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An Introduction to p-Modular Representations of p-Adic Groups *

Ramla Abdellatif

Abstract

These lecture notes offer a self-contained introduction to p-modular representations of p-adic groups, with a special focus on the general linear group $GL_2(F)$ and the special linear group $SL_2(F)$ when F is a finite extension of the field of p-adic numbers.

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1 Introduction

These notes cover the content of the lecture I gave during the second edition of the GABY summer school, which took place in Milan on June 2022. They provide a self-contained introduction to the world of p-modular representations of p-adic groups, aimed at newcomers in the domain. No specific knowledge is required beyond the standard level of classical Master courses in algebra and number theory.

The focus is made on the groups $GL_2(F)$ and $SL_2(F)$ for F a finite extension of $\mathbb{Q}_p$ on purpose. These two groups are indeed quite

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easy to manipulate as p-adic groups, but their p-modular representation theory is already much different from the corresponding complex representation theory, which is well-known (see for instance [12] for a comprehensive and self-contained study of the GL_2 case), and mysterious enough to give a good idea of what is (not) happening in more general settings. Going beyond these cases would require more technicalities and background than expected from a newcomer. The reader interested in knowing more about the general case will enjoy the last section of these notes, which provides a short survey of what is (not) known so far about p-modular representations of p-adic groups, as well as useful references to go further.

From now, we fix a prime integer p. A p-*modular representation* of a group is an action of this group on a vector space defined over a field of characteristic p. In other words, it is a linear representation of the group whose coefficients live in a field of characteristic p. Such representations appear very naturally in number theory: one can for instance consider the action of the absolute Galois group of $\mathbb{Q}$ on the p-torsion points of an elliptic curve defined over $\mathbb{Q}$. Note that this simple example is actually at the core of one of the most famous proofs in contemporary number theory, namely Andrew Wiles' proof of Fermat's Last Theorem [35]. Let us also point out that this example is closely related to the representations we will study in these lecture notes, via the so-called p-*modular Langlands correspondences* (for $GL_2(\mathbb{Q}_p)$ in this case): we will say something about this in the last section.

Regarding p-modular representations of p-adic groups, i.e. of groups of the form $G = \mathcal{G}(F)$ with $\mathcal{G}$ being a connected reductive group defined over a finite extension F of the field $\mathbb{Q}_p$ of p-adic numbers*, the story starts in the mid-nineties, with the seminal work of Laure Barthel and Ron Livné [7, 8] on p-modular representations of $GL_2(F)$. This series of two papers opened a wide range of questions that mostly remain unanswered 30 years later. The goal of these notes is to understand what are these questions and what answers have been given so far, then to give an overview of what remains unsolved and how it inspired further questions.

More precisely, we aim to explain what means the following fun-

* If you don't know what a connected reductive group is, just pick your favourite matrix group, as GL_n, SL_n or Sp_{2n} with $n \geqslant 1$, and let the coefficients of the matrices be in a finite extension of the field $\mathbb{Q}_p$. If you don't know what $\mathbb{Q}_p$ is, please refer to Section 2.1.

damental question, and how much of it has been solved so far.

Question 1.1 *Let* p *be a prime integer, let* F *be a finite extension of* $\mathbb{Q}_p$ *and let* $\overline{\mathbb{F}}_p$ *be an algebraic closure of the residue field of* F*. Can we give an exhaustive classification of (isomorphism classes of) irreducible smooth admissible representations of* G *over* $\overline{\mathbb{F}}_p$ *when* G *is either* $\mathrm{GL}_2(F)$ *or* $\mathrm{SL}_2(F)$*?*

2 Crash course on p-adic groups and related notions

This first section gathers the basic material we need about p-adic groups, general representation theory and Bruhat–Tits buildings. The reader already familiar with all these topics (for instance because they come from the world of classical Langlands correspondences) can skip this section. For others, let us give some references about the various topics dealt with in this section.

- The first subsection regards p-adic fields, which are finite extensions of the field of p-adic numbers. People who are not familiar with these objects, and more generally with local fields, can usefully read Section 7 of James Milne's lecture notes [25] or Section 4 of Helmut Koch's book [21]. A nice account on the group structure of $\mathrm{GL}_2(F)$ is given in Section 7 of the book by Colin Bushnell and Guy Henniart [12]. For further results on algebraic groups and their inner structure, a standard reference is Armand Borel's book on linear algebraic groups [9].

- The second subsection is a reminder about basic notions of representation theory. It does not require any further material than these notes. Nevertheless, readers interested in strengthening their background can usefully use [17, 29], which focus on the finite groups representation theory, or [19, Part II] for more information on the general reductive case.

- The third subsection gathers basic definitions and facts from the world of Bruhat–Tits buildings. Besides [9] aforementioned, the fifth section of the second paper coming from the seminal work of François Bruhat and Jacques Tits [11] can be useful to anyone willing to understand the general setting of buildings and parahoric subgroups.

Let us set some notations. Given any power q of the prime p we fixed, we denote by $\mathbb{F}_q$ the field with q elements. When needed, ℓ will always denote a prime *different from* p.

2.1 Local fields: arithmetical and topological properties

Let ν_p denote the standard p-adic valuation, which maps a non-zero integer n to the biggest power of p that divides n, and 0 to ∞, as well as its extension to $\mathbb{Q}$ given by the formula

$$\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b).$$

The function that maps a rational number $x \in \mathbb{Q}$ to $p^{-\nu_p(x)}$ defines a norm on $\mathbb{Q}$ called the (standard) p-*adic norm* and denoted by $\|.\|_p$. The completion of $\mathbb{Q}$ for $\|.\|_p$ is denoted by $\mathbb{Q}_p$ and called the field of p-*adic numbers*.

The unit ball of $(\mathbb{Q}_p, \|.\|_p)$ is denoted by $\mathbb{Z}_p$ and an element of this ball is called a p-*adic integer*. Note that $\mathbb{Z}_p$ is a discrete valuation ring for ν_p, hence it is a local ring with maximal ideal $p\mathbb{Z}_p$ and its residue field $\mathbb{Z}_p/p\mathbb{Z}_p$ is isomorphic to $\mathbb{F}_p$. Also notice that its field of fractions is exactly $\mathbb{Q}_p$.

Following [14, Chapter VI], we assume more generally that F is a non-Archimedean local field with residue characteristic equal to p. This means that F is a field endowed with a discrete valuation ν for which it is complete, and whose residue field is isomorphic to $\mathbb{F}_q$ for some power $q = p^f$ of p. Such a field is actually not very mysterious, thanks to the following result proven in 1916 by Alexander Ostrowski [27].

Theorem 2.1 *Any non-Archimedean local field with residue characteristic equal to p is either a finite extension of $\mathbb{Q}_p$, or a field of Laurent series $\mathbb{F}_{p^f}((t))$ for some integer $f \geqslant 1$.*

Let us now give some topological properties of F. As F is complete with respect to a discrete valuation ν, the ring O_F of elements in F having nonnegative valuation is a local ring whose maximal ideal $\mathfrak{p}_F$ is generated by any element of minimal positive valuation. Such an element is called a *uniformizing element*, or *uniformizer*, and it exists since ν is discrete.

