l(A/X, \iota_X) \xrightarrow{q \longmapsto q^{-1}} \mathcal{Z}^1_\bullet(A,X)$ is well defined.

Next, consider any $\chi \in \mathcal{Z}^1_\bullet(A,X)$. Then $(\chi \circ \iota_X)^{-1} = \mathrm{Id}_X$ and hence

$$\chi^{-1} \circ \iota_X = (\chi \circ \iota_X)^{-1} = \mathrm{Id}_X,$$

proving that $\chi^{-1} : A \to X$ satisfies (DL1). Condition (DL2) is also satisfied, as the following calculation shows

$$\begin{aligned}
\chi^{-1}(xa) &= \big(\chi(xa)\big)^{-1} && \text{[by definition of } \chi^{-1}\text{]}\\
&= \big(\chi(x)\cdot x \cdot \chi(a)\cdot x^{-1}\big)^{-1} && \text{[since } \chi \in \mathcal{Z}^1(A,X)\text{]}\\
&= \big(\chi(a)\cdot x^{-1}\big)^{-1} && \text{[since } \chi(x) = x^{-1} \text{ for all } x \in X\text{]}\\
&= x\cdot\big(\chi(a)\big)^{-1}\\
&= x\cdot\chi^{-1}(a). && \text{[by definition of } \chi^{-1}\text{]}
\end{aligned}$$

For (DL3) we have:

$$\begin{aligned}
\chi^{-1}(ab) &= \big(\chi(ab)\big)^{-1} && \text{[by definition of } \chi^{-1}\text{]}\\
&= \big(\chi(a)\cdot a\cdot\chi(b)\cdot a^{-1}\big)^{-1} && \text{[since } \chi \in \mathcal{Z}^1(A,X)\text{]}\\
&= \big(\chi(a)\cdot a\cdot(\mathrm{Id}_X^{-1}\circ\chi^{-1})(b)\cdot a^{-1}\big)^{-1} && \text{[since } \mathrm{Id}_X^{-1}\circ\chi^{-1} = \chi\text{]}\\
&= \big(\big(\chi(a)\cdot a\cdot\chi(\iota_X\circ\chi^{-1})(b)\big)\cdot a^{-1}\big)^{-1} && \text{[since } \chi\circ\iota_X = \mathrm{Id}_X^{-1}\text{]}\\
&= \big(\chi\big(a\cdot(\iota_X\circ\chi^{-1})(b)\big)\big)^{-1} && \text{[since } \chi \in \mathcal{Z}^1(A,X)\text{]}\\
&= \chi^{-1}\big(a\cdot(\iota_X\circ\chi^{-1})(b)\big). && \text{[by definition of } \chi^{-1}\text{]}
\end{aligned}$$

Thus, χ^{-1} satisfies (DL1), (DL2) and (DL3) and so lies in $\mathcal{Z}^1_l(A/X, \iota_X)$.

So that the map

$$\mathcal{Z}^1_\bullet(A,X) \xrightarrow{\chi\to\chi^{-1}} \mathcal{Z}^1_l(A/X,\iota_X)$$

is also well defined. Quite obviously, the maps $q \to q^{-1}$ and $\chi \to \chi^{-1}$ are mutually inverse maps. This completes the proof of the proposition. □

Theorem 6.7 *Let X be a normal subgroup of a group A. Then the following are equivalent:*

(i) *The short exact sequence*

$$0 \to X \xrightarrow{\iota_X} A \xrightarrow{p} A/X \to 0,$$

where $p : A \to A/X$ is the quotient map, splits; i.e., there exists a homomorphism $s : A/X \to A$ of groups with $ps = \mathrm{Id}_{A/X}$.

(ii) *X is the first component of a group factorization of A; i.e., there is a subgroup B of A such that $(X,B) \in \mathsf{FAC}(A)$.*

(iii) $\mathcal{Z}^1_l(A/X,\iota_X) \neq \emptyset$.

(iv) $\mathcal{Z}^1_\bullet(A,X) \neq \emptyset$.

Proof — It is well known (e.g., [4, Chapter 5]) that if a group homomorphism $s : A/X \to A$ splits the short exact sequence

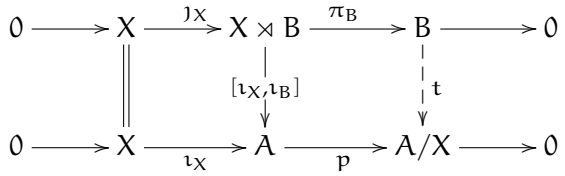
$$0 \to X \xrightarrow{\iota_X} A \xrightarrow{p} A/X \to 0,$$

then A factorizes by its subgroups X and $\mathrm{Im}(s)$. Hence (i) implies (ii). Moreover, (ii) implies (iii) by Proposition 2.5 and (iii) and (iv) are equivalent by Proposition 6.6.

Next, if $\mathcal{Z}^1_l(A/X,\iota_X) \neq \emptyset$, then we can choose an element $q : A \to X$ in $\mathcal{Z}^1_l(A/X,\iota_X)$. Then, as in the proof of Proposition 6.6, we deduce that $l = \iota_X q$ is a left quasi-projection on A and that $A^l = X$. It then follows from Theorem 5.2 that A factorizes through X and A_l. Thus, (iii) implies (ii).

Finally, if there exists a subgroup B of A such that $(X,B) \in \mathsf{FAC}(A)$, then A is isomorphic to the semi-direct product $X \rtimes B$ via $[\iota_X,\iota_B]^{-1}$

and since in the diagram



both rows are short exact sequences and since the left square commutes (see Remark 2.7), there exists an isomorphism of groups

$$t : B \to A/X$$

making the right square in the diagram commute. This implies — since π_B is a split epimorphism with splitting $\jmath_B : B \to X \rtimes B$ — that p is also a split epimorphism. This proves the implication (ii) $\Longrightarrow$ (i). □

Example 6.8 Let $\mathsf{GL}(n;\mathbb{C})$ be the group of invertible $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers. Recall that the QR-*decomposition* (also called the QR-*factorization*) of a matrix $X \in \mathsf{GL}(n;\mathbb{C})$ is a representation of the matrix as the product of a unitary matrix $Q(X)$ and a upper triangular matrix $R(X)$ with positive real entries on the main diagonal. Each invertible complex matrix has such a unique decomposition. There are several methods for actually computing the QR-decomposition. For example, one of such methods is the well-known Gram–Schmidt process.

Since QR-decomposition of any invertible complex matrix exists, the group $\mathsf{GL}(n;\mathbb{C})$ factorizes by its subgroup of the unitary matrices and the subgroup of upper triangular matrices with positive diagonal entries. It then follows that the map $X \to Q(X)$ (resp. $X \to R(X)$) sending an invertible complex $n \times n$ matrix X to the unitary matrix (resp. the upper triangular matrix with positive diagonal entries) obtained from the QR-decomposition of X is a left (resp. right) quasi-projection on the group $\mathsf{GL}(n;\mathbb{C})$. Consequently, for all $X, Y \in \mathsf{GL}(n;\mathbb{C})$ one has:

- $Q(X) \cdot Q(Y) = Q\big(Q(X) \cdot Y\big)$.
- $Q(X \cdot Y) = Q\big(X \cdot Q(Y)\big)$.
- $R(X) \cdot R(Y) = R\big(X \cdot R(Y)\big)$.
- $R(X \cdot Y) = R\big(R(X) \cdot Y\big)$.

These formulas might be helpful, say, to simplify calculation of the matrices involved in the QR-factorization of invertible complex matrices.

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