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Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p

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Abstract

During his lectures to the 1987 Singapore Group Theory Conference Otto H. Kegel proposed the following question: *"If every subgroup S of the locally finite group G contains a finite p-subgroup which is singular in S, does G then satisfy the strong Sylow Theorem for the prime p?"* In this paper we answer the question in the affirmative. The paper formed an essential part of the author's German Diplomarbeit of 1984 (the "*Charakterisierungssatz*") written before he left academia [4]. We present the *Charakterisierungssatz* as Theorem 3.9, and summarise then the result as Theorem 3.10, stating that if G is a locally finite group and p is a prime, then G satisfies the strong Sylow theorem for the prime p if and only if every subgroup S of G contains a finite p-subgroup which is singular in S. Subsequently we present a few novel concepts for Sylow theory in (locally) finite groups to encourage future research. The paper is divided in four sections: Introduction; Good Sylow p-subgroups and p-uniqueness subgroups; Basic theorems of Sylow theory in locally finite groups and our *Charakterisierungssatz*; Novel concepts for Sylow theory in (locally) finite groups.

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1 Introduction

In his four workshop lectures on Sylow theory in locally finite groups at the famed Singapore Group Theory Conference of June 1987 [10], Otto H. Kegel stated that he could not settle the following question: *if* *every subgroup* S *of the locally finite group* G *contains a finite* p*-subgroup which is singular in* S*, does* G *then satisfy the strong Sylow Theorem for the prime* p*?* Recall that the group G of arbitrary cardinality is defined to be *locally finite* if every finite subset of G is contained in a finite subgroup of G and the finite p-subgroup P of the locally finite group G is said to be *singular in* G if for every finite subgroup F of G containing P there is just a unique Sylow p-subgroup of F containing P. Here a p-group for the prime p is a group of arbitrary cardinality each of whose elements has order a finite power of p. Then a p-group is finite if and only if its order is a finite power of p. The locally finite group G is said to satisfy the *Sylow Theorem for the prime* p (or the *Sylow* p*-Theorem*) if the maximal p-subgroups of G are all conjugate in G and G satisfies the *strong Sylow Theorem for the prime* p if every subgroup of G satisfies the Sylow Theorem for the prime p. Kegel's lectures present the basics of Sylow theory in locally finite groups, give an overview of the work of Brian Hartley and Andrew Rae on Sylow theory in locally p-soluble groups, and reveal in great detail the normal structure for groups satisfying the strong Sylow Theorem for the prime p in the general case (for $p \geqslant 5$). Chapters 2 and 4 of [3] give a good overview as well but without appreciating Hartley's, Rae's and Kegel's fundamental papers properly and avoiding all their beautiful details.

In this publication we turn Kegel's question into a theorem: *If every subgroup* S *of the locally finite group* G *contains a finite* p*-subgroup which is singular in* S*, then* G *satisfies the strong Sylow Theorem for the prime* p*.* Since the converse is also true (see [4] and [10]), this characterises the locally finite groups which satisfy the strong Sylow Theorem for the prime p. The proof of our *Charakterisierungssatz* is not presented in its original form since it was written in German as the main result of the author's Diplomarbeit during 1978–1984 (see [4]). We decided against a presentation (for historical reasons) as an amalgam of English and German and translated all employed parts into English, thereby introducing a large number of corrections and embellishments, in particular Theorem 3.6.

The central discovery that enabled in those days the proof was the relationship of p-subgroups which are singular to the *good* p*-subgroups* (see [12]) and the *strongly local* p*-subgroups* (see [13]) of Andrew Rae. Let G be any locally finite group and let P be a p-subgroup of G. A *local system for* G is a family Σ of finite subgroups such that every element of G lies in a Σ-group and for every two Σ-groups there exists another Σ-group which contains both, for example, the local system of all finite subgroups of G. The p-group P is said to *reduce into a local system* Σ *for* G if for every Σ-group U we have that $P \cap U$ is a Sylow p-subgroup of U, and then P is a maximal p-subgroup of G (see below), P is said to be *good* if there exists a local system for G into which P reduces, and P is said to be *strongly local* or, as we prefer to say, *very good* if given any local system Σ for G there exists a subsystem of Σ into which P reduces. A very good p-subgroup is of course good, and, as we show below, any singular p-subgroup P of a locally finite group G is contained in a unique maximal p-subgroup of G which is very good and the existence of P enforces the conjugacy of the good Sylow p-subgroups in countable locally finite groups

We have the ambition to present not only our own results but also important known results to offer some context and a unified depiction. So when we refer to [4] it does not always mean (although it almost always means) that we present research results of ourselves.

2 Good Sylow p-subgroups and p-uniqueness subgroups

A maximal p-subgroup of a locally finite group G is called here a *Sylow* p*-subgroup* of G and we denote the set of all Sylow p-subgroups of G by $\mathrm{Syl}_p G$. If a p-subgroup of a locally finite group G reduces into a local system for G, it is a maximal p-subgroup.