From now on, we fix a uniformizing element ϖ_F and we choose to normalize ν so that $\nu(\varpi_F) = 1$. Then, for any integer n, the (fractional) ideal $\mathfrak{p}_F^n = \varpi_F^n \mathcal{O}_F$ is an open subgroup of F, and $\{\mathfrak{p}_F^n,\ n \in \mathbb{Z}_+^*\}$ is a fundamental system of open neighbourhoods of 0 in F. Note that these neighbourhoods are moreover compact subgroups of F: indeed, first notice that each quotient $\mathcal{O}_F/\mathfrak{p}_F^n$ is finite, and that the canonical map

$$\mathcal{O}_F \to \varprojlim \mathcal{O}_F/\mathfrak{p}_F^n$$

is actually an isomorphism of topological groups. This implies in particular that $\mathcal{O}_F$ is a compact group, and since $\mathfrak{p}_F^n = \varpi_F^n \mathcal{O}_F$ is topologically isomorphic to $\mathcal{O}_F$, we proved the claim. To summarize, we obtain the following statement (recall that a topological group is *locally profinite* when any open neighbourhood of the unit contains an open compact subgroup).

Proposition 2.2 *The group* $(F, +)$ *is locally profinite.*

Similarly, we see that $(F^\times, \times)$ is a locally profite group as the congruence subgroups $\{1 + \mathfrak{p}_F^n,\ n \in \mathbb{Z}_+^*\}$ provide a fundamental system of open compact neighbourhoods of 1 in $F^\times$.

Remark 2.3 For future use, we recall that locally profinite groups are in particular locally compact and totally disconnected, that *closed* subgroups of locally profinite groups are still locally profinite groups, and that the quotient of a locally profinite group by a *closed* normal subgroup is again a locally profinite group.

2.2 Topological properties of p-adic groups

Writing F as the additive group $M_1(F)$ of 1×1 matrices with coefficients in F, and $F^\times$ as the multiplicative group $GL_1(F)$ of invertible matrices in $M_1(F)$, one can wonder how the topological properties above transfer to matrices of arbitrary size $d \geqslant 1$. Given a positive integer $d \geqslant 2$, the group $M_d(F)$ is naturally isomorphic to F^{d^2}, hence it is naturally endowed with a structure of locally profinite group inherited from the product topology of F^{d^2}. Note that in this setting, the matrix multiplication is a continuous map.

Now consider $GL_d(F)$: it is an open subspace of $M_d(F)$, hence it also gets a structure of topological group (since inverting matrices is a continuous map). As in the $d = 1$ case, it is also a locally profinite group since the set of congruence subgroups

$$\left\{K(n) := I_d + \varpi_F^n M_d(F),\ n \in \mathbb{Z}_+^*\right\},$$

where I_d denotes the identity matrix, provides a fundamental system of open compact neighbourhoods of $GL_d(F)$. Let us point out that these subgroups are moreover pro-p-groups, as this will be of importance in the sequel. Furthermore, $K := GL_d(\mathcal{O}_F)$ is a maximal open compact subgroup of $GL_d(F)$, and any maximal open compact subgroup of $GL_d(F)$ is actually conjugated to K.

Remark 2.4 The latter property is really specific to $GL_d(F)$! Consider for instance the special linear group $SL_d(F) = \det^{-1}\left(\{1\}\right)$. It is a closed subgroup of $GL_d(F)$, so Remark 2.3 ensures that it is a locally profinite group (one can also check directly that it is a topological group and that a fundamental system of open compact neighbourhoods of 1 in $SL_d(F)$ is given by $\left\{K(n) \cap SL_d(F),\ n \in \mathbb{Z}_+^*\right\}$), but it is straightforward (and a good exercise for the newcomer) to see that it contains d distinct conjugacy classes of maximal open compact subgroups. One can object that these groups are conjugated in $GL_d(F)$, hence are not so different from each other: in this case, an even more striking example is given by the conjugacy classes of the p-adic group G defined by the F-points of the p-adic unitary group $U(2,1)$. Then G can be seen as a closed subgroup of $GL_3(E)$ for E/F a suitable quadratic extension. One can check as in [1, Annexe 7.2] that G has two distinct conjugacy classes of maximal open compact subgroups that are not conjugate in $GL_3(E)$.

2.3 Basic representation theory for beginners

Given a group G and a ring R, a *representation of* G *over* R is just a group action of G on an R-module. In this lecture notes, we only consider the case of a *field* R, so G acts on R-vector spaces, but representations over R-modules for more general rings are also of great interest. For instance, considering representations over $\mathcal{O}_F$ leads to some connections (via reduction modulo a uniformizing element) between p-adic (integral) representations (namely representations over F or $\mathcal{O}_F$) and p-modular representations (namely representations over an algebraic closure of k_F). The interested reader could refer for instance to [4, 16] for more information on this topic.

A representation of G over a field R is denoted by a pair (π, V), where V is an R-vector space and $\pi : G \to \mathrm{Aut}_R(V)$ is the group homomorphism defining the action of G on V. In this case, the dimension of the R-vector space V is called the *dimension of the representation* (π, V), and one-dimensional representations are called *characters* of G. We will be interested in representations of G up to isomorphism: recall that two representations (π, V) and (σ, W) of G over R

are *isomorphic* if there exists an isomorphism of R-vector spaces

$$\varphi : V \xrightarrow{\simeq} W$$

that is compatible with the action of G, i.e. such that $\varphi \circ \pi = \sigma \circ \varphi$.

From now on, we assume that G is a *locally profinite* group, as for example $\mathrm{GL}_d(F)$ or $\mathrm{SL}_d(F)$, and that C is a field. The topological nature of G makes it natural to focus on continuous representations. To take into account the locally profinite nature of G, we will consider the following notion of continuity.

Definition 2.5 Let (π, V) be a representation of G over C. We say that (π, V) is *smooth* when every vector of V has open stabilizer in G. We denote by $\mathrm{Rep}_C^\infty(G)$ the category of smooth representations of G over C.

Note that G being locally profinite ensures that it is equivalent to require that the stabilizer of any vector of V under the action of G contains an open compact subgroup of G. Also note that smoothness is really a continuity assumption, as one can check that (π, V) is smooth if, and only if, the corresponding map

$$(g, v) \in G \times V \mapsto \pi(g)(v) \in V$$

is continuous when V is endowed with the discrete topology.

Another natural condition on the representations we consider comes from number theory and arithmetic. It basically claims that these representations are locally finite.

Definition 2.6 A *smooth* representation (π, V) of G over C is *admissible* when, given any open compact subgroup H of G, the space

$$V^H := \{v \in V \mid H \subset \mathrm{Stab}_G(v)\}$$

of H-invariant vectors in V is finite-dimensional as vector space over C.

