Advances in Group Theory and Applications

13 (2022), pp. 55–59
ISSN: 2499-1287
DOI: 10.32037/agta-2022-004

Groups with Finite Hirsch Number Modulo their Hypercentres

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(*Received May 28, 2021; Accepted June 1, 2021 — Communicated by F. de Giovanni*)

Abstract

If G is a group such that G modulo its hypercentre has finite Hirsch number h, we prove that G has a normal subgroup K such that K has Hirsch number at most $h(h+3)/2$ and such that G/K is torsion-free and hypercentral. This extends a recent theorem of Dixon, Kurdachenko and Subbotin.

Mathematics Subject Classification (2020): 20F19, 20F14, 20E99
Keywords: Hirsch number; hypercentre of a group

1 Introduction

Dixon, Kurdachenko and Subbotin's paper [1] is devoted to proving the following very interesting theorem. If G is a group such that G modulo its hypercentre has finite Hirsch number h, then G has a normal subgroup K such that K has Hirsch number at most $h(5h^2+5h+1)/2$ and G/K is torsion-free and hypercentral.

A group G has finite Hirsch number the integer h if G has an ascending series (running from $\langle 1\rangle$ to G) such that exactly h of the factors are infinite cyclic, the remaining factors being locally finite (note that finite Hirsch number is equivalent in the terminology of [1] to finite 0-rank with all periodic sections locally finite). Here we prove the following.

Theorem *If G is a group such that G modulo its hypercentre has finite Hirsch number h, then G has a normal subgroup K such that K has Hirsch number at most $h(h+3)/2$ and G/K is torsion-free and hypercentral.*

Notation For any group G, $\tau(G)$ denotes the locally finite radical of G, $\{\zeta_i(G)\}_{i\geqslant 0}$ the upper central series of G, $\zeta(G) = \cup_{i\geqslant 0}\zeta_i(G)$ the hypercentre of G and $\{\gamma^{i+1}G\}_{i\geqslant 0}$ the lower central series of G; here i denotes an arbitrary ordinal. Further $hn(G)$ denotes the Hirsch number of G, either finite or undefined. Thus the hypothesis of the theorem is $hn(G/\zeta(G)) = h < \infty$ and the conclusion yields $hn(K) \leqslant h(h+3)/2$.

2 Proofs

In the proof below we use the following, of which Proposition 1 and Proposition 2 are well known.

Proposition 1 *If G is a group with finite Hirsch number, then G has normal subgroups $T \leqslant N \leqslant M$ with $T = \tau(G)$ locally finite, N/T torsion-free nilpotent of finite rank, M/N free abelian of finite rank and G/M finite. In particular $G/\tau(G)$ has finite rank. If G is also finitely generated, then $G/\tau(G)$ is minimax.*

Proposition 2 *If G is a group with $\tau(G) = \langle 1\rangle$, then $\tau(G/\zeta_i(G)) = \langle 1\rangle$ for each ordinal i.*

From Theorems 1 and 2 of [5] we have the following.

Proposition 3 *Let G be a group and let h and k be integers such that $hn(G/\zeta_k(G)) \leqslant h$. Then the Hirsch number $hn(\gamma^{k+1}G)$ is finite and $hn(\gamma^{h+2k+1}G) \leqslant h(h+3)/2$.*

Proof of the Theorem — Clearly we may assume that $\tau(G) = \langle 1\rangle$ and hence by Proposition 2 that $\tau(G/\zeta(G)) = \langle 1\rangle$. Consider first the case where G is finitely generated, so by Proposition 1 the group $G/\zeta(G)$ is minimax. We prove that such G have finite central heights (so we can apply Proposition 3). Assume otherwise. Suppose the ordinal i is such that if $j < i$, whenever $G/\zeta_j(G)$ is minimax for any such G, then G has finite central height. Set $Z = \zeta_i(G)$ and suppose G/Z is minimax. If $j = i-1$ exists, then $hn(G'\zeta_j(G)/\zeta_j(G))$ is finite by Proposition 3, so $hn/G/\zeta_j(G))$ is finite, $G/\zeta_j(G)$ is minimax by Propositions 1 and 2 and therefore G has finite central height. Consequently i is a limit ordinal. Note that if $G/\zeta_j(G)$ has finite central height k for some $j < i$, then $Z = \zeta_{j+k}(G)$, $j + k < i$ and G has finite central height.

By Proposition 1 there exists N normal in G with $Z \leqslant N$, N/Z torsion-free nilpotent of finite rank and G/N abelian-by-finite. We now prove that G has finite central height by induction on the (nilpotency) class of N/Z. If $Z = N$ then G/Z is finitely presented. But then Z is finitely G-generated, $Z = \zeta_j(G)$ for some $j < i$ and G has finite central height. Suppose $N > Z$. Set $M/Z = \zeta_1(N/Z)$. Then M/Z is torsion-free abelian of finite rank s where $1 \leqslant s \leqslant h$. Note also that N/M is torsion-free.