Lemma 2.1 (see [4]) *Let* p *be a prime and let* P *be a* p*-subgroup of a locally finite group* G*. If there exists a local system* Σ *for* G *into which* P *reduces, then* P *is a Sylow* p*-subgroup of* G*.*

Proof — Let $S \in \mathrm{Syl}_p G$ with $P \leqslant S$. Suppose, $P \neq S$. Then there exists an element $x \in S \backslash P$. Let $U \in \Sigma$ with $x \in U$. It follows that $\langle P \cap U, x\rangle$ is a p-subgroup of U with $P \cap U < \langle P \cap U, x\rangle \leqslant S$. This contradicts the prerequisite $P \cap U \in \mathrm{Syl}_p U$. □

Notice that the above result is proved in [3], Lemma 2.2.10, only for nested local systems and in a more complicated way. The local system Σ for the locally finite group G is said to be *nested* (in German *geschachtelt*) if there is a sequence $\{U_n \mid n \in \mathbb{N}\}$ of finite subgroups of G such that $U_n \leqslant U_{n+1}$ for all $n \in \mathbb{N}$ and $\Sigma = \{U_n \mid n \in \mathbb{N}\}$. If G is a countable locally finite group and $\{x_n \mid n \in \mathbb{N}\}$ an enumeration of G, let $U_n := \langle x_1, x_2, \dots, x_n\rangle$ $(n \in \mathbb{N})$. Then $\{U_n \mid n \in \mathbb{N}\}$ is a nested local system for G. If the locally finite group G has a nested local system, then G is countable. We can identify all the good Sylow p-subgroups of countable locally finite groups by means of nested local systems for them.

Lemma 2.2 (see [4]) *Let* G *be a countable locally finite group.*

a) *If* Σ *is a local system for* G*, then* Σ *contains a local subsystem* Σ_1 *which is nested.*

b) *Let* $\Sigma = \{U_n \mid n \in \mathbb{N}\}$ *be a nested local system for* G*. Then there exist with respect to (w.r.t.)* Σ *good Sylow* p*-subgroups of* G*. In particular,* G *contains at least one good Sylow* p*-subgroup.*

Proof — a) Let Σ be a local system for G and $\{x_n \mid n \in \mathbb{N}\}$ an enumeration of G. For $x, y \in G$, we define $U_x \in \Sigma$ with $x \in U_x$ and $\langle U_x, U_y \rangle \leqslant U_{xy}$ as follows: let $U_{x_1} \in \Sigma$ with $x_1 \in U_{x_1}$; if subgroups $U_{x_1\,x_2\,x_3\,\dots\,x_n} \in \Sigma$ are already defined with

$$x_1, x_2, x_3, \dots, x_n \in U_{x_1\,x_2\,x_3\,\dots\,x_n} \ (n \in \mathbb{N}),$$

let $U_{x_{n+1}} \in \Sigma$ with $x_{n+1} \in U_{x_{n+1}}$ and $U_{x_1\,x_2\,x_3\,\dots\,x_n\,x_{n+1}} \in \Sigma$ with

$$\langle U_{x_1\,x_2\,x_3\,\dots\,x_n}, U_{x_{n+1}} \rangle \leqslant U_{x_1\,x_2\,x_3\,\dots\,x_n\,x_{n+1}} \ (n \in \mathbb{N}).$$

Then the countable subset $\Sigma_1 := \{U_{x_1\,x_2\,x_3\,\dots\,x_n} \mid n \in \mathbb{N}\}$ of Σ is a nested local system for G.

b) Let $P_1 \in \mathrm{Syl}_p U_1$. If

$$P_1 \leqslant P_2 \leqslant \dots \leqslant P_n$$

are already finite p-subgroups of G with $P_i \in \mathrm{Syl}_p U_i$ $(1 \leqslant i \leqslant n)$, let $P_{n+1} \in \mathrm{Syl}_p U_{n+1}$ with $P_n \leqslant P_{n+1}$ $(n \in \mathbb{N})$. Define $S := \bigcup_n P_n$. Then S is a p-subgroup of G, which reduces into Σ, and so is good with $S \in Syl_p G$ by Lemma 2.1. □

Another argument for proving Lemma 2.2 b) comes from Kegel's Lemma 1.1 of [10] and is very similar to that of Lemma 2.1. Note also that Lemmata 2.1 and 2.2 a) are (and were) well-known but we presented slick improved proofs and did not find Lemma 2.2 a) in the literature. For Lemma 2.2 b) see also [12], 1.11.

We can now introduce the p-uniqueness subgroups and present the close relationship between them and the good Sylow p-subgroups.

In [4] we call p-*dominant* a p-subgroup of the locally finite group G if it is finite and is contained in a unique Sylow p-subgroup S of G, and call then S *singular* (in German *einzigartig* or *einmalig* or *singulär*, in a double sense). Although "dominant" in German is "dominant" in English we now find it smarter to define such a p-subgroup of G as a p-*uniqueness subgroup* (in German, quite a bit unwieldy, p-*Einzigartigkeitsuntergruppe* or p-*Einmaligkeitsunterguppe*) *of* G *for* S or *w.r.t.* S. We observe that there is no danger of confounding our p-uniqueness subgroups with the p-uniqueness subgroups which play a major role in the classification of the finite simple groups (see page 82 of [5]).