Remark 2.7 We impose smoothness in the definition of admissibility, though it does not seem a priori necessary. Nevertheless, we will see in the sequel that these two notions are strongly related, and that the relevant representations on a number-theoretic or arithmetic point of view (e.g. for Langlands correspondences) are always smooth, so this is not an important restriction.

We end up this subsection by recalling a fundamental notion that defines the elementary pieces on which to build all representations of a given group. Let us already point out that the way to recover all representations from the irreducible ones heavily depends on the field C on which they are defined.

Definition 2.8 A representation (π, V) of G over C is *irreducible* when V contains *exactly* two subspaces that are stable under the action of G (namely $\{0\}$ and V).

Remark 2.9 By definition, irreducible representations of G are non-zero.

2.4 Parahoric and parabolic subgroups via Bruhat–Tits theory

Recall that the groups we are interested in are of the form $\mathcal{G}(F)$ for some connected reductive group $\mathcal{G}$ defined over F (if you don't know what this means, stick to your favorite matrix group example, as GL_d or SL_d). For simplicity, and because the examples we study here fit in this framework, we assume that $\mathcal{G}$ is *split* over F, which means that its maximal torii* are split over F.

Definition 2.10 A *parabolic subgroup* of $\mathcal{G}$ is a Zariski-closed subgroup $\mathcal{P}$ such that $\mathcal{G}/\mathcal{P}$ is a projective variety. A *Borel subgroup* of $\mathcal{G}$ is a parabolic subgroup $\mathcal{B}$ that is minimal among parabolic subgroups of $\mathcal{G}$.

Remark 2.11 As classically done in the litterature, we abusively call a *parabolic* (*resp. Borel*) *subgroup of* G any group of the form $\mathcal{P}(F)$ for $\mathcal{P}$ some parabolic (resp. Borel) subgroup of $\mathcal{G}$. The same will be done for torii of G, defined as the groups of F-rational points of torii of $\mathcal{G}$.

Example 2.12 When $\mathcal{G} = GL_d$, parabolic subgroups are (up to conjugacy) block-upper triangular matrices of the following form:

$$\begin{pmatrix} GL_{n_1} & \star & \star & \star \\ 0 & GL_{n_2} & \star & \star \\ 0 & 0 & \ddots & \star \\ 0 & 0 & 0 & GL_{n_r} \end{pmatrix}$$

* A *torus* is just a subgroup of $\mathcal{G}$ isomorphic to some power of the multiplicative group $\mathbb{G}_m$.

with $r \in \mathbb{Z}_+^*$ and $n_1, \ldots, n_r \in \mathbb{Z}_+^*$ satisfying $\sum_{i=1}^r n_i = d$. They are minimal (i.e. provide Borel subgroups) iff $r = d$, which means that all n_i's are equal to 1.

From now on, $\mathcal{G}$ will either be GL_2 or SL_2. Let us fix a first set of notations that will be constantly used in the sequel.

- $\mathcal{B}$ denotes the Borel subgroup of upper triangular matrices in $\mathcal{G}$.
- $\mathcal{U}$ denotes the unipotent radical of $\mathcal{B}$, which consists in upper triangular matrices in $\mathcal{G}$ whose diagonal coefficients are all equal to 1.
- $\mathcal{T}$ denotes the maximal (split) torus of diagonal matrices in $\mathcal{G}$.
- $B = \mathcal{B}(F)$, $U = \mathcal{U}(F)$ and $T = \mathcal{T}(F)$.

Let us point out that B has a semi-direct product decomposition of the form $B = TU$, where T acts on U by conjugation. This is a special case of *Levi decomposition*, which allows to write any parabolic subgroup as a semi-direct product involving its unipotent radical. Such decompositions are of importance to initiate the study of irreducible smooth representations of G. Precise definitions in the general case can be found in [9, Section 11] or in [19, II.1].

We end up this section by introducing parahoric subgroups by the mean of the Bruhat–Tits building of G. We will use the very nice lattice construction given by Jean-Pierre Serre in [30], which can be extended to some other groups (see for instance the case of the quasi-split group $U(2,1)$ explained by Jacques Tits in [31] and fully detailed in [1, Chapter 4]) but does unfortunately not hold for arbitrary $\mathcal{G}$.

We let $\mathcal{X}$ denote the graph defined as follows:

- the vertices of $\mathcal{X}$ correspond to the homothety classes $[L]$ of $\mathcal{O}_F$-lattices L in the two-dimensional F-vector space $F \oplus F$;
- two vertices $[L_0]$ and $[L_1]$ are connected by an edge iff there exist representatives L_0 and L_1 of the associated homothety classes that satisfy $\varpi_F L_1 \subset L_0 \subset L_1$ (recall that ϖ_F is a fixed uniformizing element of F).

It is a good exercise to check that $\mathcal{X}$ is actually a tree and that G acts on $\mathcal{X}$ via its natural action on $\mathcal{O}_F$-lattices. Note that this action is transitive on vertices when $G = GL_2(F)$, but has two distinct orbits when $G = SL_2(F)$. One can easily check that the stabilizer of the standard vertex $[\mathcal{O}_F^2]$ is $K := \mathcal{G}(\mathcal{O}_F)$ and that (the set of vertices of) $\mathcal{X}$ is endowed with a natural distance defined as the minimal number of edges required to connect two vertices of $\mathcal{X}$. Another subgroup of G that will matter in the sequel is the pointwise stabilizer of the edge that connects $[\mathcal{O}_F^2]$ to $[\mathcal{O}_F \oplus \mathfrak{p}_F]$. It is called an *Iwahori subgroup of* G and is equal to

$$I = \left\{ M \in K \mid M \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \bmod \varpi_F \right\} .$$

Its pro-p-radical is the set of matrices whose reduction modulo ϖ_F has diagonal coefficients equal to 1: it is called a *pro-*p*-Iwahori subgroup of* G.

Remark 2.13 The groups K and I are instances of *parahoric subgroups* of G, defined as (pointwise) stabilizers of facets of the (affine) Bruhat–Tits building of G^{ad}. To know more about the general setting, please refer to [11]. As an exercise, one can try to define the same data for the unitary group $U(2,1)$, following for instance [1, Chapter 4] or [31].

3 Non-supercuspidal representations of p-adic groups

The goal of this section is to initiate the classification of irreducible smooth (admissible) representations of G over an algebraically closed field C of characteristic p. This consists in the study of parabolically induced representations, which play here the same role as Eisenstein series do in the theory of modular forms: they provide the first examples to consider and to understand quite easily, but they are not the most exciting objects of the theory.

Remark 3.1 Since we choose to focus on GL_2 and SL_2, we do not give the most general possible setting for parabolically induced representations. The reader interested in understanding parabolic induction for arbitrary groups can refer for instance to [1, Section 2.2.1] or to [33, I.5.7]. Let us nevertheless mention that extending the results of this section to other p-adic groups is not so difficult, and essentially requires technical arrangements (see [34] or [1, Chapter 5]). This will not be the case for Section 4.

