On Groups whose Subnormal Abelian Subgroups are Normal

L.A. Kurdachenko — J. Otal — I.Ya. Subbotin

Abstract

In the current paper we study the groups, whose subnormal abelian subgroups are normal. We obtained a quite detailed description of such hyperabelian groups with a periodic Baer radical. The description of hyperabelian Lie algebras, whose abelian subideals are ideals, is also obtained.

1 Introduction

Not very many relationships between subgroups in groups are transitive. Of the most essential, here you can specify the relations "to be a subgroup", "to be a subnormal subgroup", "to be an ascendant subgroup". At the same time, such important relationships as "to be a normal subgroup", to be a subnormal subgroup with fixed defect", "to be a permutable subgroup", "to be a pronormal subgroup", "to be an abnormal subgroup" no longer possess the transitivity property. Moreover, the transitivity property turned out to be a rather strict restriction, which in many cases made it possible to obtain an observable description of the corresponding groups (see [2],[8],[7],[9], [14],[21],[22],[27],[28],[29],[34]).

If G is a group, in which the relation "to be a normal subgroup" is transitive, then every subnormal subgroup of G is normal. This simple circumstance leads us to problems of the following type: to study groups in which not all, but only subnormal subgroups with some natural fixed property are normal. For example, the paper [15] has considered the groups, whose infinite subnormal subgroups are normal; the paper [11] has considered the groups, whose subnormal subgroups of infinite special rank are normal.

I.N. Abramovskii in [1] initiated the study of groups in which the transitivity condition is imposed on only abelian normal subgroups. Actually, he studied the groups with the transitivity of normality for Dedekind normal subgroups, but since he considered such locally finite groups with abelian Sylow p-subgroups, those Dedekind subgroups are abelian. It turns out that the class of such groups in which this transitivity inherited by subgroups coincides with the class of locally finite groups with the transitivity for all normal subgroups.

It is clear that the study of groups in which all abelian subnormal subgroups are normal makes sense in those classes of groups in which there exist nontrivial subnormal abelian subgroups. One of these classes is the class of hyperabelian groups. Recall that a group G is called *hyperabelian* if G has an ascending series of normal subgroups, whose factors are abelian.

It is clear that in groups in which the relation "to be a normal subgroup" is transitive (such groups are called T-*groups*) every subnormal abelian subgroup is normal. The soluble T-groups were studied in detail in [2],[14], [16],[27],[28],[29]. In particular, it turned out that in finite solvable groups the property "to be a T-group" is inherited by subgroups. This, however, is no longer the case for infinite groups. A group G is called $\overline{\overline{T}}$-*group*, if every subgroup of G is a T-group. The description of soluble $\overline{\overline{T}}$-groups is in the paper [27]. The following type of groups is connected with $\overline{\overline{T}}$-groups.

A subgroup H of a group G is called *transitively normal* in G, if H is normal in every subgroup S such that H is subnormal in S (see [23]). It is not hard to see that a group G is a $\overline{\overline{T}}$-group if and only if every subgroup of G is transitively normal in G. It is natural to ask if in the previous equivalence one can replace "every" with "every abelian". From the results of the paper [18] it is possible to obtain that a soluble group G is a $\overline{\overline{T}}$-group if and only if every abelian subgroup of G is transitively normal in G. In contrast, groups whose subnormal abelian subgroups are normal differs significantly from T-groups. These differences can already be seen in the following example.

Let $\{p_n \mid n \in \mathbb{N}\}$ be the set of all odd primes and let $C_n = \langle c_n \rangle$ be the cyclic group of order $p_n^n, n \in \mathbb{N}$. For any $n \in \mathbb{N}$, C_n has an automorphism x_n of p_n^{n-1}, so we can consider the natural semidirect product $C_n \rtimes \langle x_n \rangle$. In the Cartesian product $\mathrm{Cr}_{n \in \mathbb{N}} \left(C_n \rtimes \langle x_n \rangle \right)$ we choose $G = T \rtimes \langle x \rangle$, where $T = \mathrm{Dr}_{n \in \mathbb{N}} C_n$ and $x = (x_n)_{n \in \mathbb{N}}$. It is not hard to prove that every abelian subnormal subgroup of G is normal, but G is not a T-group.

We will need the following concepts. Recall that group G is called a *Dedekind group* if every subgroup of G is normal. The study of such groups was started by R. Dedekind in the paper [12], and therefore these groups were called Dedekind. Later in the paper [4], R. Baer obtained a complete description of such groups, which is as follows:

If G is non-abelian Dedekind, then $G = Q \times E \times B$ where Q is a quaternion group, E is an elementary abelian 2-subgroup and B is a periodic abelian $2'$-subgroup.

Let G be a group. Denote by $B(G)$ the subgroup generated by subnormal cyclic subgroups. The subgroup $B(G)$ is called the *Baer radical* of a group G. The subgroup $B(G)$ is locally nilpotent and every its finitely generated subgroup is subnormal in G [6]. The group G is called a *Baer group* if $G = B(G)$.

Let G be a group and B, C be the normal subgroups of G such that $B \leqslant C$. The factor C/B is called G-*central*, if $C_G(C/B) = G$. The factor C/B is called G-*eccentric*, if $C_G(C/B) \neq G$.

Let G be a group and A be a normal subgroup of G. We construct the *upper* G-*central series* of A as

$$\langle 0 \rangle = A_0 \leqslant A_1 \leqslant \ldots A_\alpha \leqslant A_{\alpha+1} \leqslant \ldots A_\gamma,$$

where

$$A_1 = \zeta_G(A) = \left\{ a \in A \mid [a, g] = 1 \text{ for all elements } g \in G \right\},$$

$A_{\alpha+1}/A_\alpha = \zeta_G(A/A_\alpha)$, for all ordinals $\alpha < \gamma, A_\lambda = \cup_{\beta < \lambda} A_\beta$ for all limit ordinals $\lambda < \gamma$, and $\zeta_G(A/A_\gamma) = \langle 0 \rangle$. We note that every subgroup of this series is G-invariant. The last term A_γ of this series is called the *upper* G-*hypercenter of* A and will be denoted by $\zeta_{G^\infty}(A)$. If $A = A_\gamma$, then A is called G-*hypercentral*; if γ is finite, then A called G-*nilpotent*.

A normal subgroup A of G is said to be G-*hypereccentric*, if it has

an ascending series

$$\langle 0 \rangle \leqslant C_0 \leqslant C_1 \leqslant \ldots C_\alpha \leqslant C_{\alpha+1} \leqslant \ldots C_\gamma = C$$

of G-invariant subgroups of A such that each factor $C_{\alpha+1}/C_\alpha$ is a G-eccentric and G-chief, for every $\alpha < \gamma$.

We say that the normal abelian subgroup A of a group G has the $Z(G)$-*decomposition*,s if

$$A = \zeta_G^\infty(A) \oplus \eta_G^\infty(A)$$

where $\eta_G^\infty(A)$ is the maximal G-hypereccentric G-invariant subgroup of A. This concept was introduced by D.I. Zaitsev [35]. It is not hard to see that in this case, $\eta_G^\infty(A)$ includes every G-hypereccentric G-invariant subgroup of A, in particular, it is unique.

The main results of this paper are the following one.

Theorem A *Let* G *be a hyperabelian group, whose subnormal abelian subgroups are normal,* B *be the Baer radical of* G, L *be the locally nilpotent radical of* G, *and* R *be the locally nilpotent residual of* G. *Suppose that the Baer radical of* G *is periodic. Then* G *satisfies the following conditions:*

  (i) G/B *is abelian and residually finite, every subgroup of* B *is G-invariant, in particular* B *is a Dedekind group;*

 (ii) $R \leqslant B, R$ *is G-hypereccentric, the factor-group* G/R *is hypercentral,* $2 \notin \Pi(R)$ *and* $\Pi(R) \cap \Pi(L/R) = \emptyset$,

(iii) $C_G(R) \leqslant L$;

(iv) *if* p *is an odd prime, then the Sylow* p*-subgroups of* G *are nilpotent;*

 (v) *if* p *is an odd prime, then the Sylow* p*-subgroup of* B *is the Sylow* p*-subgroup of locally nilpotent radical.* $\Pi(L/B)$ *is a subset of the set* $\{2\}$, *and, in particular, the Sylow* p*-subgroups of* L *are abelian for odd primes* p.

(vi) $\mathrm{Tor}(L) = R \times \mathrm{Tor}(Z)$ *where* Z *is the upper hypercenter of* G.