Proposition 2.3 *Let* G *be a locally finite group and let* p *be a prime. Let* P *be a finite* p*-subgroup of* G*. The following properties are equivalent:*

1) P *is a* p*-uniqueness subgroup of* G*.*

2) P *is singular in* G*.*

3) *Whenever* P_1 *and* P_2 *are finite* p*-subgroups of* G *with* $P \leqslant P_1 \cap P_2$ *then* $\langle P_1, P_2 \rangle$ *is a* p*-group.*

Proof — 1) $\Rightarrow$ 2) Suppose P is not singular in G. Then we have a finite subgroup F of G such that P is contained in at least two Sylow p-subgroups P_1 and P_2 of F. Let S_i be a Sylow p-subgroup of G which contains P_i $(i = 1, 2)$. If $S_1 = S_2$ then $\langle P_1, P_2 \rangle \leqslant \langle S_1, S_2 \rangle \cap F$ is a p-group which contradicts $P_1 \in \mathrm{Syl}_p F$ and $P_2 \in \mathrm{Syl}_p F$. Thus $S_1 \neq S_2$. Therefore P is not a p-uniqueness subgroup of G.

2) $\Rightarrow$ 3) Let $P \leqslant P_1 \cap P_2$ where P_1 and P_2 are finite p-subgroups of G and suppose that $F := \langle P_1, P_2 \rangle$ is not a p-group. Then $P \leqslant F$ and since $\langle P_1, P_2 \rangle$ is not a p-group there are two distinct Sylow p-subgroups Q_1 and Q_2 of F containing P_1 and P_2, respectively. But then $P \leqslant Q_1 \cap Q_2$ and so P is not singular in G.

3) $\Rightarrow$ 1) Suppose that 3) holds and that P is not a p-uniqueness subgroup of G. Then there are distinct Sylow p-subgroups Q_1 and Q_2 of G such that $P \leqslant Q_1 \cap Q_2$. Let $x \in Q_1 \setminus Q_2$ and $y \in Q_2 \setminus Q_1$. It follows that $P_1 := \langle P, x \rangle$ and $P_2 := \langle P, y \rangle$ are finite p-groups and that $\langle P_1, P_2 \rangle$ is not a p-group, contradicting 3). □

Kegel discovered insight gaining equivalent conditions for the conjugacy of good Sylow p-subgroups in countable locally finite groups. We expandedly restate and improvedly reprove his result in our terminology thereby adding the property of the existence of a p-uniqueness subgroup. We also notice hat Kegel's argument for 2) $\Rightarrow$ 4) on page 6 and following of [10] is really not fully convincing.

Theorem 2.4 (see [10], Theorem 1.2) *For the countable locally finite group G and the prime p the following properties are equivalent:*

1) *There exists a nested local system $\{G_i \mid i \in \mathbb{N}\}$ for G and an index i_0 such that for every pair $j \geqslant i \geqslant i_0$ of indices every Sylow p-subgroup P_i of G_i lies in a unique Sylow p-subgroup P_j of G_j.*

2) *There exists a finite p-subgroup P_0 of G which is singular in G.*

3) *There exists a p-uniqueness subgroup P_0 of G.*

4) *Any two good Sylow p-subgroups of G are conjugate in G.*

Proof — 1) $\Rightarrow$ 2) Choose $P_{i_0} \in \mathrm{Syl}_p G_{i_0}$ and put $P_0 := P_{i_0}$. Let F be any finite subgroup of G containing P_0. For every index j such that $F \leqslant G_j$, the unique Sylow p-subgroup of G_j containing P_0 must contain a Sylow p-subgroup of F, and no other Sylow p-subgroup of F can contain P_0. Clearly 2) $\Rightarrow$ 1). From Proposition 2.3 follow 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 2). To show 4) $\Rightarrow$ 1) assume that for any nested local system $\{G_i \mid i \in \mathbb{N}\}$ for G and any index i_0, there are infinitely many pairs $j \geqslant i \geqslant i_0$ of indices for which some (and hence any by conjugation) Sylow p-subgroup of G_i is contained in at least two Sylow p-subgroups of G_j. We then can construct, similar to Theorem 3.2 or Theorem 3.8 below, $2^{\aleph_0}$ maximal p-subgroups of G which are good by Lemma 2.2 and cannot all be conjugate in G. Thus 4) entails 1), and hence 2). It remains to show 3) $\Rightarrow$ 4). Let P and Q be good Sylow p-subgroups of G obtained as two unions of Sylow p-subgroups of nested local systems $\{G_i \mid i \in \mathbb{N}\}$ and $\{H_i \mid i \in \mathbb{N}\}$ for G (see Lemma 2.2) and let S_0 be the unique Sylow p-subgroup of G containing P_0; we show that P is conjugate to S_0 and S_0 is conjugate to Q, and therefore P is conjugate to Q; if P and S_0 are not conjugate then one of them must have property ($\star$) of Theorem 3.1 (see below) which means in particular that it is not singular; so P has property ($\star$); now P reduces into $\{G_i \mid i \in \mathbb{N}\}$, that is, $P \cap G_i \in \mathrm{Syl}_p G_i$ for all $i \in \mathbb{N}$; there exists an index i_0 such that $P_0 \leqslant G_{i_0}$; then $P_0 \leqslant P_{i_0}$ for some unique $P_{i_0} \in \mathrm{Syl}_p G_{i_0}$; now, by Sylow's classical theorem, let x be an element of G_{i_0} such that $P_{i_0}^x = P \cap G_{i_0}$; then $P_{i_0}^x$ is a finite p-subgroup of P which is contained in just only one Sylow p-subgroup of G thereby contradicting property ($\star$) of P; for exactly the same reasons S_0 is conjugate to Q; therefore P must be conjugate to Q. □