3.1 Parabolically induced representations

Let us first define what are parabolically induced representations of G. Recall that B denotes the parabolic subgroup of G made of upper-triangular matrices and that T denotes the subgroup of diagonal matrices in G. Since $B = TU$ with U being a locally pro-p-group, irreducible smooth representations of B actually come (by inflation) from irreducible smooth representations of T. In our setting, T is either isomorphic to $(F^\times)^2$ or to $F^\times$ (depending on whether $\mathcal{G}$ is either GL_2 or SL_2), thus its irreducible smooth representations are one-dimensional, defined as follows by a pair (χ_1, χ_2) of smooth C-valued characters of $F^\times$ (resp. by one smooth character $\chi : F^\times \to C$):

$$\forall\, g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B,\ (\chi_1 \otimes \chi_2)(g) := \chi_1(a)\chi_2(c) \qquad (3.1)$$
$$\big(\text{resp. } \chi(g) := \chi(a)\big)$$

For the sake of clarity, we let $\mathcal{C}^\infty(G)$ be the set of functions $f : G \to C$ such that there exists an open compact subgroup K_f of G satisfying $f(gx) = f(g)$ for any pair $(g, x) \in G \times K_f$. Note that $\mathcal{C}^\infty(G)$ is naturally endowed with an action of G by left-translations (said otherly, this action is given by

$$\pi_\infty(x)(f) := \big[g \mapsto f(gx)\big]$$

for all $(g, x) \in G \times G$), which defines (by construction) a smooth representation $\big(\pi_\infty, \mathcal{C}^\infty(G)\big)$ of G over C. Parabolically induced representations of G are defined as certain subrepresentations of $\big(\pi_\infty, \mathcal{C}^\infty(G)\big)$ made of B-equivariant functions, as follows.

Definition 3.2 Let σ denote an irreducible smooth representation of T over C, as well as its inflation to an irreducible admissible representation of B.

We set

$$\mathrm{Ind}_B^G(\sigma) := \big\{ f \in \mathcal{C}^\infty(G) \mid \forall (b, g) \in B \times G,\ f(bg) = \sigma(b) \cdot f(g) \big\}$$

Then $\mathrm{Ind}_B^G(\sigma)$ defines a (smooth) subrepresentation of $\big(\pi_\infty, \mathcal{C}^\infty(G)\big)$ over C, called *parabolically induced from* (B, σ).

More generally, a smooth representation of G is called *parabolically induced* if it is isomorphic to $\left(\pi_\infty, \mathrm{Ind}_B^G(\sigma)\right)$ for some irreducible smooth representation σ of T over C.

This construction actually defines a functor

$$\mathrm{Ind}_B^G : \mathrm{Rep}_C^\infty(T) \to \mathrm{Rep}_C^\infty(G)$$

that is a right-adjoint to the natural restriction functor, as stated by the next proposition.

Proposition 3.3 (Smooth Frobenius reciprocity) *Let σ be an irreducible smooth representation of T over C and π be a smooth representation of G over C. Then the evaluation at I_2 induces an isomorphism of C-vector spaces*

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_B^G(\sigma)\right) \xrightarrow{\simeq} \mathrm{Hom}_B\left(\pi|_B, \sigma\right)$$

that is functorial in both $\pi \in \mathrm{Rep}_C^\infty(G)$ and $\sigma \in \mathrm{Rep}_C^\infty(T)$.

This strong connection is a key tool to prove the following result, which fully characterizes the representation $\mathrm{Ind}_B^G(\sigma)$. It is due to Laure Barthel and Ron Livné [7, 8] for GL_2 and to Ramla Abdellatif [2] for SL_2.

Theorem 3.4 *Let σ and η be irreducible smooth representations of T over C.*

1) *The representation $\mathrm{Ind}_B^G(\sigma)$ is irreducible iff σ cannot be extended to a smooth representation of G.*

2) *Letting $\mathbf{1}$ denote the trivial representation, $\mathrm{Ind}_B^G(\mathbf{1})$ is a length 2 representation of G that fits into the following non-split short exact sequence:*

$$0 \longrightarrow \mathbf{1} \longrightarrow \mathrm{Ind}_B^G(\mathbf{1}) \longrightarrow \mathrm{St}_G \longrightarrow 0\ ,$$

where St_G is called the Steinberg representation.

3) *We have: $\mathrm{Ind}_B^G(\sigma) \simeq \mathrm{Ind}_B^G(\eta)$ iff $\sigma \simeq \eta$.*

We postpone the proof of this theorem to the next subsection, and rather make now some remarks on its contents. First, let us explicit in a more concrete way the condition given on σ in the first point of the theorem — we kept the general formulation of [1, Théorème 5.1.2], since this theorem actually holds for arbitrary (quasi-split) groups of semi-simple rank 1. We saw previously (see (3.1)) that irreducible smooth representations of T are characters, so saying that σ extends to a smooth representation of G means that σ is the restriction to T of a smooth character of G. Since the derived group of $\mathcal{G}$ is SL_2, any smooth character of G factors through the determinant map

$$\det : G \to F^\times.$$

Thus we obtain the following reformulation of our irreducibility condition on $\mathrm{Ind}_B^G(\sigma)$:

- if $G = SL_2(F)$, then $\mathrm{Ind}_B^G(\sigma)$ is irreducible iff σ is not the trivial character $\mathbf{1}$;
- if $G = GL_2(F)$, then $\mathrm{Ind}_B^G(\sigma)$ is irreducible iff σ is not of the form $\chi \circ \det$ for some smooth character $\chi : F^\times \to C$. With the notations of (3.1), this is equivalent to say that $\chi_1 \neq \chi_2$.

When $\sigma = \chi \circ \det$ with $\chi : F^\times \to C$ is a smooth character, it is straightforward to check that $\mathrm{Ind}_B^G(\sigma)$ is isomorphic to $(\chi \circ \det) \otimes \mathrm{Ind}_B^G(\mathbf{1})$, thus the second assertion of Theorem 3.4 takes care of any reducible parabolically induced representation of G.

Remark 3.5 Theorem 3.4 is clearly wrong when C is not of characteristic p. Indeed:

- over the field $\mathbb{C}$ of complex numbers, there exists a smooth character of G whose restriction to T induces an irreducible representation of G, hence our irreducibility criterion becomes wrong. Moreover, the Steinberg representation can occur as subrepresentation of some parabolically induced representation of G over $\mathbb{C}$, what does not happen for p-modular representations (as appears from the second assertion of the theorem). More details on these phenomena are given in [12, page 65];
- over a field of prime characteristic $\ell \neq p$, parabolically induced representations can be of length strictly greater than 2, unlike what claims the second assertion of our theorem. For instance, when $p = 3$ and $\ell = 3$, then $\mathrm{Ind}_B^G(\mathbf{1})$ is of length 3, as explained in [33, page 99].

3.2 Proving Theorem 3.4

Since parabolically induced representations are by construction defined from representations of B, a natural idea to initiate their study is to consider their restrictions as representations of B and see what happens. Since the elements of the underlying vector spaces are functions over G that are B-equivariant, they can be seen as functions over the quotient $B\backslash G$.