There exists an s-generator subgroup A of M with M/AZ periodic. If $a, b \in A$, then $[a, b] \in \zeta_j(G)$ for some $j < i$ and A is finitely generated. Hence by passing to $G/\zeta_j(G)$ for large enough $j < i$, we may assume that A has been chosen abelian. Hence A and AZ/Z are now both free abelian of rank s so $A \cap Z = \langle 1 \rangle$. If $a \in A$ and $g \in G$ there exists a positive integer m with $(a^g)^m \in AZ$, so there exists some $j < i$ with $(a^g)^m \in A\zeta_j(G)$. Then there exists some fixed $j < i$ such that this holds for all a in some finite generating set of A and for all g or g^{-1} in some finite generating set of G. We may pass to $G/\zeta_j(G)$ and assume that $j = 0$. Let B denote the isolator of A in the torsion-free locally nilpotent group M. Note that B like A is torsion-free abelian of rank s. The above shows that $B^G \leqslant B$ and hence that B is normal in G.

Set $T/B = \tau(G/B)$. If $x \in T \cap Z$ there is a positive integer m with $x^m \in A \cap Z = \langle 1 \rangle$ and Z torsion-free. Therefore $x = 1$ and $T \cap Z = \langle 1 \rangle$. Now MT/T is torsion-free and M/BZ is periodic. Clearly $ZT/T \leqslant \zeta_i(G/T)$. Thus MT/T modulo $\zeta_i(G/T)$ is torsion-free and periodic and hence $MT/T \leqslant \zeta_i(G/T)$. Therefore by induction on the class of N/Z we may assume that G/T has finite central height, k say. Then $[Z,_k G] \leqslant T \cap Z = \langle 1 \rangle$ and consequently G has finite central height.

We now drop the assumption that G is finitely generated. Recall $h = hn(G/\zeta(G)) < \infty$. Let $X \leqslant Y$ be finitely generated subgroups of G. By Propositions 3 and 1 and the above $X/\tau(X)$ is soluble-by-finite, minimax and of finite central height. Thus X has a unique normal subgroup K_X minimal subject to $\tau(X) \leqslant K_X$ and X/K_X torsion-free nilpotent. By Proposition 3 there exists M normal in X with $\tau(X) \leqslant M$, $hn(M) \leqslant h(h+3)/2$ and X/M nilpotent. Clearly we may choose M with X/M torsion-free (replace M by M_1, where $M_1/M = \tau(X/M)$). Thus $K_X \leqslant M$ and $hn(K_X) \leqslant h(h+3)/2$.

Clearly $K_X \leqslant X \cap K_Y$. Set $K = \cup_X K_X$. Then K is a normal subgroup of G with G/K locally nilpotent. If $x \in G$ with $x^m \in K$, then there exists X with $x \in X$ and $x^m \in K_X$. Consequently $x \in K_X \leqslant K$ and $H = G/K$ is torsion-free. Then $H/\zeta(H)$ is torsion-free (by Proposition 2), locally nilpotent with $hn(H/\zeta(H))$ finite. By a theorem of Mal'cev, see [2], Theorem 6.36, $H/\zeta(H)$ is nilpotent and so H is hypercentral.

Since $hn(K/\zeta(K))$ is finite, K has an ascending series whose factors are locally finite or infinite cyclic. If at least $1 + h(h+3)/2$ of these factors are infinite cyclic, there exists $X \leqslant G$ finitely generated with $hn(K_X) > h(h+3)/2$. This final contradiction shows that $hn(K) \leqslant h(h+3)/2$ and the theorem follows. □

The converse to the theorem does not hold; that is, if K is a normal subgroup of a group G with $hn(K) < \infty$ and G/K torsion-free locally nilpotent, then $G/\zeta(G)$ need not have finite Hirsch number. For example, let G be the split extension of the additive group of the rationals by the multiplicative group of the rationals; here $\zeta(G) = \langle 1 \rangle$ and G has a free abelian image of infinite rank. However if G is finitely generated the converse does hold. The following recovers immediately all the information on the finitely generated case discovered during the proof of the theorem from the theorem itself.

Proposition 4 *Let G be a finitely generated group with a normal subgroup K such that $hn(K) < \infty$ and G/K is nilpotent. Then $hn(G)$ is finite, $G/\tau(G)$ is minimax and soluble-by-finite, $G/\tau(G)$ is isomorphic to a linear group over the rationals and $G/\tau(G)$ has finite central height.*

Proof — G/K is finitely generated and nilpotent. Hence $hn(G/K)$ and $hn(G)$ are finite. Then $G/\tau(G)$ is minimax, soluble-by-finite and (torsion-free)-by-finite by Proposition 1 and hence $G/\tau(G)$ is isomorphic to a linear group over the rationals (e.g. by [4], Corollary 1.3). Every linear group over the rationals has finite central height (see for instance [3], Corollary 8.8 and Theorem 9.33). □

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