Let S be a Sylow p-subgroup of the locally finite group G. A finite subgroup F of G is called *S-dominant* if S reduces into every subgroup U of G which contains F, that is, $S \cap U \in \mathrm{Syl}_p U$ for all subgroups U of G such that $F \leqslant U$.

Lemma 2.5 (see [4]) *Let* G *be a locally finite group,* p *a prime,* $S \in \mathrm{Syl}_p G$ *and* F *a finite subgroup of* G*. The following properties are equivalent:*

1) F *is* S*-dominant.*

2) *For each finite subgroup* U *of* G *with* $F \leqslant U$ *we have* $S \cap U \in \mathrm{Syl}_p U$*.*

Proof — 1) ⇒ 2) is clear, so we only need to prove that 2) implies 1). Since F is finite, there exists a local system Σ for G such that for each Σ-group U we have $F \leqslant U$. Let V be a subgroup of G with $F \leqslant V$. Then $\Sigma_1 := \{V \cap U \mid U \in \Sigma\}$ is a local system for V into which $S \cap V$ reduces. Therefore from Lemma 2.1 follows $S \cap V \in \mathrm{Syl}_p V$. □

Lemma 2.6 (see [4]) *Let* G *be a locally finite group and* $S \in \mathrm{Syl}_p G$*. The following properties are equivalent:*

1) S *is very good.*

2) *There exists an* S*-dominant subgroup of* G*.*

Proof — 1) ⇒ 2) Suppose no S-dominant subgroup of G exists. Then, according to Lemma 2.5, to every finite subgroup F of G there exists one finite subgroup U_F of G with $F \leqslant U_F$ and $S \cap U_F \notin \mathrm{Syl}_p U_F$. Then $\Sigma := \{U_F \mid F \text{ finite subgroup of } G\}$ is a local system for G that possesses no local subsystem into which S reduces.

2) ⇒ 1) Let F be an S-dominant subgroup of G and Σ a local system for G. Let $\Sigma_1 := \{U \mid U \in \Sigma \text{ and } F \leqslant U\}$. Then Σ_1 is, because of the S-dominance of F, a local subsystem of Σ into which S reduces. □

Lemma 2.7 (see [4]) *Let* G *be a locally finite group and let* p *be a prime.*

a) *If* F *is a* p*-uniqueness subgroup of* G *and* S *is the singular Sylow* p*-subgroup of* G *with* $F \leqslant S$*, then* F *is an* S*-dominant subgroup of* G*.*

b) *Every singular Sylow* p*-subgroup of* G *is very good.*

Proof — Since b) follows from a) and Lemma 2.6 we only need to prove a). Let U be a subgroup of G with $F \leqslant U$. Let $P \in \mathrm{Syl}_p U$ and $T \in \mathrm{Syl}_p G$ with $F \leqslant S \cap U \leqslant P \leqslant T$. From $F \leqslant S$ and the p-uniqueness of F follows $T = S$. Therefore $S \cap U \geqslant S \cap P = P$. □

The following consequence of this lemma is a relevant insight.

Theorem 2.8 (see [4]) *Let* p *be a prime and* P *be a* p*-uniqueness subgroup of the locally finite group* G *(or, equivalently by Proposition 2.3, let* P *be a singular* p*-subgroup of* G*). Then the singular Sylow* p*-subgroup* S *of* G *containing* P *is very good.*

We can now summarise the relationship between good Sylow p-subgroups and p-uniqueness subgroups together with the Sylow p-subgroups containing them as follows:

- singular Sylow p-subgroups are very good;
- p-uniqueness subgroups are singular, and conversely;
- in countable locally finite groups good Sylow p-subgroups are identified by nested local systems;
- in countable locally finite groups the existence of a p-uniqueness subgroup compels the conjugacy of all good Sylow p-subgroups.

We end the discussion of good Sylow p-subgroups by pointing out that there exist 1) countable locally finite groups with Sylow p-subgroups which are not good (see the note at page 5 of [10]: "It may be worthwhile to point out that a countable infinite locally finite group may have maximal p-subgroups which" are not good) and 2) locally finite groups of cardinality $2^{\aleph_0}$ without good Sylow p-subgroups.