Now recall that Bruhat decomposition states that $G = B \sqcup Bw_0U$, where

$$w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in K$$

is a representative of a generating element of the Weyl group

$$W \simeq \mathbb{Z}/2\mathbb{Z}$$

of G. All this ensures that, given an irreducible smooth representation σ of T over C, the map that sends an element of $\mathrm{Ind}_B^G(\sigma)$ to its value at I_2 is a surjective linear map $\mathrm{Ind}_B^G(\sigma) \to C$ that is compatible with the action of B on both sides when C is endowed with the action defined by σ. In other words, it provides a morphism φ_σ of $\mathrm{Rep}_C^\infty(B)$ from $\mathrm{Ind}_B^G(\sigma)$ to σ. It is straightforward to check that its kernel $\ker\varphi_\sigma$ is the subspace of elements of $\mathrm{Ind}_B^G(\sigma)$ whose support lies in Bw_0U. The keystone of the proof of Theorem 3.4 lies in the following proposition.

Proposition 3.6 *The representation of* B *carried by* $\ker\varphi_\sigma$ *is irreducible.*

To prove this result, we first note that any element of $V_\sigma := \ker\varphi_\sigma$ is a smooth function, hence has support in a double coset of the form Bw_0U_f, where U_f is an *open compact* subgroup of U (that depends of course of the function). This said, it implies that, as a representation of B, V_σ is isomorphic to $\big(\pi_\sigma, \mathcal{C}_c^\infty(U)\big)$, where $\mathcal{C}_c^\infty(U)$ denotes the space of compactly supported smooth functions from U to C, and π_σ is the action of B on $\mathcal{C}_c^\infty(U)$ defined as follows: $\forall b = tu \in B = TU$, $\forall f \in \mathcal{C}_c^\infty(U)$, $\forall v \in U$,

$$\big(\pi_\sigma(f)\big)(v) := \sigma\big(w_0 t w_0^{-1}\big) f\big(t^{-1} v t u\big). \tag{3.2}$$

One can make this formula very explicit as G is a subgroup of $GL_2(F)$, but this is not very enlightening so we choose to avoid this.

We are hence reduced to show that $\big(\pi_\sigma, \mathcal{C}_c^\infty(U)\big)$ is an irreducible representation of B. This relies on the following proposition and lemma, left as an exercise to the reader (as well as extensively proven in [1, Chapters 3 and 5] for the proposition, and [1, Lemma 2.1.9] for the lemma, in case the reader does not have the time or the will to try solving this exercise). Note that the lemma will also play a crucial role in the study of supercuspidal p-modular representations of G (see Section 4).

Proposition 3.7 1) U *admits an exhaustive increasing filtration by open compact pro-*p*-subgroups.*

2) *Any open compact subgroup of* U *provides (by conjugation by powers of a well-chosen* $t_0 \in T$*) a decreasing sequence of open compact neighbourhoods of* I_2 *in* U.*

3) *For any open compact subgroup* U_0 *of* U*, the space* $\mathcal{C}_c^\infty(U_0)^{U_0}$ *of* U_0*-invariant elements of* $\mathcal{C}_c^\infty(U)$ *supported in* U_0 *is a* 1*-dimensional* C*-vector space (that is generated by the characteristic function* $\mathbf{1}_{U_0}$ *of* U_0*).*

Lemma 3.8 *Any non-zero smooth representation of a pro-*p*-group over* C *has non-zero fixed vectors.*

Indeed, assume that W is a non-zero B-subrepresentation of π_σ and pick a non-zero element f in W. By the first point of Proposition 3.7, there exists an open compact pro-p-subgroup U_0 that contains the support of f. Lemma 3.8 and the third point of Proposition 3.7 then imply that the characteristic function of U_0 belongs to the U_0-subrepresentation of π_σ generated by f. A fortiori, this proves that $\mathbf{1}_{U_0}$ belongs to W, and one can easily check (using the two first statements of Proposition 3.7) that the B-subrepresentation of π_σ generated by $\mathbf{1}_{U_0}$ is actually π_σ, so we have $W = \pi_\sigma$ by double inclusion.

This proves Proposition 3.6, which ensures that the B-representation $\mathrm{Ind}_B^G(\sigma)|_B$ is of length 2 and fits into the following short exact sequence:

$$0 \longrightarrow V_\sigma \longrightarrow \mathrm{Ind}_B^G(\sigma)|_B \longrightarrow \sigma \longrightarrow 0\ . \tag{3.3}$$

Using Proposition 3.7, one can prove (as in [2, Proposition 2.6] for SL_2 and in [1, Proposition 5.3.9] for a more general framework that contains GL_2 and SL_2) the following statement.

* For $G = SL_2(F)$, we have for instance $t_0 = \begin{pmatrix} \varpi_F^{-1} & 0 \\ 0 & \varpi_F \end{pmatrix}$.

Proposition 3.9 *The short exact sequence* (3.3) *splits if, and only if,* σ *extends to a smooth character of* G.

We can now finish the proof of Theorem 3.4. If $\mathrm{Ind}_B^G(\sigma)$ is a reducible representation of G, then it must be of length 2 (since its restriction to B is) and have a subquotient V of dimension 1 (whose restriction to B is given by σ), so σ extends to a smooth representation V of G. The converse holds thanks to Proposition 3.9, thus the two first assertions of Theorem 3.4 follow. The third point follows directly from (3.3) by smooth Frobenius reciprocity (Proposition 3.3) and is left as an exercise to the reader (see also [1, Chapter 5]).

$\rightsquigarrow$ At this stage, we have built all irreducible smooth representations of G that come as subquotients of a parabolically induced representations. As proven in a uniform way in [1, Sections 5.4 and 5.5], there is no non-trivial isomorphism among them, which means that we have proven the following classification result.

Theorem 3.10 *Any irreducible subquotient of a parabolically induced representation is isomorphic to exactly one of the following representations:*

1) *a smooth character of* G;

2) *a parabolically induced representation* $\mathrm{Ind}_B^G(\sigma)$, *where* σ *is a smooth character of* T *that does not extend to a smooth character of* G;

3) *a twist* $\mathrm{St}_G \otimes \chi$ *of the Steinberg representation by a smooth character* χ *of* G.

Note that the only finite-dimensional representations in this list are the smooth characters of G. In analogy with the complex setting, the representations of the second point are sometimes called *principal series representations*, while those of the third point may be called *special series representations*.

Now that we are done with parabolic induction, we are left with the following question: are there other irreducible smooth representations of G? If so, can we classify them all? Before trying to solve these problems in the next section, let us introduce some useful vocabulary.

Definition 3.11 A *supercuspidal representation* of G over C is an irreducible admissible (hence smooth by definition) representation π of G for which there is no pair (P, σ), where $P = MU$ is a proper parabolic subgroup of G with unipotent radical U and Levi factor M, and σ is an irreducible admissible representation of M over C, such that π is isomorphic to a subquotient of $\mathrm{Ind}_P^G(\sigma)$.

Remark 3.12 The admissibility assumption is crucial here on both sides.