(vii) *if the Sylow 2-subgroup of* G *is nilpotent, then* $L = D$, *moreover if the orders of elements of Sylow 2-subgroup* B_2 *of* B *are not bounded, then the Sylow 2-subgroups of* G *are abelian, if the orders of elements of Sylow 2-subgroup* B_2 *of* B *are not bounded, then the Sylow 2-subgroups of* G *nilpotent.*

(viii) *if* $p \notin \Pi(R)$ *and the orders of elements of* B_p *are bounded, then the hypercenter of* G *of finite number includes* B_p; *if the orders of elements of* B_p *are not bounded, then the hypercenter of* G *of the number* ω *includes* B_p;

(ix) *the hypercentral length of* G/R *is at most* $\omega + 1$;

(x) *if the Sylow 2-subgroup* D *of* G *is not nilpotent, then* D *satisfies the following conditions:*

(xa) $B \cap D = B_2$ *is abelian and orders of elements of* B_2 *are not bounded;*

(xb) $D = C\langle a \rangle$ *where* $C = C_D(B_2), a^2 \in B$ *and* $b^a = b^{-1}$ *for every element* $b \in B_2$;

(xc) $B_2 \cap C_D(a)$ *is an elementary abelian 2-subgroup;*

(xd) $[C, C] \leqslant B_2 \cap C_D(a)$, *in particular,* $[C, C]$ *is an elementary abelian 2-subgroup;*

(xe) D *includes a normal subgroup* $S \geqslant B_2$ *such that the factor-group* D/S *is elementary abelian and* $S = C_S(a)B_2$;

(xi) *if the Sylow 2-subgroup* L_2 *of* L *is not nilpotent, then* L_2 *satisfies the following conditions:*

(xia) *the Sylow 2-subgroup* B_2 *of* B *is abelian and orders of elements of* B_2 *are not bounded;*

(xib) $L_2 = B_2\langle a \rangle, a^2 \in B_2$ *and* $x^a = x^{-1}$ *for every element* $x \in B_2$;

(xic) $C_{B_2}(a)$ *is an elementary abelian 2-subgroup;*

(xid) *either* $a^2 = 1$ *or* $a^4 = 1$.

Conversely, in every such group each subnormal abelian subgroup is normal.

Corollary A1 *Let* G *be a hyperabelian group, whose subnormal abelian subgroups are normal and let* B *be the Baer radical of* G. *Suppose that the Baer radical of* G *is periodic. If* G *is locally nilpotent, then* G *satisfies the following conditions:*

(i) *if* p *is an odd prime, then the Sylow* p-*subgroup of* G *is abelian;*

(ii) *if the Sylow 2-subgroup* D *of* G *is not nilpotent, then* D *satisfies the following conditions:*

(iia) *the Sylow 2-subgroup B_2 of B is abelian and orders of elements of B_2 are not bounded;*

(iib) $D = B_2\langle a\rangle$, $a^2 \in B_2$, *and* $x^a = x^{-1}$ *for every element* $x \in B_2$;

(iic) $C_{B_2}(a)$ *is an elementary abelian 2-subgroup;*

(iid) *either* $a^2 = 1$ *or* $a^4 = 1$.

Corollary A2 *Let G be a periodic hyperabelian group, whose subnormal abelian subgroups are normal, let B be the Baer radical of G, L be the locally nilpotent radical of G. Suppose that the set $\Pi(G) = \{p_1, p_2, \ldots, p_k\}$ is finite, $k > 1$, and let $p_1 > p_2 > \ldots > p_k$. Then G satisfies the following conditions:*

(i) B *is a Dedekind group, then every subgroup of B is G-invariant, $C_G(B) \leqslant B$;*

(ii) *the factor-group G/B is abelian and finite;*

(iii) B *includes the Sylow p_1-subgroup of G, the Sylow p_j-subgroup of G/B has a special rank at most $j - 1$, $2 \leqslant j \leqslant k - 1$;*

(iv) *if $p_k \neq 2$, then the Sylow p_k-subgroup of G/B has a special rank at most $k - 1$, if $p_k = 2$, then the Sylow 2-subgroup of G/B has a special rank at most $k + 1$;*

(v) *if R is a locally nilpotent residual of G, then $G = R \rtimes S$ where S is a hypercentral subgroup, moreover, Sylow $2'$-subgroup of S is nilpotent;*

(vi) *if S_1 is a subgroup of G such that $G = R \rtimes S_1$, then the subgroups S and S_1 are conjugate.*

Corollary A3 *Let G be a periodic hyperabelian group, whose subnormal abelian subgroups are normal. Suppose that $\Pi(G) = \{p, q\}$ where $p < q$. If G is not locally nilpotent, then G satisfies the following conditions:*

(i) *the Sylow q-subgroup Q of G is abelian, $G = Q \rtimes P$ where P is a Sylow p-subgroup of G;*

(ii) *the Sylow p-subgroups of G are conjugate.*

(iii) p *divides $q - 1$;*

(iv) *a factor-group* $P/C_P(Q)$ *is cyclic and* $C_P(Q) \times Q$ *is the Baer radical of* G;

 (v) *if* $p \neq 2$ *and* P *is non-abelian, then the orders of elements of* P *are bounded and* P *is nilpotent*;

(vi) *if* $p = 2$ *and* P *is not Dedekind, then* $P/C_P(Q)$ *has order 2 and* P *satisfies the following conditions:*

    (via) $C_P(Q) = B$ *is abelian and orders of elements of* B *are not bounded;*

    (vib) $P = B\langle a \rangle$, $a^2 \in B$ *and* $x^a = x^{-1}$ *for every element* $x \in B$;

    (vic) $C_B(a)$ *is an elementary abelian 2-subgroup;*

    (vid) *either* $a^2 = 1$ *or* $a^4 = 1$.

This topic is not specific only to group theory. Lie algebras, whose subideals are ideals, have also been studied (see [32] and [33]). In the current work, the following description of hyperabelian Lie algebras, whose abelian subideals are ideals, is obtained.

Theorem B *Let* L *be a hyperabelian Lie algebra. Suppose that* L *is non-abelian. If every abelian subideal of* L *is an ideal, then* $L = A \oplus Fd$ *where* A *is an abelian ideal of* L *and* $[d, a] = a$ *for all elements* $a \in A$. *In particular, every subideal of* L *is an ideal.*

Corollary B1 *Let* L *be a hyperabelian Lie algebra. Then every abelian subideal of* L *is an ideal if and only if every subideal of* L *is an ideal.*

An essential generalization of Lie algebras are Leibniz algebras. We note that the Leibniz algebras, whose subideal are ideals, have been studied in papers [20],[24]. However, the situation with abelian subalgebras of Leibniz algebras is much more complicated. In contrast to Lie algebras, their classes such as solvable, nilpotent, and those which are close to them may be completely unsaturated with abelian subalgebras. In the current paper, we give an example of a nilpotent Leibniz algebra of nilpotency class 2, which has a unique abelian subalgebra, and the dimension of this subalgebra is 1. Its construction shows that it is possible to construct examples of this type with very diverse properties. This shows that abelian subalgebras of Leibniz algebras have a very weak effect on the structure of Leibniz algebras, which cannot be said about the influence of nilpotent subalgebras.

2 Groups whose subnormal abelian subgroups are normal

We begin with some elementary properties.

Lemma 2.1   *Let* G *be a group, whose subnormal abelian subgroups are normal, and* L *be the locally nilpotent radical of* G. *Then* L *includes the* G-*invariant subgroup* K *such that every subgroup of* K *is* G-*invariant and every subnormal subgroup* R *of* G *such that* $K \leqslant R$ *and* $K \neq R$ *is not nilpotent.*

PROOF — Let S be a subnormal nilpotent subgroup of L. If x is an arbitrary element of S, then the subgroup $\langle x \rangle$ is subnormal in L. Then $\langle x \rangle$ is subnormal in G, so that $\langle x \rangle$ is normal in G. It follows that every subgroup of S (and S also) is normal in G. In particular, S is Dedekind. It follows that S is nilpotent of nilpotency class at most 2. Since it is true for each nilpotent subnormal subgroup of L, the subgroup K, generated by all nilpotent subnormal subgroup of L, is nilpotent of nilpotency class at most 2. By what we proved above, every subgroup of K is G-invariant.   □

Corollary 2.2   *Let* G *be a group, whose subnormal abelian subgroups are normal. Then the Baer radical* B(G) *of* G *includes every nilpotent subnormal subgroup and every subgroup of* B(G) *is* G-*invariant (in particular,* B(G) *is a Dedekind group).*

Corollary 2.3   *Let* G *be a group, whose subnormal abelian subgroups are normal. Then the factor-group* $G/C_G\big(B(G)\big)$ *is abelian.*

PROOF — By Corollary 2.2, every subgroup of B(G) is G-invariant. By Corollary 2.2, every subgroup of B is G-invariant. Then $G/C_G(B)$ is abelian (see, for example, [30], Theorem 1.5.1).   □

Corollary 2.4   *Let* G *be a group, whose subnormal abelian subgroups are normal. If* G *is hyperabelian, then every subgroup of* B(G) *is* G-*invariant (in particular,* B(G) *is a Dedekind group) and* B(G) *includes* $C_G\big(B(G)\big)$.