First, we let G be a finite group with $|\mathrm{Syl}_p G| \geqslant 2$, e.g. the symmetric group $\underline{S}^{2p}$ of degree 2p for the prime p for which we know surely that

$$|\mathrm{Syl}_p \underline{S}^{2p}| \geqslant 2p - 2 \geqslant 2.$$

Consider the $\mathbb{N}$*-fold cartesian power*

$$\begin{aligned} G^{[\mathbb{N}]} &:= \prod\{G_i \mid G_i := G \ \text{ for all } \ i \in \mathbb{N}\} \\ &= \{(x_1, x_2, \dots) \mid x_i \in G_i \ \text{ for all } \ i \in \mathbb{N}\} \end{aligned}$$

of G and notice that *it satisfies the Sylow* p*-Theorem*.

Proof — For $S, T \in \mathrm{Syl}_p G^{[\mathbb{N}]}$ there are $S_i, T_i \in \mathrm{Syl}_p G_i = \mathrm{Syl}_p G$ $(i \in \mathbb{N})$ such that S, resp. T, is the cartesian product of the S_i's, resp. the T_i's. If $x_i \in G_i = G$ with $S_i^{x_i} = T_i$ $(i \in \mathbb{N})$ and $x := (x_i)_{i \in \mathbb{N}}$, then $S^x = T$. □

The group $G^{[\mathbb{N}]}$ contains the $\mathbb{N}$*-fold direct power*

$$G^{(\mathbb{N})} := \prod{}^0 \{(x_i)_{i \in \mathbb{N}} \in G^{[\mathbb{N}]} \mid x_i = 1 \text{ for almost all } i \in \mathbb{N}\},$$

which *does not satisfy the Sylow* p*-Theorem*.

PROOF — Let $S, T \in \mathrm{Syl}_p G^{(\mathbb{N})}$. If there is an $x \in G^{(\mathbb{N})}$ with $S^x = T$, then $S^{x\pi_i} = T^{\pi_i}$ for almost all $i \in \mathbb{N}$. Thus for $P, Q \in \mathrm{Syl}_p G$ with $P \neq Q$, the groups $P^{(\mathbb{N})}$ and $Q^{(\mathbb{N})}$ are not in $G^{(\mathbb{N})}$ — but in $G^{[\mathbb{N}]}$ — conjugate Sylow p-subgroups of $G^{(\mathbb{N})}$. Alternatively, it follows from $|G^{(\mathbb{N})}| = \aleph_0$ and $|\mathrm{Syl}_p G^{(\mathbb{N})}| = 2^{\aleph_0}$ — since $|\mathrm{Syl}_p G| \geqslant 2$ we can refer to Theorems 3.1 and 3.2 (see below) — that not all Sylow p-subgroups of $G^{(\mathbb{N})}$ can be conjugate. □

The example $G^{(\mathbb{N})} \leqslant G^{[\mathbb{N}]}$ shows that in uncountable locally finite groups the Sylow p-Theorem is not inherited by normal subgroups.

Moreover, $G^{[\mathbb{N}]}$ contains *the diagonal subgroup*

$$D := \big\{(x_i)_{i\in\mathbb{N}} \in G^{[\mathbb{N}]} \mid (\exists\, x \in G)(\forall\, i \in \mathbb{N})\, x_i = x\big\} \simeq G$$

via the isomorphism

$$\delta : D \longrightarrow G, \quad \big((x_i)_{i\in\mathbb{N}}\big)^{\delta} := x,$$

from D onto G with $D \cap G^{(\mathbb{N})} = \langle 1 \rangle$. Since $G^{(\mathbb{N})}$ is a normal subgroup of $G^{[\mathbb{N}]}$, we have $\langle G^{(\mathbb{N})}, D\rangle = DG^{(\mathbb{N})}$; this is a countable subgroup of $G^{[\mathbb{N}]}$. The Sylow p-subgroups of $G^{[\mathbb{N}]}$ (resp. of $G^{(\mathbb{N})}$) are cartesian (resp. direct) products of the Sylow p-subgroups of the G_i's ($i \in \mathbb{N}$), namely $\prod\{S_n^{\pi_i} \mid i \in \mathbb{N}\}$ $\big($resp. $\prod^0\{S_n^{\pi_i} \mid i \in \mathbb{N}\}\big)$ for $S_n \in \mathrm{Syl}_p G$ ($n \in \mathbb{N}$), where π_i: $G^{[\mathbb{N}]} \to G_i$ is the projection $\pi_i\big((x_k)_{k\in\mathbb{N}}\big) := x_i$ on the factor G_i ($i \in \mathbb{N}$). Any $P \in \mathrm{Syl}_p D$ normalises exactly one Sylow p-subgroup $S(P)$ of $G^{[\mathbb{N}]}$ (resp. exactly one Sylow p-subgroup $S^0(P)$ of $G^{(\mathbb{N})}$), namely $S(P) = \prod\{P^{\pi_i} \mid i \in \mathbb{N}\}$ $\big($resp. $S^0(P) = \prod^0\{P^{\pi_i} \mid i \in \mathbb{N}\}\big)$. Therefore every Sylow p-subgroup of D is a p-uniqueness subgroup of $DG^{(\mathbb{N})}$, and $P\,S^0(P)$, for $P \in \mathrm{Syl}_p D \simeq \mathrm{Syl}_p G$, is a singular Sylow p-subgroup of $DG^{(\mathbb{N})}$ and so is good, even very good, by Theorem 2.8; these Sylow p-subgroups are conjugate: if $P_1, P_2 \in \mathrm{Syl}_p D$ and $P_1^x = P_2$ with $x \in D$, then

$$\begin{aligned}\big(P_1\, S^0(P_1)\big)^x &= \big(P_1 \textstyle\prod^0\{P_1^{\pi_i} \mid i \in \mathbb{N}\}\big)^x \\ &= P_2 \textstyle\prod^0\{P_2^{\pi_i} \mid i \in \mathbb{N}\} = P_2\, S^0(P_2)\end{aligned}$$