- Regarding G since, unlike the complex case (see below), there exists irreducible smooth representations of $GL_2(F)$ that are NOT admissible. For instance, Daniel Lê proved in [24] that this happens when F is an unramified cubic extension of $\mathbb{Q}_p$.
- Regarding M, because one can actually prove that any irreducible representation of G is a subquotient of $\mathrm{Ind}_B^G(\sigma)$ for some representation σ of G, see [6, VI.1]

This is clearly different from the classical setting of complex representations, where it is well known [12, Page 73] that irreducible smooth representations are *automatically* admissible. This automatic admissibility also holds for ℓ-modular representations (with $\ell \neq p$) of $GL_n(F)$, see [33, Page 102].

4 From supercuspidality to supersingularity

The goal now is to classify all supercuspidal representations of G, as defined above (Definition 3.11). Note that we are not looking after an empty set, as proven by Florian Herzig, Karol Kozioł and Marie-France Vignéras in [18]. The title of the latter reference suggests that the actual property we study for p-modular representations is not supercuspidality, but *supersingularity*. Following the pioneering work of Laure Barthel and Ron Livné, itself inspired from the ideas developed by Colin Bushnell and Phil Kutzko in their seminal work for complex representations [13], we will define supersingularity for representations of our group G and see that, for irreducible admissible representations, it is equivalent to supercuspidality.

4.1 From parabolic to parahoric: motivation

The upshot is that instead of working with parabolic induction, we consider parahoric induction, which means that we will induce representations of K or I, instead of B, and see what happens. From this idea originated the theory of types, which allowed Colin Bushnell and Phil Kutzko to classify all irreducible smooth (hence admissible) complex representations of $GL_N(F)$ and to draw a road-map for arbitrary G that still motivates many current trends in representation theory.

In the p-modular setting, this idea makes sense without referring to the complex case. Indeed, recall that Lemma 3.8 ensures in particular that there exists no non-trivial irreducible smooth representation for pro-p-groups. As a consequence, any irreducible smooth representation of K (resp. of I) must be trivial on its pro-p-radical (i.e. its largest normal pro-p-subgroup), hence factors through an irreducible representation of $K/K(1)$ (resp. of $I/I(1)$). Good news is that this quotient is a finite group, respectively given by $\mathcal{G}(k_F)$ or $\mathcal{T}(k_F)$ in our case, for which we know very well the irreducible representations over C.

- Letting $q = p^f$ denote the cardinality of k_F, any irreducible representation of $\mathcal{G}(k_F)$ is isomorphic to $\mathrm{Sym}^{\vec{r}}(C^2) \otimes (\chi \circ \det)$ for some f-tuple $\vec{r}$ of nonnegative integers strictly smaller than p and some character χ of $k_F^\times$; see [1] or [20] for an explicit definition of $\mathrm{Sym}^{\vec{r}}(C^2)$.
- Since $\mathcal{T}(k_F) \simeq (k_F^\times)^a$ with $a=1$ (resp. 2) and $\mathcal{G}=SL_2$ (resp. GL_2), irreducible representations of $\mathcal{T}(k_F)$ are just given by an a-tuple of characters of $k_F^\times$, the latter matching with $(q-1)^{\text{th}}$ roots of unity in C as $k_F^\times$ is a cyclic group.

Remark 4.1 For once, things are easier in the p-modular setting than in the complex one! Indeed, irreducible smooth complex representations of K are much harder to classify than what we did above: see for instance [12, §6] for the group GL_2, which already covers 4 full pages.

4.2 Compact induction: when our world collapses

Parahoric induction is a special instance of compact induction. We will define right now for an arbitrary open subgroup H of G that is compact mod center, i.e. such that H/Z is compact (where Z denotes the center of G).

Definition 4.2 Let (σ, V) be a smooth representation of H over C. We let $\mathrm{ind}_H^G(\sigma)$ denote the space of functions $f : G \to V$ satisfying the two following conditions:

- the support of f in G is compact mod center;
- there exists an open compact subgroup K_f of G such that

$$f(hgk) = \sigma(h)\big(f(g)\big)$$

for any triple $(h, g, k) \in H \times G \times K_f$.

This space is naturally endowed with a smooth action of G by right translations. The resulting representation of G is said to be *compactly induced from* (H, σ).

As their parabolic counterpart, compactly induced representations satisfy an adjunction property with respect to the restriction functor.

Proposition 4.3 (Compact Frobenius reciprocity) *Let* σ *be an irreducible smooth representation of* H *over* C *and* π *be a smooth representation of* G *over* C*. Then the evaluation at* I_2 *induces an isomorphism of* C*-vector spaces*

$$\mathrm{Hom}_G\left(\mathrm{ind}_H^G(\sigma), \pi\right) \xrightarrow{\simeq} \mathrm{Hom}_H\left(\sigma, \pi|_H\right)$$

that is functorial in both $\pi \in \mathrm{Rep}_C^\infty(G)$ *and* $\sigma \in \mathrm{Rep}_C^\infty(H)$*.*

When working with complex representations, compactly induced representations give the missing pieces of the classification. We do not explain this in detail here, since this is very nicely done for $GL_2(F)$ in the first fifteen sections of [12]. We only want to stress out the main tools used to get there, and explain how they dramatically fail for p-modular representations. First, we already saw that irreducible smooth complex representations are always admissible, and that this is false for p-modular representations. Another key result is the following, which comes from [12, Theorem 11.4] for $GL_2(F)$ and will later be proven false for p-modular representations.

Theorem 4.4 *Let* σ *be an irreducible smooth representation of* KZ *such that:*

$$\forall\, g \in G,\ \left(\mathrm{Hom}^G_{K^g \cap K}(\sigma^g, \sigma) \neq \{0\}\right) \iff (g \in K)\ .$$

Then $\mathrm{ind}_K^G(\sigma)$ *is irreducible and* (*super*)*cuspidal.*

Regarding the tools rather than the results, the existence of Haar measures (defined from non-linear S-valued functionals on $\mathcal{C}_c^\infty(G)$ for representations over S, see [12, page 26] for more precise statements) is crucial in the study of complex representations to define Hecke algebras and to use the theory of matrix coefficients. It is not difficult to see that index p subgroups of G would lead to a terrible non-sense such as $0 = 1$ if C-valued Haar measures were to exist.

Finally, another key tool in the classical theory of types is given by the *level* of an irreducible smooth representation, defined as the smallest nonnegative integer n for which non-zero $K(n+1)$-fixed vectors exist. In particular, level 0 representations (also called *unramified* representations) are crucial, but what matters more is the existence of representations of positive level. For p-modular representations, this obviously fails because of Lemma 3.8, which explicitly states that there is no positive level representation.

Now that we stated what we cannot do, let us see what we can do.

4.3 Spherical Hecke algebras and cokernel representations

The big step towards the development of p-modular representation theory took place in the mid-nineties, when Laure Barthel and Ron Livné [7, 8] used parahoric induction to characterize supercuspidal representations by their eigenvalues under certain spherical Hecke actions. Let us first define the Hecke algebras involved in this context.