PROOF — By Corollary 2.2, every subgroup of B(G) is G-invariant. Put B = B(G) and $C = C_G\big(B(G)\big)$, and suppose that B does not include C. Then CB/B is not trivial. Since G is hyperabelian, CB/B includes non-trivial abelian G-invariant subgroup A/B. The choice of A shows that A is nilpotent. But Corollary 2.2 implies that, in this case, B must include A, and we obtain a contradiction. This contradiction proves the inclusion $C_G\big(B(G)\big) \leqslant B(G)$.   □

Corollary 2.5 *Let* G *be a group, whose subnormal abelian subgroups are normal, and let* B *the Baer radical of* G. *If* G *is hyperabelian, then the factor-group* G/B *is abelian.*

PROOF — By Corollary 2.2, every subgroup of B is G-invariant. Then $G/C_G(B)$ is abelian by Corollary 2.3. By Corollary 2.4, $C_G(B) \leqslant B$, and therefore G/L is abelian.   □

Corollary 2.6 *Let* G *be a group, whose subnormal abelian subgroups are normal. If* G *is hyperabelian, then* G *is hypercyclic.*

PROOF — By Corollary 2.2, every subgroup of $B(G)$ is G-invariant, so that $B(G)$ has an ascending series of G-invariant subgroups

$$\langle 1 \rangle = C_0 \leqslant C_1 \leqslant \ldots C_\alpha \leqslant C_{\alpha+1} \leqslant \ldots C_\beta = B(G),$$

whose factors are cyclic. Since the factor-group $G/B(G)$ is abelian by Corollary 2.5, this series can be extended to an ascending series of normal subgroups

$$\langle 1 \rangle = C_0 \leqslant C_1 \leqslant \ldots C_\alpha \leqslant C_{\alpha+1} \leqslant \ldots C_\beta = B(G) \leqslant C_{\beta+1}$$
$$\leqslant \ldots C_\lambda \leqslant C_{\lambda+1} \leqslant \ldots C_\gamma = G,$$

whose factors are cyclic.   □

Corollary 2.7 *Let* G *be a group, whose subnormal abelian subgroups are normal. If* G *is hyperabelian, then* G *is locally supersoluble.*

PROOF — We must only note that every hypercyclic group is locally supersoluble (see [5], Theorem 1).   □

Proposition 2.8 *Let* G *be a group, whose subnormal abelian subgroups are normal and let* B *be the Baer radical of* G. *Suppose that* B *is not periodic. If* G *is hyperabelian, then* G *satisfies the following conditions:*

(i) B *is abelian;*

(ii) $G = B\langle a \rangle$ *and* $x^a = x^{-1}$ *for every element* $x \in B$;

(iii) $C_B(a)$ *is an elementary abelian 2-subgroup;*

(iv) *either* $a^2 = 1$ *or* $a^4 = 1$.

Conversely, in every such group each subnormal abelian subgroup is normal.

Proof — By Corollary 2.2, B is a Dedekind group. Being non-periodic, B is abelian. Using Corollary 2.2 we obtain that every subgroup of B is G-invariant. Then $G/C_G(B)$ has order 2 (see, for example, [30], Theorem 1.5.7). Corollary 2.4 shows also that $C_G(B) = B$, so that $G = B\langle a \rangle$ for some element $a \in G$ such that $a^2 \in B$. Moreover, $x^a = x^{-1}$ for every element $x \in L$ (see, for example, [30], Theorem 1.5.7). If $y \in C_B(a)$, then we obtain that $y = y^a = y^{-1}$. It follows that $y^2 = 1$. Suppose that $a^2 \neq 1$. Since $a^2 \in C_L(a)$, we have $(a^2)^2 = 1$.

Conversely, suppose that a group G satisfies all above conditions. Let A be an arbitrary subnormal abelian subgroup of G. If B includes A, then by (ii) every cyclic subgroup of A is G-invariant. It follows that A is a normal subgroup of G. Suppose now that B does not include A. Since B is a maximal subgroup of G, we have $G = BA$. But in this case, G is nilpotent (see [17], Lemma 4). However, the fact that B is not periodic together with condition (ii) show that it is impossible. This contradiction shows that B includes A.   □

Lemma 2.9 *Let* G *be a group, whose subnormal abelian subgroups are normal, and let* B *be the Baer radical of* G. *Suppose that* B *is periodic. Then every abelian subgroup of* $\zeta(B)$ *has the* $Z(G)$*-decomposition.*

Proof — Let A be an arbitrary subgroup of $\zeta(B)$. By Corollary 2.2, A and each its subgroup are G-invariant. By Corollary 2.3, the factor-group $G/C_G(B)$ is abelian. Let S be an arbitrary finite subgroup of A. The inclusion $C_G(B) \leqslant C_G(S)$ implies that $G/C_G(S)$ is abelian. Then a subgroup S has the $Z(G)$-decomposition

$$S = \zeta_G^\infty(S) \oplus \eta_G^\infty(S)$$

(see [13], Corollary 1.6.5). Let K be a finite subgroup of $\zeta(B)$ including S. Again K is G-invariant. As proved above, K has the $Z(G)$-decomposition $K = \zeta_G^\infty(K) \oplus \eta_G^\infty(K)$. Clearly, $\zeta_G^\infty(S) \leqslant \zeta_G^\infty(S)$. As noted above, $\eta_G^\infty(K)$ includes every G-hypereccentric G-invariant subgroup of K, in particular, $\eta_G^\infty(S) \leqslant \eta_G^\infty(K)$. Taking into account these arguments, it is not difficult to show that $A = \zeta_G^\infty(A) \oplus \eta_G^\infty(A)$.   □

Corollary 2.10 *Let* G *be a group, whose subnormal abelian subgroups are normal, and let* B *be the Baer radical of* G. *Suppose that* B *is periodic. If* $p \in \Pi(B), p \neq 2$, *and* P *is the Sylow* p*-subgroup of* B, *then either the upper hypercenter of* G *includes* P, *or* P *is* G*-hypereccentric.*

PROOF — Since $p \neq 2$, the structure of a Dedekind group shows that $P \leqslant \zeta(B)$. Lemma 2.9 implies that P has the $Z(G)$-decomposition

$$P = \zeta_G^\infty(P) \oplus \eta_G^\infty(P).$$

Now suppose the contrary, let both subgroups $\zeta_G^\infty(P), \eta_G^\infty(P)$ are not trivial. Then we can choose $1 \neq c \in \zeta_G(P)$, and $1 \neq d \in \eta_G^\infty(P)$ such that $|c| = |d| = p$. Since subgroup $\langle d \rangle$ is G-invariant, $G/C_G(\langle d \rangle)$ is a non-trivial cyclic group and its order divides $p - 1$. Let g be an element such that

$$G/C_G(\langle d \rangle) = \langle g C_G(\langle d \rangle) \rangle.$$

Then $d^g = d^k$ where $1 < k < p$. We have $c^g = c$. Clearly, $cd \notin \zeta_G(P)$. The subgroup $\langle cd \rangle$ is G-invariant, so that

$$cd \neq (cd)^g = (cd)^m = c^m d^m$$

where $1 < m < p$. On the other hand, $c^m d^m = (cd)^g = c^g d^g = cd^k$. It follows that $m \equiv 1 \pmod p$ and $m \equiv k \pmod p$, and we obtain a contradiction, which proves the result. $\square$

Lemma 2.11 *Let G be a group, whose subnormal abelian subgroups are normal and let B be the Baer radical of G. Suppose that B is periodic. Then the upper hypercenter of G includes the Sylow 2-subgroup of B.*

PROOF — Denote by D the Sylow 2-subgroup of B. By Corollary 2.2, D is a Dedekind group. Suppose first that D is abelian. Let

$$D_k = \Omega_k(D) = \left\{ d \mid d \in D \text{ and } |d| \text{ divides } p^k \right\}.$$

Consider a factor D_{k+1}/D_k. Since every subgroup of D is G-invariant, by Corollary 2.2, every subgroup of D_{k+1}/D_k is also G-invariant. We note that the center of each group includes normal cyclic subgroup of order 2. Since D_{k+1}/D_k is elementary abelian 2-group, we obtain that the factor D_{k+1}/D_k is G-central. Since it is true for every positive integer k, the equality $D = \cup_{k \in \mathbb{N}} D_k$ shows that the upper hypercenter of G includes D.