(see also Theorem 2.4). The countable group $G^{(\mathbb{N})}$ also has by Lemma 2.2 good Sylow p-subgroups, which are not conjugate, and we are able to designate some distinguished of them explicitly: let

$$U_i := G_1 \times G_2 \times \ldots \times G_i \quad (i \in \mathbb{N});$$

then $\Sigma := \{U_i \mid i \in \mathbb{N}\} \cap G^{(\mathbb{N})}$ is a nested local system for $G^{(\mathbb{N})}$; if $P_i \in \mathrm{Syl}_p G_i = \mathrm{Syl}_p G$ $(i \in \mathbb{N})$, then

$$P^0 := \big(P_1 \times P_2 \times \ldots\big) \cap G^{(\mathbb{N})}$$

is a p-subgroup of $G^{(\mathbb{N})}$ which reduces into Σ and thus is a good Sylow p-subgroup of $G^{(\mathbb{N})}$ by Lemma 2.1.

The group $DG^{(\mathbb{N})}$ has indeed also (many) Sylow p-subgroups, which are not good: since $|\mathrm{Syl}_p G| \geqslant 2$ we can construct using the method employed in the proof of Theorem 3.2 or that employed in the proof of Theorem 3.8 an infinitely ($\aleph_0$) high tree of finite p-subgroups of $DG^{(\mathbb{N})}$ with $\langle 1 \rangle$ as the root which branches properly at each location with proper inclusions and where two immediate successors of each point do not generate a p-group; this tree has $2^{\aleph_0}$ branches which constitute $2^{\aleph_0}$ many ascending unions of finite p-subgroups and thus $2^{\aleph_0}$ many p-subgroups P_ι where any two of them do not generate a p-group; choosing for each P_ι a Sylow p-subgroup S_ι of $DG^{(\mathbb{N})}$ containing P_ι now gives $2^{\aleph_0}$ Sylow p-subgroups of $DG^{(\mathbb{N})}$ $(1 \leqslant \iota \leqslant 2^{\aleph_0})$ on the treetop; since the good Sylow p-subgroups of the countable group $DG^{(\mathbb{N})}$ are conjugate (Theorem 2.4), at most $\aleph_0$ of these $2^{\aleph_0}$ Sylow p-subgroups can be good; there remain (with or without the continuum hypothesis) at least $2^{\aleph_0} - \aleph_0$ many Sylow p-subgroups in the treetop which are not good and too many to be conjugate in $DG^{(\mathbb{N})}$. We note that Rae [12] constructs, by introducing the unwieldy concept of "weakly goodness" and by referring to another group he constructed (see [12], 5.11), a countable locally soluble group possessing a Sylow p-subgroup which is not good (see [12], 5.31). This example is much more complicated than ours.

Second, let p and q be primes with $q \equiv 1 \pmod p$ and

$$A := \langle a, b \mid a^p = b^q = (ab)^p = 1 \rangle.$$

Then $|A| = pq$ and A has q Sylow p-subgroups and a normal Sylow q-subgroup, so is metabelian. If $(p, q) = (2, 3)$, then $A = \underline{S}^3$ is the symmetric group of degree 3. The group A contains the elements a and $a' := ab$ of order p which are not p-*consonant*, that is, they do not generate a p-group. The $\mathbb{N}$-fold cartesian power $A^{[\mathbb{N}]}$ of A is locally finite and metabelian of exponent pq. László G. Kovács, Bernhard H. Neumann and Hugo de Vries constructed, based on the elements a and a' (and exemplarily for $(p, q) = (2, 3)$), an $\mathbb{N}$-fold interdirect power H of A, that is, $A^{(\mathbb{N})} \leqslant H \leqslant A^{[\mathbb{N}]}$, with the following properties (see [11], Theorem 3.7): H is metabelian of exponent q and order $2^{\aleph_0}$ with a countable Sylow p-subgroup and a Sylow p-subgroup of order $2^{\aleph_0}$ (hence without Sylow Theorem for the prime p). They also constructed, using again a and a', an $\mathbb{N}$-fold interdirect power H of A with the following amazing properties (see [11], Theorem 4.4, and also [12], 1.13): H has order $2^{\aleph_0}$, each Sylow p-subgroup of H is countable, H has a countable normal (hence unique) Sylow q-subgroup, which has no complement in H, and each Sylow p-subgroup has a complement in H, which is normal in H and contains elements of order p. No Sylow p-subgroup of H can be good: suppose a Sylow p-subgroup S of H reduces into a local system Σ for H; we then choose a Σ-group U containing an element x of order p of a complement of S, and a $P \in \mathrm{Syl}_p U$ containing x; since $S \cap U \in \mathrm{Syl}_p U$ there is a $y \in U$ with $P^y = S \cap U$; then $\langle x \rangle^y \leqslant S$ whereas $\langle x \rangle^y$ belongs to the normal complement of S, which is a contradiction.