Definition 4.5 Let H be an open compact mod center subgroup of G and σ be a smooth representation of H. The *Hecke algebra attached to* (H, σ) is the endomorphism algebra

$$\mathcal{H}(G, H, \sigma) := \mathrm{End}_{C[G]}\big(\mathrm{ind}_H^G(\sigma)\big).$$

We know from compact Frobenius reciprocity (Proposition 4.3) that, given a smooth representation π of G, its space

$$\pi^H \simeq \mathrm{Hom}_H(\mathbf{1}, \pi)$$

of H-invariant vectors is isomorphic (as a C-vector space) to

$$\mathrm{Hom}_G\big(\mathrm{ind}_H^G(\mathbf{1}), \pi\big),$$

thus is naturally endowed with a structure of right $\mathcal{H}(G, H, \mathbf{1})$-module. In particular, if H is the pro-p-radical of a parahoric subgroup (as K or I), it follows from Lemma 3.8 that π^H is a *non-zero* right $\mathcal{H}(G, H, \mathbf{1})$-module, and the structure of this module can actually provide interesting information on the representation π. This is for instance the case when:

- $H = K(1)$ is the pro-p-radical of K, with $\mathcal{H}\big(G, K(1), \mathbf{1}\big)$ being closely related to $\mathcal{H}(G, K, \sigma)$ for σ some irreducible smooth representation of K (the latter being called *spherical Hecke algebras*);
- $H = I(1)$ is the pro-p-radical of I, with $\mathcal{H}\big(G, I(1), \mathbf{1}\big)$ being closely related to $\mathcal{H}(G, I, \sigma)$ for σ some semi-simple smooth representation of I (the latter being called *Iwahori-Hecke algebras*).

By lack of space and time, we will not address the second case in these notes. Let us only mention that it has been the source of many interesting results that help to understand how different the theory of p-modular representations is from its complex and ℓ-modular counterparts, and that it is currently a motivation to several fruitful researches led in the domain. Let us also point out that, when G is GL_2, we actually consider the Hecke algebra attached to KZ, where $Z \simeq F^\times$ denotes the center of G, with σ being extended (by the choice of any smooth character of Z that extends the central character of σ) to a smooth representation of KZ. To keep uniform notations, we will denote by $\mathcal{K}$ the group K (resp. KZ) when $\mathcal{G}$ is SL_2 (resp. GL_2) and still write σ for the corresponding representation of $\mathcal{K}$.

From now on, we focus on the spherical Hecke case, meaning that we look at right $\mathcal{H}(G, \mathcal{K}, \sigma)$-modules for σ an irreducible smooth representation of $\mathcal{K}$ over C. The next proposition, proven in [7, Proposition 8] for GL_2 and in [2, Corollaire 3.9] for SL_2, shows how nice these algebras are. As can be seen in the aforementioned references, the proof is basically based on Cartan decomposition of G with respect to K and on explicit matrix calculations.

Proposition 4.6 *Let σ be an irreducible smooth representation of K, extended to an irreducible smooth representation of KZ when G is $\mathrm{GL}_2(F)$. Then there exists an operator $T_\sigma \in \mathcal{H}(G, \mathcal{K}, \sigma)$ such that the spherical Hecke algebra $\mathcal{H}(G, \mathcal{K}, \sigma)$ is exactly the polynomial algebra $C[T_\sigma]$.*

Note that the action of T_σ on elements of $\mathrm{ind}_{\mathcal{K}}^{G}(\sigma)$ is explicit and can be very nicely described, at least regarding the support of the functions involved in the game, on the Bruhat–Tits tree of G. These descriptions are given in [7, Section 5] for GL_2 and [2, Section 3.2.3] for SL_2.

The great idea of Laure Barthel and Ron Livné has been to introduce the following cokernel representations as replacement for the usual compactly induced representations.

Definition 4.7 For σ an irreducible smooth representation of $\mathcal{K}$ and $\lambda \in C$, we define $\pi(\sigma, \lambda)$ as the smooth representation of G given by

$$\pi(\sigma, \lambda) := \mathrm{Coker}\,(T_\sigma - \lambda) = \frac{\mathrm{ind}_{\mathcal{K}}^{G}(\sigma)}{(T_\sigma - \lambda\,\mathrm{id})\left(\mathrm{ind}_{\mathcal{K}}^{G}(\sigma)\right)}.$$

The next theorem, due to Laure Barthel and Ron Livné for $GL_2(F)$ and to Ramla Abdellatif for $SL_2(F)$ (see respectively [7, Theorem 33] and [2, Théorèmes 3.18 and 3.36]), shows how important the understanding of these cokernel representations is to classify p-modular representations of G. Also note that it proves the reducibility of $\mathrm{ind}_K^G(\sigma)$ in general, even under the assumptions of Theorem 4.4. Once more, we see how far from each other are complex and p-modular representation theories for p-adic groups.

Theorem 4.8

1) *Any irreducible admissible representation of* G *over* C *is a quotient of* $\pi(\sigma, \lambda)$ *for some pair* (σ, λ) *as in* Definition 4.7.

2) *If* λ *is non-zero, then any subquotient of* $\pi(\sigma, \lambda)$ *is non supercuspidal.*

The second statement of this theorem can actually be made more precise, since $\pi(\sigma, \lambda)$ can be fully described as soon as λ is non-zero. As proven in [7, Theorem 33] and [2, Théorème 3.18], it is generically a parabolically induced representation, and it provides a non-trivial extension of a smooth character of G by the corresponding twist of the Steinberg representation in the remaining cases. We hence obtain that supercuspidal representations will appear as quotients of $\pi(\sigma, 0)$ for σ covering all irreducible smooth representations of K. This leads to the following notion of supersingularity.

Definition 4.9 An irreducible admissible representation of G is *supersingular* when it is isomorphic to a quotient of $\pi(\sigma, 0)$ for some irreducible smooth representation σ of K.

Said differently, supersingular representations of G are those on which a suitable spherical Hecke algebra acts as trivially as possible (since its generator T_σ acts by 0 on $\pi(\sigma, 0)$), which is consistent with other notions of supersingularity that appear in number theory. What we noticed after stating Theorem 4.8 can therefore be rephrased as follows: *supercuspidality implies supersingularity*. The converse is actually true, but it is a highly non-trivial result (see [2, Corollaire 3.41] and [7, Corollary 36]).

Theorem 4.10 *For irreducible admissible representations of G, being supercuspidal is equivalent to begin supersingular.*

Remark 4.11 The notion of supersingularity depends a priori on the choice of a maximal open compact subgroup K of G. For the groups considered in these notes (namely GL_2 and SL_2), it is not too difficult to check (independently from Theorem 4.10, which provides another proof of this claim) that this choice does not impact our results, meaning that we recover exactly the same representations for any choice of K. For other groups, this question is more subtle to solve and requires much more work to prove that the notion of supersingularity (extended to arbitrary groups in the colossal work [6] of Noriyuki Abe, Guy Henniart, Florian Herzig and Marie-France Vignéras) is actually independent of the choice of K when one consider irreducible *admissible* representations of G, and is also equivalent to supercuspidality.