Suppose that D is not abelian. Then $D = Q \times A$ where Q is a quaternion group and A is an elementary abelian 2-subgroup. Then

$$\zeta(D) = \zeta(Q) \times A,$$

in particular, $\zeta(D)$ and $D/\zeta(D)$ are elementary abelian 2-groups. As it was done above, we can prove that $\zeta(D)$ and $D/\zeta(D)$ are G-central, and we obtain that the upper hypercenter of G includes D.   □

Proposition 2.12   *Let* G *be a group whose subnormal abelian subgroups are normal,* B *the Baer radical of* G, *and* R *the locally nilpotent residual of* G. *Suppose that* B *is periodic. If* G *is hyperabelian, then* B *includes* R, $2 \notin \Pi(R)$, $\Pi(R) \cap \Pi(B/R) = \emptyset$. *Furthermore,* B *is* G-*hypereccentric and the factor-group* G/B *is hypercentral.*

PROOF — We have $B = D \times S$ where D is a Sylow 2-subgroup of D and S is a Sylow $2'$-subgroup of B. Denote by π the set of primes $p \in \Pi(B)$ such that the upper hypercenter of G includes the Sylow p-subgroup of B. Lemma 2.11 shows that $2 \in \pi$. By Corollary 2.10, we have $B = A \times S$ where S is a Sylow π-subgroup of B and A is a Sylow π'-subgroup of B. By Corollary 2.5, the factor-group G/B is abelian. It follows that the factor-group G/A is hypercentral. It follows that $R \leqslant A$.

Suppose that $R \neq A$. Then there exists a prime $q \in \Pi(B) \backslash \pi$ such that for the Sylow q-subgroup Q of B we have $R \cap Q \neq Q$. Since every subgroup of Q is G-invariant, every subgroup of QR/R is G-invariant. The fact that G/R is residually locally nilpotent implies that every subgroup of G/R, having prime order q, is G-central. It follows that QR/R has the non-trivial G-central factors. The G-isomorphism

$$Q/(R \cap Q) \simeq_G QR/R$$

shows that $Q/(R \cap Q)$ has non-trivial G-central factors. However the choice of Q and Corollary 2.10 imply that every G-chief factor of $Q/(R \cap Q)$ must be G-eccentric. This contradiction proves the equality $A = R$.   □

Lemma 2.13   *Let* G *be a hyperabelian group, whose subnormal abelian subgroups are normal, and let* B *be the Baer radical of* G. *Suppose that* B *is periodic. The following assertions hold:*

(i)   *if* p *is an odd prime, then the Sylow* p-*subgroups of* G *are nilpotent;*

(ii)   *if* p *is an odd prime, then the Sylow* p-*subgroup of* B *is the Sylow* p-*subgroup of the locally nilpotent radical* L. *In particular, the Sylow* p-*subgroup of the locally nilpotent radical* L *of* G *is abelian.*

PROOF — Let p be an odd prime, and P be the Sylow p-subgroup of G. Corollary 2.2 shows that every subgroup of $P \cap B = B_p$ is G-invariant, in particular, B_p is a Dedekind group. The fact that p is

odd implies that B_p is abelian. If the orders of elements of B_p are bounded, then the factor-group $P/C_P(B_p)$ is finite cyclic p-group (see, for example, [30], Theorem 1.5.6). Let x be an element such that $P = \langle x \rangle C_P(B_p)$. Since

$$P/B_p = P/(P \cap B) \simeq PB/B$$

is abelian, both subgroups $C_P(B_p)$ and $\langle x \rangle B_p$ are normal in P. The subgroup $C_P(B_p)$ is nilpotent, and the subgroup $\langle x \rangle B_p$ is also nilpotent (see, for example [10], Corollary 1.77). Being a product of the two normal nilpotent subgroups $\langle x \rangle B_p$ and $C_P(B_p)$, P is nilpotent.

If the orders of elements of B_p are not bounded, then the factor-group $P/C_P(B_p)$ is isomorphic to some subgroup of the multiplicative group of the ring of p-adic integers (see, for example, [30], Theorem 1.5.6). But the last group has no elements of order p (see, for example, [31], Chapter II, §3, Theorem 2). Thus $P = C_P(B_p)$, so P is nilpotent.

Let P_1 be the Sylow p-subgroup of L. As it was proved above, P_1 is nilpotent. Then Corollary 2.2 implies that B includes P_1, in particular, P_1 is abelian. □

Lemma 2.14 *Let* G *be a hyperabelian group, whose subnormal abelian subgroups are normal, let* B *be the Baer radical of* G *and* D *be the Sylow 2-subgroup of* G. *If* D *is not nilpotent, then* D *satisfies the following conditions:*

(i) $B \cap D = B_2$ *is abelian and orders of elements of* B_2 *are not bounded;*

(ii) $D = C \langle a \rangle$ *where* $C = C_D(B2_)$, $a^2 \in B$, *and* $b^a = b^{-1}$ *for every element* $b \in B_2$;

(iii) $B_2 \cap C_D(a)$ *is an elementary abelian 2-subgroup;*

(iv) $[C, C] \leqslant B_2 \cap C_D(a)$; *in particular,* $[C, C]$ *is an elementary abelian 2-subgroup;*

(v) D *includes a normal subgroup* $S \geqslant B_2$ *such that the factor-group* D/S *is elementary abelian and* $S = C_S(a)B_2$.

PROOF — By Corollary 2.2, B_2 is a Dedekind group, moreover, every subgroup of B_2 is G-invariant. Suppose first that B_2 is abelian and the orders of its elements are bounded. Then $D/C_D(B_2)$ is a

direct product of two finite cyclic groups (see, for example, [30], Theorem 1.5.6). Put $C = C_D(B_2)$ and let $D/C = \langle aC \rangle \times \langle bC \rangle$. Since

$$D/B_2 = D/(B \cap D) \simeq DB/B,$$

the fact that G/B is abelian implies that D/B_2 is also abelian. Then the subgroups $C, \langle a \rangle B_2$ and $\langle b \rangle B_2$ are normal in D. A subgroup C is clearly nilpotent and the subgroups $\langle a \rangle B_2, \langle b \rangle B_2$ are nilpotent (see, for example [10], Corollary 1.77). It follows that $D = \left(\langle a \rangle B_2 \right) \left(\langle b \rangle B_2 \right) C$ is also nilpotent.

If B_2 is not abelian, then as we noted above, $B_2 = Q \times E$ where Q is a quaternion group and E is an elementary abelian 2-subgroup. Since, by Corollary 2.2, every cyclic subgroup of E is G-invariant, the center of G includes $E \times \zeta(Q)$. By the same reason, the center of $G/\left(E \times \zeta(Q) \right)$ includes $B_2/\left(E \times \zeta(Q) \right)$. It follows that the second hypercenter of D includes $B_2 = B \cap D$. The isomorphism

$$D/(B \cap D) \simeq DB/B$$

and the fact that G/B is abelian imply that D/B_2 is abelian. It follows that D is nilpotent. The contradictions, which have been obtained in both above cases, show that B_2 must be abelian and the orders of its elements are not bounded.

By Corollary 2.2, every subgroup of B_2 is G-invariant, therefore

$$D/C = \langle aC \rangle$$

is a cyclic group of order 2 and $b^a = b^{-1}$ for each element $b \in B_2$ (see, for example, [30], Theorem 1.5.6, or [31], Chapter II, §3, Theorem 2). If $y \in C_B(a)$, then we obtain that $y = y^a = y^{-1}$. It follows that $y^2 = 1$.

Let c_1, c_2 be arbitrary elements of C. The fact that D/B_2 is abelian implies that $c_1^a = c_1 b_1, c_2^a = c_2 b_2$ for some elements $b_1, b_2 \in B_2$. We have

$$[c_1, c_2]^a = [c_1^a, c_2^a] = [c_1 b_1, c_2 b_2] = b_1^{-1}[c_1, c_2 b_2] b_1 [b_1, c_2 b_2]$$

$$= [c_1, c_2 b_2][b_1, c_2 b_2] = [c_1, b_2] b_2^{-1}[c_1, c_2] b_2 [b_1, b_2] b_2^{-1}[b1, c_2] b_2$$

$$= [c_1, c_2].$$

Since D/B_2 is abelian, $[C, C] \leqslant B_2$, and using (iii) we obtain that $[C, C]$

is an elementary abelian subgroup.