In the following section we shall point out that there exist countable locally finite groups 3) without singular Sylow p-subgroups, 4) with good Sylow p-subgroups which are not very good, and 5) with very good Sylow p-subgroups which are not singular.

3 Basic theorems of Sylow theory in locally finite groups and our "Charakterisierungssatz"

In this section we first present — with quite considerably improved proofs — the basics of Sylow theory in locally finite groups (Theorem 3.1 to Theorem 3.5) and subsequently prepare and carry out the proof of our *Charakterisierungssatz* (Theorem 3.6 to Theorem 3.9) which, in turns, allows us to prove very easily our main theorem (Theorem 3.10).

In the following statement, the property $(\star)$ means that S is not singular; see the same property $(*)$ on page 8 of [10]. This property was for the first time discovered by Ali O. Asar [1].

Theorem 3.1 (see [4], and Theorem 3.6 below for a generalisation) *Any locally finite group G which does not satisfy the Sylow Theorem for the prime p contains a Sylow p-subgroup S with the following property:*

($\star$) *Every finite subgroup of S lies in at least two Sylow p-subgroups of G.*

Proof — Let S and T be two Sylow p-subgroups of G which are not conjugate (in G). If T is not singular, that is, T does have property ($\star$), the result is immediate, so suppose that T is singular and let Y be a p-uniqueness subgroup for T. We show that then S has property ($\star$), that is, S is not singular. To this end let X be an arbitrary finite subgroup of S. Then $\langle X, Y\rangle$ is a finite group. According to the Sylow p-Theorem for finite groups there is an $x \in G$ such that X and Y^x lie in the same Sylow p-subgroup of $\langle X, Y\rangle$. Then $\langle X, Y^x\rangle$ is a p-group. From the assumption on Y it now follows that $\langle Y^x, X\rangle \leqslant T^x$. Hence X lies in at least the two Sylow p-subgroups S and T^x of G. Therefore X is not a p-uniqueness subgroup for S. □

We now prepare an alternative proof of the basic theorem of Sylow theory known as the "Asar-Hartley theorem" (see [1] and [3], Theorem 2.3.11, for the original proof). Our proofs of Theorem 3.2 a) and b) with reference to a) are much clearer and more detailed than the original proof by Asar, which may be considered rather cumbersome. Note also that in [10], Theorem 1.3, Kegel sagely combines Theorem 3.1 with Theorem 3.2 c).

Theorem 3.2 (see [4]) *Let G be a locally finite group and let P be a p-subgroup of G for the prime p.*

a) *Suppose P has the following property: ($\dagger$) To every finite subgroup F of P there exists an $x = x\,(F) \in G$ with $F^x \leqslant P$ such that $\langle P, P^x\rangle$ is not a p-group. Then there are $2^{\aleph_0}$ infinite ascending chains*

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

of finite p-subgroups of G with indices $i_k \in \{0, 1\}$ $(k \in \mathbb{N})$ such that for all $n \in \mathbb{N}$ and each choice of indices i_k $(1 \leqslant k \leqslant n)$, the group $\langle X_{i_1 i_2 \ldots i_n 0}, X_{i_1 i_2 \ldots i_n 1}\rangle$ is not a p-group.

b) *Let $P \in \mathrm{Syl}_p G$ with the property ($\star$). Then P has property ($\dagger$).*

c) *Let* $P \in \mathrm{Syl}_p G$ *with the property* $(\star)$ *and let* X *be a finite subgroup of* P. *Then there are* $2^{\aleph_0}$ *many infinite ascending chains*

$$X < X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

with the properties from point a).

Proof — a) Let X be a finite subgroup of P and y an element of G such that: 1) $\langle P, P^y \rangle$ is not a p-group, and 2) $X^y \leqslant P$. Because of the first property there exists a finite subgroup X_0 of P with $X \leqslant X_0$ such that $\langle X_0, X_0^y \rangle$ is not a p-group, and because of the second property we have $\langle X, X^y \rangle \leqslant P$, hence $X_0 \neq X \neq X_0^y$. If we substitute in the last two sentences X by X_0, we get two finite p-subgroups X_{00} and X_{01} of G with $X_0 < X_{00}$ and $X_0 < X_{01}$ such that $\langle X_{00}, X_{01} \rangle$ is not a p-group. Since P^y has the property $(\dagger)$, too, we can quite analogously substitute X by $X_1 := X_0^y$ and so get two finite p-subgroups X_{10} and X_{11} of G with $X_1 < X_{10}$ and $X_1 < X_{11}$ such that the subgroup $\langle X_{10}, X_{11} \rangle$ is not a p-group. We now have constructed four ascending chains

$$X < X_0 < X_{00}, \quad X < X_0 < X_{01}, \quad X_1 < X_{10} \quad \text{and} \quad X_1 < X_{11}$$