4.4 When $F = \mathbb{Q}_p$: some classification results

As it will clearly appear in Section 5.1 below, there is a huge gap between what happens when $F = \mathbb{Q}_p$, which is now well understood, and what happens for other values of F, which remains very mysterious. The main breakthrough has been provided in 2001 by Christophe Breuil, who proved the following result [10, Théorème 1.1] for $GL_2(\mathbb{Q}_p)$.

Theorem 4.12 *For any irreducible smooth representation σ of $GL_2(\mathbb{Z}_p)$, the cokernel representation $\pi(\sigma, 0)$ is an irreducible admissible representation of $GL_2(\mathbb{Q}_p)$.*

In particular, this result states that supercuspidal representations of $GL_2(\mathbb{Q}_p)$ are exactly the supersingular cokernel representations. Breuil also studied the existence of non-trivial isomorphisms between such representations [10, Théorème 1.3] and managed to get a complete classification of isomorphism classes of irreducible admissible representations of $GL_2(\mathbb{Q}_p)$. This allowed him to state the first p-modular semi-simple local Langlands correspondence ever [10, Définition 4.2.4].

In 2010, Ramla Abdellatif proved [2, Théorème 3.26 and Corollaire 4.8] the counterpart of this statement for $\mathrm{SL}_2(\mathbb{Q}_p)$. As can be noticed below, there is already a huge difference in the behaviour of the cokernel representations, which are not irreducible at all, even when $F = \mathbb{Q}_p$.

Theorem 4.13 *Let σ be an irreducible smooth representation of $\mathrm{SL}_2(\mathbb{Z}_p)$ over C.*

1) *The cokernel representation $\pi(\sigma, 0)$ fits into a non-split short exact sequence of representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ of the following form:*

$$0 \longrightarrow \pi_\sigma^\alpha \longrightarrow \pi(\sigma, 0) \longrightarrow \pi_\sigma \longrightarrow 0 \ ,$$

where $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ is an element of $\mathrm{GL}_2(\mathbb{Q}_p)$ that conjugates the two classes of maximal open compact subgroups in $\mathrm{SL}_2(\mathbb{Q}_p)$.

2) *The representation π_σ is irreducible admissible, and π_σ^α is isomorphic to π_{σ^α}.*

Proving this theorem allowed her to provide a full classification of irreducible smooth (admissible, see [2, Section 3.7.5] on how to drop this assumption) representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ and to formulate the first p-modular semisimple local Langlands correspondence involving actual L-packets of representations [2, Définition 4.13]. For the sake of completeness, let us mention that Chuangxun Cheng proved independently some of these results [15].

5 What comes next? Some open questions

Let us finish these notes by some interesting facts and open questions that remain very mysterious, even for GL_2 and SL_2. This list is certainly not exhaustive, but these simple questions may already fully convince the reader of how wildly surprising and badly understood remain p-modular representations of p-adic groups.

5.1 Regarding GL_2 and SL_2

The classification results we gave above when $F = \mathbb{Q}_p$ are very specific to this case. Indeed, when F is a non-trivial extension of $\mathbb{Q}_p$, many results have been proven to deny what was expected to hold, based on the observations done in the $F = \mathbb{Q}_p$ case. For instance:

- we already mentioned Daniel Lê's work about the existence of non-admissible irreducible smooth representations of $\mathrm{GL}_2(F)$ for F being an unramified cubic extension of $\mathbb{Q}_p$, what is in contrast with what we saw for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see Theorem 4.12).

- Benjamin Schraen [28] proved that, when F is a quadratic extension of $\mathbb{Q}_p$, then the cokernel representations $\pi(\sigma, 0)$ of $\mathrm{GL}_2(F)$ are not of finite presentation. This result has been later extended to the case of arbitrary finite extensions of $\mathbb{Q}_p$ by his student Zhixiang Wu [36]. Note that the compatibility between representations of $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ given by the natural restriction map (proven by Ramla Abdellatif in [2, Corollaires 3.19 and 3.26]) ensures that the same statements hold for cokernel representations of $\mathrm{SL}_2(F)$.

Let us also emphasize once again the difference with the complex setting by a striking example. It directly follows from Maschke's theorem that smooth representations of K over $\mathbb{C}$ are semisimple, i.e. are isomorphic to a direct sum of irreducible smooth representations of K. This statement is totally false for p-modular representations of G, even when $F = \mathbb{Q}_p$, as proven by Stefano Morra for $\mathrm{GL}_2(\mathbb{Q}_p)$ and by him and Ramla Abdellatif for $\mathrm{SL}_2(\mathbb{Q}_p)$ (see respectively [26] and [5]).

Finally, let us note that most of the current work done in this setting regards the case where F is of characteristic 0 (i.e. a finite extension of $\mathbb{Q}_p$), but the original setting is about F being a non-Archimedean local field of residue characteristic p. Actually, beyond the results of [2, 7, 8], nothing is known about the equal characteristic case, even when $F = \mathbb{F}_p((t))$. Understanding how supercuspidal representations of $\mathrm{GL}_2\left(\mathbb{F}_p(t)\right)$ and $\mathrm{SL}_2\left(\mathbb{F}_p(t)\right)$ behave is for instance a question that remains wide open for now.

5.2 Regarding other groups

Going beyond the cases of GL_2 and SL_2 remains a very hazardous task that produced very little results so far. Some results have been proven for groups of F-semisimple rank 1: the most general statements are due to Ramla Abdellatif [1, Chapter 5] and to her and Julien Hauseux [3], while some special cases of unramified unitary groups have been studied by Ramla Abdellatif [1, Chapter 4] and by Karol Kozioł alone [22] or with Peng Xu [23]. For the sake of completeness, let us also mention that Xu reproved in his PhD thesis some of the results of [1, Chapter 4]. In any case, the main question remains unanswered so far: we have no clue about how to classify supercuspidal (or, equivalently, supersingular) representations of such groups, even when $F = \mathbb{Q}_p$. The most promising step in this direction comes from [3], which suggests a new idea to construct supersingular representations, namely starting from irreducible smooth representations of minimal parabolic subgroups (as studied for instance in Matthieu Vienney's PhD thesis on representations of the Borel subgroup of $GL_2(F)$, see [32]).

To conclude in direction of most ambitious people willing to know what happens for arbitrary reductive groups, or more humbly for $GL_n(F)$ when n is bigger than 2, let us say that the only result that is actually known is that for $\mathcal{G}$ being a connected reductive group over F, irreducible admissible representations of G are classified up to supersingular representations of Levi factors of parabolic subgroups of G (see [6]). This basically means that we are reduced to study supersingular representations, but as can be guessed from above, getting a complete classification seems for now out of reach.

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Ramla Abdellatif
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
(UMR CNRS 7235)
Université de Picardie Jules Verne
33, rue Saint-Leu
80 039 Amiens Cedex 1 (France)
e-mail: ramla.abdellatif@math.cnrs.fr