If $b \in B_2$, then
$$[a, b] = a^{-1} b^{-1} a\, b = b^2,$$
so $[a, B_2] = B_2^2$. Put $E = B_2^2$. Put $Z/E = C_{C/E}(aE)$. Since D/E is nilpotent of nilpotency class at most 2, the subgroup Z/E is normal in D/E. The fact that B_2/E is an elementary abelian 2-subgroup implies that Z includes B_2. The choice of Z/E shows that $[a, Z] \leqslant E$. Since Z includes B_2 and $[a, B_2] = E$, we obtain the equality $[a, Z] = [a, B_2]$. If z_1, z_2 are arbitrary elements of Z, then

$$[a, z_1 z_2] = [a, z_2] z_2^{-1} [a, z_1] z_2.$$

The inclusions $Z \leqslant C$ and $[a, B_2] \leqslant B_2$ imply that

$$z_2^{-1} [a, z_1] z_2 = [a, z_1],$$

so

$$[a, z_1 z_2] = [a, z_1][a, z_2].$$

Thus we obtain the equality $[a, Z] = \big\{ [a, z] \,|\, z \in Z \big\}$.

Let now z be an arbitrary element of Z. The equality $[a, Z] = [a, B_2]$ implies that there exists an element $b \in B_2$ such that $[a, z] = [a, b]$. It follows that $z^{-1} a z = b^{-1} a b$ or $z b^{-1} \in C_Z(a)$. Thus we obtain the equality $Z = C_Z(a) B_2$.

Consider the mapping

$$\xi_a : C/E \longrightarrow B_2/E$$

defined by the rule $\xi_a(xE) = [aE, xE], x \in C$. It is not hard to see that ξ_a is an endomorphism of C/E such that

$$\mathrm{Ker}(\xi_a) = Z/E \quad \text{and} \quad \mathrm{Im}(\xi_a) = [aE, C/E].$$

The isomorphism

$$(C/E)/(Z/E) = (C/E)/\mathrm{Ker}(\xi_a) \simeq \mathrm{Im}(\xi_a) = [aE, C/E]$$

implies that

$$C/Z \simeq (C/E)/(Z/E)$$

is an elementary abelian 2-group. The equality $D = \langle a \rangle C$ implies that

$$D/Z = \langle aZ \rangle (C/Z).$$

Put $S = \langle a, Z \rangle$. Then $S = \langle a \rangle C_Z(a) B_2 = C_S(a) B_2$. Clearly, the factor-group D/S is elementary abelian. □

Corollary 2.15 *Let* G *be a hyperabelian group, whose subnormal abelian subgroups are normal, let* B *be the Baer radical of* G *and* L_2 *be the Sylow 2-subgroup of the locally nilpotent radical* L *of* G. *If* L_2 *is not nilpotent, then* L_2 *satisfies the following conditions:*

  (i) *the Sylow 2-subgroup* B_2 *of* B *is abelian and orders of elements of* B_2 *are not bounded;*

 (ii) $L_2 = B_2 \langle a \rangle, a_2 \in B_2$ *and* $x^a = x^{-1}$ *for every element* $x \in B_2$;

(iii) $C_{B_2}(a)$ *is an elementary abelian 2-subgroup;*

(iv) *either* $a^2 = 1$ *or* $a^4 = 1$.

PROOF — If we suppose that the orders of elements of B_2 are bounded, then Lemma 2.13 implies that the Sylow 2-subgroup D of the group G is nilpotent. In this case, L_2 is nilpotent too, and we obtain a contradiction. This contradiction shows that the orders of elements of B_2 must be not bounded.

   Let $C = C_D(B_2)$. The subgroup $C \cap L_2$ is nilpotent, and using again Corollary 2.2 we obtain that B includes $C \cap L_2$. Since L_2 is not nilpotent, C can not include L_2. This means that $C \cap L_2 = B_2$. Thus we can choose an element a such that $a \in L_2$. Lemma 2.13 proves that $L_2 = B_2 \langle a \rangle, a^2 \in C \cap L_2 = B_2$ and $x^a = x^{-1}$ for every element $x \in B_2$.

   Lemma 2.13 shows also that $C_{B_2}(a)$ is an elementary abelian 2-subgroup. We have that $a^2 \in B_2$, and since $a^2 \in C_{B_2}(a)$ either $a^2 = 1$ or $a^4 = 1$. □

Lemma 2.16 *Let* Q *be a quaternion group of order 8. Then* $\mathrm{Pot}(Q)$ *of all power automorphisms of* Q *is an elementary abelian 2-group of order 4.*

   This result is well know.

PROOF OF THEOREM A — To prove condition (i), we note that the fact that G/B is abelian follows from Corollary 2.5, and the fact that

every subgroup of B is G-invariant follows from Corollary 2.2. We have

$$B = \operatorname*{Dr}_{p \in \Pi(B)} B_p$$

where is the Sylow p-subgroup of B for every prime p. Since B is a Dedekind group, B_p is abelian for odd prime p. A subgroup B_2 is abelian or $B_2 = Q \times E$ where Q is a quaternion group and E is an elementary abelian 2-subgroup. If orders of elements of B_p are not bounded, then we obtain that $G/C_G(B_p)$ is isomorphic to a subgroup of the multiplicative group of the ring of p-adic integers (see, for example, [30], Theorem 1.5.6). This group is residually finite. If the orders of elements of B_p are bounded and $p \neq 2$, then $G/C_G(B_p)$ is a finite cyclic group (see, for example, [30], Theorem 1.5.6). If B_2 is abelian and the orders of elements of B_2 are bounded, then $G/C_G(B_2)$ is a direct product of two finite cyclic group (see, for example, [30], Theorem 1.5.6). If B_2 is not abelian, then as we noted above, $B_2 = Q \times E$ where Q is a quaternion group and E is an elementary abelian 2-subgroup. Since every cyclic subgroup of E is G-invariant, the center of G includes E. It follows that $C_G(B_2) = C_G(Q)$, therefore $G/C_G(B_2)$ is finite. We have

$$C_G(B) = \bigcap_{p \in \Pi(B)} C_G(B_p).$$

The Remak's theorem shows that $G/C_G(B)$ is isomorphic to a subgroup of $\operatorname*{Cr}_{p \in \Pi(B)} G/C_G(B_p)$, in particular, $G/C_G(B)$ is residually finite. Corollary 2.4 shows that $C_G(B) \leqslant B$. If B_2 is abelian, then B is also abelian, so that, in this case, $C_G(B) = B$ and G/B is residually finite. If B_2 is not abelian, then

$$C_G(B) = B \cap C_G(B) = E \times \zeta(Q) \times \operatorname*{Dr}_{p \in \Pi(B) \setminus \{2\}} B_p.$$

It follows that $B/C_G(B) = QC_G(B)/C_G(B)$, in particular, $B/C_G(B)$ is finite. We note that finite subgroups of residually finite group are closed in profinite topology. It implies that G/B is residually finite, and (i) is proved,

Proposition 2.12 implies that $R \leqslant B, 2 \notin \Pi(R)$, R is G-hypereccentric and G/R is hypercentral, $\Pi(R) \cap \Pi(B/R) = \emptyset$. The fact that $2 \notin \Pi(R)$ together with Proposition 2.12 imply that $\Pi(R) \cap \Pi(L/R) = \emptyset$.

Let $g \in C_G(R)$. The choice of g yields that $B \leqslant \zeta(\langle g, B \rangle)$. The fact

that $\langle g, B \rangle / B$ is hypercentral shows that $\langle g, B \rangle$ is hypercentral. Since the factor-group G/B is abelian, $\langle g, B \rangle$ is a normal subgroup of G. Then $\langle g, B \rangle \leqslant L$, and (iii) is proved.

Assertions (iv) and (v) follows from Lemma 2.13.

Let $\pi = \Pi(R)$, $\sigma = \Pi(L) \, \Pi(R)$. Since G is hyperabelian, $\Pi(L) = \Pi(B)$. Let $p \in \sigma$, then by Corollary 2.10 the upper hypercenter Z of G includes B_p. If p is an odd prime, then by (v) B_p is the Sylow p-subgroup of L. If B does not includes the Sylow 2-subgroup L_2 of L, then Corollary 2.15 implies that L_2/B_2 has order 2. Then the centre of G/B_2 includes L_2/B_2. This means that Z includes L_2. In other words, Z includes $\mathrm{Dr}_{p \in \sigma} L_p$, and (vi) is proved.

Let D be the Sylow 2-subgroup of G. Then Sylow 2-subgroup L_2 of L is nilpotent. Corollary 2.2 implies that, in this case, B includes L_2. By (v), B includes every Sylow p-subgroup of L for all odd primes p. Hence $L = B$. If the orders of elements of B_2 are not bounded, then we can apply the arguments from the first part of the proof of Lemma 2.13 to obtain that D is abelian. Suppose the orders of elements of B_2 are bounded and B_2 is not abelian. Then $B_2 = Q \times E$ where Q is a quaternion group and E is an elementary abelian 2-subgroup. Since every cyclic subgroup of E is G-invariant, the center of G includes $E \times \zeta(Q)$. By the same reason, the center of $G/(E \times \zeta(Q))$ includes $B_2 = (E \times \zeta(Q))$. It follows that the second hypercenter of D includes $B_2 = B \cap D$. The fact that D/B_2 is abelian implies that D is nilpotent and (vii) is proved.

Let p be an odd prime and P_1 be the Sylow p-subgroup of L. By (iii), P_1 is nilpotent. By Corollary 2.2 B includes P_1, in particular, P_1 is abelian, and (iv) is proved.

Suppose now that $p \notin \Pi(R)$. By Corollary 2.10, the upper hypercenter of G includes B_p. Corollary 2.2 implies that, in this case, every factor $\Omega_{j+1}(B_p)/\Omega_j(B_p)$ is G-central, thus the series

$$\langle 1 \rangle \leqslant \Omega_1(B_p) \leqslant \Omega_2(B_p) \leqslant \ldots \leqslant \Omega_j(B_p) \leqslant \Omega_{j+1}(B_p)$$
$$\leqslant \ldots \Omega_\omega(B_p) = \cup_{j \in \mathbb{N}} \Omega_j(B_p)$$

is G-central, and (viii) is proved.

By (viii), the ωth hypercenter of G/R includes B/R. Since G/B is abelian, the hypercentral length of G/R is at most $\omega + 1$.

Assertions (ix) and (x) follows from Lemma 2.14 and Corollary 2.15.

Conversely, suppose that a group G satisfies all above condition.

Let A be an arbitrary subnormal abelian subgroup of G. If B includes A, then by (i) every cyclic subgroup of A is D-invariant. It follows that A is a normal subgroup of G.

Suppose now that B does not include A. Since G/B is abelian, the product BA is normal subgroup of G. We note that the subgroup BA is nilpotent (see [17], Lemma 4). But, in this case, the Baer radical of G includes BA, and we obtain a contradiction. This contradiction proves that B must include A.   □

PROOF OF COROLLARY A1 — Let p be an odd prime and let P be the Sylow p-subgroup of G. Theorem A implies that P is nilpotent. Using Corollary 2.2 we obtain that P must be abelian. The assertion (ii) follows from Theorem A.   □

PROOF OF COROLLARY A2 — Assertion (1) follows from Theorem A. The fact that G/B is abelian also follows from Theorem A. Corollary 2.7 implies that a group G is locally supersoluble. Then G has a series of normal subgroups

$$\langle 1 \rangle = S_0 \leqslant S_1 \leqslant S_2 \leqslant \ldots \leqslant S_{k-1} \leqslant S_k = G$$

where S_1 is the Sylow p_1-subgroup, S_2 is the Sylow $\{p_1, p_2\}$-subgroup, $\ldots, S_{k-1}$ is the Sylow $\{p_1, p_2, \ldots, p_{k-1}\}$-subgroup of G.

Let $B_1 = B \cap S_1, C_1 = C_G(B_1)$. By Corollary 2.2, every subgroup of B_1 is G-invariant, in particular, B_1 is a Dedekind group. Since $k > 1$, $p_1 \neq 2$. It implies that B_1 is abelian. The Sylow p_1-subgroup $S_1 \cap C_1$ of C_1 is normal in G. Now,

$$S_1/B_1 = S_1/(B \cap S_1) \simeq S_1 B/B$$

is abelian, so $S_1 \cap C_1$ is nilpotent. Corollary 2.2 shows that B_1 includes $S_1 \cap C_1$. Then $S_1/B_1 = S_1/(S_1 \cap C_1) \simeq S_1 C_1/C_1 \leqslant G/C_1$. If the orders of elements of B_1 are at most $p_1^{t_1}$ for some positive integer t_1, then the factor-group G/C_1 is isomorphic to a subgroup of the multiplicative group of the ring $Z/p_1^{t_1}Z$ (see, for example, [30], Theorem 1.5.6). Then S_1/B_1 is finite cyclic p_1-group. In this case, the subgroup S_1 is nilpotent (see, for example, [10], Corollary 1.77). But Corollary 2.2 shows that, in this case, B must include S_1, i.e. $S_1 = B_1$, and in particular, S_1 is abelian.

If the orders of elements of B_1 are not bounded, then G/C_1 is isomorphic to a subgroup of the multiplicative group of the ring

of p_1-adic integers (see, for example, [30], Theorem 1.5.6). But the last group has no elements of order p_1 (see, for example, [31], Chapter II, §3, Theorem 2). Thus again $S_1 = B_1$. We note that in both cases G/C_1 is a cyclic, whose order divides $p_1 - 1$. In particular, G/C_1 does not contain the p_1-elements.

Denote by B_2 the Sylow p_2-subgroup of B and let $C_2 = C_G(B_2) \cap C_1$. Since G is locally supersoluble, the Sylow p_2-subgroup Q_2/S_1 of C_1/S_1 is normal. By its choice, Q_2 is locally nilpotent, so that $Q_2 = P_2 \times S_1$, where P_2 is the Sylow p_2-subgroup of Q_2, and hence the Sylow p_2-subgroup of C_1. In particular, we obtain that P_2 is normal in G. By Corollary 2.2, every subgroup of B_2 is G-invariant, in particular, B_2 is a Dedekind group. Suppose that $p_2 \neq 2$. Then B_2 is abelian. If the orders of elements of B_2 are at most $p_2^{t_2}$ for some positive integer t_2, then the factor C_1/C_2 is isomorphic to a subgroup of the multiplicative group of the ring $Z/p_2^{t_2}Z$ (see, for example, [30], Theorem 1.5.6). Then $P_2/(P_2 \cap C_2)$ is a finite cyclic p_2-group. Since P_2/B_2 is abelian, $P_2 \cap C_2$ is nilpotent. The fact that $P_2 \cap C_2$ is normal in G together with Corollary 2.2 yields that $B \supseteq P_2 \cap C_2$, so we have $P_2 \cap C_2 = B_2$. Thus P_2/B_2 is a finite cyclic p_2-group. In this case, the subgroup P_2 is nilpotent (see, for example [10], Corollary 1.77), and, using again Corollary 2.2, we obtain that $P_2 = B_2$.

If the orders of elements of B_2 are not bounded, then C_1/C_2 is isomorphic to a subgroup of the multiplicative group of the ring of p_2-adic integers (see, for example, [30], Theorem 1.5.6). But the periodic part of the last group is a cyclic group, whose order is $p_2 - 1$ (see, for example, [31], Chapter II, §3, Theorem 2). Thus again $P_2 = B_2$. It follows that C_1/C_2 is a cyclic group, whose order divides $p_2 - 1$. In particular, the Sylow p_2-subgroup of G/C_2 is cyclic.

Suppose now that $p_2 = 2$. If B_2 is abelian and orders of elements of B_2 are bounded, then using the above arguments we obtain that the Sylow 2-subgroup of C_1 coincides with B_2. It follows that the Sylow 2-subgroup of G/C_2 is a direct product of at most three cyclic groups.

Suppose now that B_2 is abelian and the orders of its elements are not bounded. Then the factor C_1/C_2 is isomorphic to a subgroup of the multiplicative group of the ring of 2-adic integers (see, for example, [30], Theorem 1.5.6). But the periodic part of the last group has order 2 (see, for example, [31], Chapter II, §3, Theorem 2). Thus there exists an element a such that $C_1 = \langle a \rangle C_2$ and $b^a = b^{-1}$ for each element $b \in B_2$. It follows that the Sylow 2-subgroup of G/C_2

has special rank at most 2.

Suppose now that B_2 is not abelian. Since every subgroup of B_2 is G-invariant, B_2 is a Dedekind group. Then, as we noted, $B_2 = Q \times E$ where Q is a quaternion group and E is an elementary abelian 2-subgroup. Since every cyclic subgroup of E is G-invariant, the center of G includes E. It follows that $C_G(B_2) = C_G(Q)$. In this case, Lemma 2.16 shows that the Sylow 2-subgroup of C_1/C_2 is elementary abelian and has order at most 4. Hence, in this case, the Sylow 2-subgroup of C_1 is an extension of an elementary abelian subgroup by a finite 2-group. Hence, it is nilpotent (see, for example [10], Corollary 1.77), so that the Sylow 2-subgroup of C_1 coincides with B_2. It follows that the Sylow 2-subgroup of G/C_2 is a direct product of at most three cyclic groups.

If $\Pi(G) = \{p_1, p_2\}$, then $C_2 = C_G(B_2) \cap C_G(B_1) = C_G(B) \leqslant B$ so that G/B is a finite p_2-group.

If $k > 2$, then we can apply the above arguments and ordinary induction to prove the assertions (ii)–(iv).

Let R be the locally nilpotent residual of G. By Theorem A, R is G-hypereccentric, $B = R \times Z$ where $\Pi(R) \cap \Pi(Z) = \emptyset$, and Z is a subgroup of the upper hypercenter of G. In the factor-group G/Z the subgroup B/Z is G-hypereccentric, and $(G/Z)/(B/Z)$ is finite and abelian. Then $G/Z = B/Z \rtimes S/Z$, and every complement to B/Z in G/Z is conjugate to S/Z (see [19], Proposition 2.5). The equality $B = R \times Z$ shows that $G = R \rtimes S$.

Let S_1 be another subgroup such that $G = R \rtimes S_1$. We have

$$B = R \times (B \cap S_1).$$

Let $\pi = \Pi(B)$, then $B \cap S_1$ is a Sylow π'-subgroup of B. But B is abelian and its Sylow π'-subgroup coincides with Z. Hence $B \cap S_1 = Z$, in particular, S_1 includes Z. In the factor-group G/Z we have

$$G/Z = B/Z \rtimes S_1/Z.$$

By what we noted above, S_1/Z and S/Z conjugate. Then the subgroups S_1 and S conjugate. □

Corollary 2.17 *Let G be a periodic hyperabelian group, whose subnormal abelian subgroups are normal. Suppose that $\Pi(G) = \{p, q\}$ where $p < q$. If G is not locally nilpotent, then G satisfies the following conditions:*

(i) *the Sylow q-subgroup Q of G is abelian, $G = Q \rtimes P$ where P is*

a Sylow p-*subgroup of* G;

(ii) *the Sylow* p-*subgroups of* G *conjugate.*

(iii) p *divides* q − 1;

(iv) *the factor-group* P/C_P(Q) *is cyclic, and* C_P(Q) × Q *is the Baer radical of* G;

(v) *if* p ≠ 2 *and* P *is non-abelian, then the orders of elements of* P *are bounded, and* P *is nilpotent;*

(vi) *if* p = 2 *and* P *is not a Dedekind group, then* P/C_P(Q) *has order* 2, *and* P *satisfies the following conditions:*

  (via) C_P(Q) = B *is abelian, and orders of elements of* B *are not bounded;*

  (vib) P = B⟨a⟩, a^2 ∈ B, *and* x^a = x−1 *for every element* x ∈ B;

  (vic) C_B(a) *is an elementary abelian 2-subgroup;*

  (vid) *either* a^2 = 1 *or* a^4 = 1.

PROOF OF COROLLARY A3 — From Corollary A2 it follows that Q is abelian and the Baer radical of G includes Q. Corollary 2.10 and the fact that G is not locally nilpotent imply that Q must be G-hypereccentric. Since G/Q is a p-group, Q coincides with locally nilpotent residual of G. By Corollary A2, G = Q ⋊ P and, clearly, in this case, P is the Sylow p-subgroup of G. Moreover, Corollary A2 implies that the Sylow p-subgroups of G conjugate.

If the orders of elements of Q are at most q^t for some positive integer t, then the factor-group G/C_G(Q) ≃ P/C_P(Q) is isomorphic to a subgroup of the multiplicative group of the ring Z/q^tZ (see, for example, [30], Theorem 1.5.6). The fact that q ≠ 2 implies that the last group is cyclic, and p divides q − 1. If the orders of elements of Q are not bounded, then the P/C_P(Q) is isomorphic to a subgroup of the multiplicative group of the ring of q-adic integers (see, for example, [30], Theorem 1.5.6). But the periodic part of the last group is a cyclic, having order q − 1.

Let B be the Sylow p-subgroup of the Baer radical of G. Since the Baer radical of G is a Dedekind group, then B ⩽ C_P(Q). If p is odd, then Theorem A implies that P is nilpotent. Then C_P(Q) is

nilpotent and Corollary 2.2 implies that the Baer radical of G includes $C_P(Q)$. Thus $B = C_P(Q)$ and $C_P(Q) \times Q$ is the Baer radical of G.

Suppose that $p = 2$. If P is nilpotent, then $C_P(Q) \times Q$ is the Baer radical of G. If P is not nilpotent, then the result follows from assertion (xi) of Theorem A. □

3 Lie algebras whose abelian subideals are ideals

It is almost obvious that if L is a Lie algebra, whose subalgebras are ideals, then L is abelian.

Lemma 3.1 *Let* L *be a Lie algebra, whose abelian subideals are ideals. Then* L *includes an abelian ideal* K *such that every subalgebra of* K *is an ideal of* L *and every subideal* R *of* L *such that* $K \leqslant R$ *and* $K \neq R$ *is not nilpotent.*

PROOF — Let S be a nilpotent subideal of L. If x is an arbitrary element of S, then a subalgebra $\langle x \rangle = Fx$ is a subideal of S (see [3], Lemma 3.7). Then $\langle x \rangle$ is a subideal of L, so that a subalgebra $\langle x \rangle$ is an ideal of L. Since it is true for each nilpotent subideal of L, in the subalgebra K generated by all nilpotent subideals of H, every cyclic subalgebra is an ideal of L. It follows that every subalgebra of K (and K itself) is an ideal of L. The choice of K shows that K includes every nilpotent subideal of L. By what we noted above, K is abelian.   □

Let L be a Lie algebra over a field F, M be non-empty subset of L and H be a subalgebra of L. Put

$$\mathrm{Ann}_H(M) = \{a \in H \,|\, [a, M] = \langle 0 \rangle\}.$$

The subset $\mathrm{Ann}_H(M)$ is called the *annihilator* or the *centralizer* of M in subalgebra H.

It is not hard to see that $\mathrm{Ann}_H(M)$ is a subalgebra of L. Moreover, if M is an ideal of L, then $\mathrm{Ann}_L(M)$ is an ideal of L.

Lemma 3.2 *Let* L *be a Lie algebra over a field* F, *and let* A *be an abelian ideal of* L. *If every subalgebra of* A *is an ideal of* L, *then the factor-algebra* $L/\mathrm{Ann}_L(A)$ *has dimension* 1, *and for every element* $x \in L$ *there exists an element* $\sigma_x \in F$ *such that* $[x, a] = \sigma_x a$ *for all elements* $a \in A$.

This assertion is (for example) a partial case of Lemma 2.2 of the paper [25].

The Lie algebra L is called hyperabelian M if L has an ascending series of ideals whose factors are abelian.

PROOF OF THEOREM B — By Lemma 3.1, L has the greatest nilpotent ideal A of L. Being hyperabelian, L includes a non-zero abelian ideal, so that A is non-zero. Let $C = Ann_L(A)$, and suppose that A does not include C. Then $(C + A)/A$ is non-zero. Since L is hyperabelian, $(C + A)/A$ includes a non-zero abelian ideal S/A. The inclusion $S \leqslant C$ shows that S is nilpotent. But Lemma 3.1 implies that in this case, A must include S, and we obtain a contradiction. This contradiction proves the inclusion $Ann_L(A) \leqslant A$. Since A is abelian, $Ann_L(A) = A$ and every subalgebra of A is an ideal of L. Lemma 3.2 implies that L/A has dimension 1, so that $L = A \oplus Fb$ for some element $b \in L$. Using again Lemma 3.2, we obtain that there exists an element $\beta \in F$ such that $[b, a] = \beta a$ for all elements $a \in A$. If we suppose that $\beta = 0$, then L is abelian, and we obtain a contradiction. Hence $\beta \neq 0$. Put $d = \beta^{-1}b$, then

$$[d, a] = [\beta^{-1}b, a] = \beta^{-1}[b, a] = \beta^{-1}\beta a = a \quad \text{for all elements } a \in A.$$

In conclusion, we now give an example of a nilpotent Leibniz algebra of nilpotency class 2, which has a unique abelian subalgebra, and the dimension of this subalgebra is equal to 1.   □

Example 3.3 Let n be an arbitrary positive integer and let V be a vector space over a field Q of rational numbers, having dimension n. Let us define on the space V a positive defined bilinear form Φ. In other words, we can choose a basis $\{v_1, \ldots, v_n\}$ of the space V such that $\Phi(v_k, v_k) = 1$ for all $k, 1 \leqslant k \leqslant n$, and $\Phi(v_k, v_j) = 0$ whenever $k \neq j$. Put $L = V \oplus Qc$, and define the operation $[,]$ on L for the elements of the basis by the following rule: $[v, c] = [c, v] = [c, c] = 0$, and $[v_k, v_j] = \Phi(v_k, v_j)c, 1 \leqslant k, j \leqslant n$, and expand it bilinearly to all elements of L. Put $Z = Qc$. Then the centre of L includes Z and the factor-algebra L/Z is abelian. It follows that L is a nilpotent Leibniz algebra, moreover, its nilpotency class is 2.

Let x, y be the elements of L. Then

$$x = \xi c + \sum_{1 \leqslant j \leqslant n} \alpha_j v_j, y = \eta c + \sum_{1 \leqslant k \leqslant n} \beta_k v_k, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in F.$$

We have

$$[x, y] = \left[\xi c + \sum_{1 \leqslant j \leqslant n} \alpha_j v_j, \eta c + \sum_{1 \leqslant k \leqslant n} \beta_k v_k\right]$$

$$= \sum_{1 \leqslant j \leqslant n} \sum_{1 \leqslant k \leqslant n} \alpha_j \beta_k [v_j, v_k] = \left(\sum_{1 \leqslant j \leqslant n} \alpha_j \beta_j\right) c.$$

It follows that

$$[x, x] = (\alpha_1^2 + \ldots + \alpha_n^2) c.$$

In particular, if $x \notin Z$, then $(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)$ and, therefore, $[x, x] \neq 0$. Now if S is a subalgebra of L such that Z does not include S, then S contains an element $x \notin Z$. Then $[x, x] \neq 0$, and hence a subalgebra S is not abelian. This means that Z is the only non-trivial abelian subalgebra of L.

REFERENCES

[1] I.N. Abramovskij: "Subgroups of direct products of groups", in: Some Questions on the Theory of Groups and Rings (Russian), *Inst. Fiz. im. Kirenskogo Sibirsk. Otdel. Akad. Nauk SSSR*, Krasnoyarsk (1973), 3–8, 173.

[2] I.N. Abramovskij – M.I. Kargapolov: "Finite groups with a property of transitivity for normal subgroups", *Uspekhi Mat. Nauk* 13, No. 3 (1958), 242–243.

[3] R.K. Amayo – I. Stewart: "Infinite-Dimensional Lie Algebras", *Noordhoff*, Leyden (1974).

[4] R. Baer: "Situation der Untergruppen und Struktur der Gruppe", *S.-B. Heidelberg Akad.* 2 (1933), 12–17.

[5] R. Baer: "Supersoluble groups", *Proc. Amer. Math. Soc.* 6 (1955), 16–32.

[6] R. Baer: "Nil-Gruppen", *Math. Z.* 62 (1955), 402–437.

[7] A. Ballester-Bolinches – L.A. Kurdachenko – J. Otal – T. Pedraza: "Local characterizations of infinite groups whose ascendant subgroups are permutable", *Forum Math.* 22 (2010), 187–201.

[8] A. Ballester-Bolinches – L.A. Kurdachenko – T. Pedraza: "On periodic radical groups in which permutability is a transitive relation", *J. Pure Appl. Algebra* 210 (2007), 665–671.

[9] J.C. Beidleman – B. Brewster – D.J.S. Robinson: "Criteria for permutability to be transitive in finite groups", *J. Algebra* 222 (1999), 400–412.

[10] C. Casolo: "Groups with all subgroups subnormal", *Note Mat.* 28 (2008), 1–149.

[11] M. De Falco – F. de Giovanni – C. Musella – Ya.P. Sysak: "Groups of infinite rank in which normality is a transitive relation", *Glasgow Math. J.* 56 (2014), 387–393.

[12] R. Dedekind: "Über Gruppen deren sämtliche Teiler Normalteiler sind", *Math. Ann.* 48 (1897), 548–561.

[13] M.R. Dixon – L.A. Kurdachenko – I.Ya. Subbotin: "Rank of Groups: the Tools, Characteristics and Restrictions" *Wiley*, New Jersey (2017).

[14] W. Gaschütz: "Gruppen in denen das Normalteilersein transitiv ist", *J. Reine Angew. Math.* 198 (1957), 87–92.

[15] F. de Giovanni – S. Franciosi: "Groups in which every infinite subnormal subgroup is normal", *J. Algebra* 96 (1985), 566–580.

[16] F. de Giovanni – M. Trombetti: "Large characteristic subgroups in which normality is a transitive relation", *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 30 (2019), 255–268.

[17] H. Heineken – L.A. Kurdachenko: "Groups with subnormality for all subgroups that are not finitely generated", *Ann. Mat. Pura Appl.* 169 (1995), 203–232.

[18] L.A. Kurdachenko – J. Otal: "On the influence of transitively normal subgroups on the structure of some infinite groups", *Publ. Mat.* 57 (2013), 83–106.

[19] L.A. Kurdachenko – J. Otal – I.Ya. Subbotin: "Some criteria for existence of supplements to normal subgroups and their applications", *Internat. J. Algebra Comput.* 20 (2010), 689–719.

[20] L.A. KURDACHENKO – N.N. SEMKO – I.YA. SUBBOTIN: "The Leibniz algebras whose subalgebras are ideals", *Open Math.* 15 (2017), 92–100.

[21] L.A. KURDACHENKO – I.YA. SUBBOTIN: "On transitivity of pronormality", *Comment. Math. Univ. Carolin.* 43 (2002), 583–594.

[22] L.A. KURDACHENKO – I.YA. SUBBOTIN: "Abnormal subgroups and Carter subgroups in some infinite groups", *Algebra Discrete Math.* (2005), 63–77.

[23] L.A. KURDACHENKO – I.YA. SUBBOTIN: "Transitivity of normality and pronormal subgroups", *Contemp. Math.* 421 (2006), 201–212.

[24] L.A. KURDACHENKO – I.YA. SUBBOTIN – V.S. YASHCHUK: "Leibniz algebras, whose subideals are ideals", *J. Algebra Appl.* 17 (2018), 1850151, 15pp.

[25] L.A. KURDACHENKO – I.YA. SUBBOTIN – V.S. YASHCHUK: "Leibniz algebras, whose subalgebras are left ideals", *Serdica Math. J.* 46 (2020), 175–194.

[26] B.I. PLOTKIN: "Radical groups", *Math. Sb.* 37 (1955), 507–526.

[27] D.J.S. ROBINSON: "Groups in which normality is a transitive relation", *Proc. Cambridge Philos. Soc.* 60 (1964), 21–38.

[28] D.J.S. ROBINSON: "A note on finite groups in which normality is transitive", *Proc. Amer. Math. Soc.* 19 (1968), 933–937.

[29] D.J.S. ROBINSON: "Groups whose homomorphic images have a transitive normality relation", *Trans. Amer. Math. Soc.* 176 (1973), 181–213.

[30] R. SCHMIDT: "Subgroup Lattices of Groups", *de Gruyter*, Berlin (1994).

[31] J.-P. SERRE: "Cours d'Arithmetique", *Presses Universitaires de France*, Paris (1970).

[32] I.N. STEWART: "Subideals of Lie Algebras", *PhD Thesis*, University of Warwick (1969).

[33] V.R. VAREA: "On Lie algebras in which the relation of being an ideal is transitive", *Comm. Algebra* 13 (1985), 1135–1150.

[34] G. ZACHER: "I gruppi risolubili in cui i sottogruppi di composizione coincidono con i sottogruppi quasinormali", *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* 37 (1964), 150–154.

[35] D.I. Zaitsev: "The hypercyclic extensions of abelian groups", in: Groups Defined by Properties of Systems of Subgroups, *Akad. Nauk Ukrain. SSR*, Kiev (1979), 16–37.

L.A. Kurdachenko
Department of Algebra
Dnipro National University
Gagarin Prospect 70
Dnipro 10 (Ukraine)
e-mail: lkurdachenko@i.ua

J. Otal
Departmento de Matematikas
Universidad de Zaragoza
Pedro Cerbuna 12
50009 Zaragoza (Spain)
e-mail: otal@unizar.es

I.Ya. Subbotin
Department of Mathematics
National University
5245 Pacific Concourse Drive
Los Angeles, CA 90045 (USA)
e-mail: isubboti@nu.edu