of finite p-subgroups of G such that the subgroups $\langle X_0, X_1 \rangle, \langle X_{00}, X_{01} \rangle$ and $\langle X_{10}, X_{11} \rangle$ are not p-groups. Now let $n \in \mathbb{N}$ with $n \geqslant 2$ and let already be constructed 2^n ascending chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n}$$

of finite p-subgroups of G with indices $i_k \in \{0, 1\}$ $(1 \leqslant k \leqslant n)$ such that for each $m \in \mathbb{N}$ with $m \leqslant n-1$ and each choice of indices i_k $(1 \leqslant k \leqslant m)$ the subgroup $\langle X_{i_1 i_2 \ldots i_m 0}, X_{i_1 i_2 \ldots i_m 1} \rangle$ of G is not a p-group. Whilst repeating the construction of the first two sentences successively with the 2^n groups $X_{i_1 i_2 \ldots i_n}$ in place of X, we get, because each conjugate of P possesses the property $(\dagger)$, in each case two p-subgroups $X_{i_1 i_2 \ldots i_n 0}$ and $X_{i_1 i_2 \ldots i_n 1}$ of G such that

$$X_{i_1 i_2 \ldots i_n} < X_{i_1 i_2 \ldots i_n 0} \cap X_{i_1 i_2 \ldots i_n 1}$$

and

$$\langle X_{i_1 i_2 \ldots i_n 0}, X_{i_1 i_2 \ldots i_n 1} \rangle$$

is not a p-group. Therewith we now have constructed 2^{n+1} ascending chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < X_{i_1 i_2 \ldots i_n i_{n+1}}$$

having the requested properties. Therefore we can w.r.t. inclusion recursively construct a tree of height $\aleph_0$ of finite p-subgroups of G, which branches properly at each location with proper inclusions, hence must contain $2^{\aleph_0}$ infinite branches. Also any two immediate successors of an arbitrary point do not generate a p-group. These branches are just the required chains.

b) Let F be a finite subgroup of P and R be a Sylow p-subgroup of G with $F \leqslant R \neq P$. Then there is an element x in R with $x \notin P$ and the group $\langle F, x\rangle$ is a finite p-group. Let $Y := \langle F, x\rangle \cap P$. Then we have $Y \neq \langle F, x\rangle$. It is well-known that as a finite p-group $\langle F, x\rangle$ satisfies the normaliser condition. Therefore Y is a proper subgroup of $N_{\langle F,x\rangle}(Y)$. Let y be an element in $\langle F, x\rangle$, but not in Y, which normalises Y. Then $y \notin P$. Since y is a p-element and P by assumption a Sylow p-subgroup of G, it follows that $y \notin N_G(P)$ and that $\langle P, P^y\rangle$ is not a p-group. This is the property (†) from point a) for P.*

c) We combine the proofs of point a) and point b). Let $R \in \mathrm{Syl}_p G$ with $X \leqslant R \neq P$, $x \in R \backslash P$ and $T := P \cap \langle X, x\rangle$. Being a finite p-group, $\langle X, x\rangle$ satisfies the normaliser condition. Hence there exists a $t \in \langle X, x\rangle \backslash T$ with $t \in N_{\langle X,x\rangle}(T)$. Then $\langle P, P^t\rangle$ is not a p-group, since else $t \in P$, and so there exists a finite subgroup X_0 of P with $X \leqslant X_0$ such that with $X_1 := X_0^t$ the group $\langle X_0, X_1\rangle$ is not a p-group. Thus, we have $X_0 \neq X \neq X_1$ since $\langle X, X^t\rangle \leqslant T$ is a p-group. Of course, $X \leqslant X_0$, but also $X \leqslant X_1$ because of $t \in X$. We can repeat this construction whilst replacing X by X_0 and also by its conjugate X_1. Thereby we construct subgroups $X_{00}, X_{01}, X_{10}, X_{11}$ and four ascending chains

$$X < X_0 < X_{00},\ X < X_0 < X_{01},\ X < X_1 < X_{10} \text{ and } X < X_1 < X_{11}$$

of finite p-subgroups of G. We subsequently repeat this construction with each of the $X_{i_1 i_2}$'s and whilst doing this infinitely often we construct $2^{\aleph_0}$ many chains

$$X < X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

* Asar [1, Lemma 1] (unwieldy) considers instead of R a p-subgroup Y of G such that $Y < U$ $(= P)$, chooses $y \in Y \backslash U$, defines $F^* := U \cap \langle F, y\rangle$ with $F \leqslant U \cap Y$, finds $F \leqslant F^*$ and $N_{\langle F,\, y\rangle}(F^*) > F^*$, and finally concludes $N_G(F^*) < N_G(U)$, since U is the unique maximal p-subgroup of $N_G(U)$ and $N_{\langle F,\, y\rangle}(F^*) < U$.

of finite p-subgroups of G with the properties from point a). So we can, starting from an arbitrary subgroup X of P as a "minimal point" or a "root", recursively w.r.t. inclusion construct a tree of height $\aleph_0$ of finite p-subgroups of G, which branches properly at each location, hence must contain $2^{\aleph_0}$ infinite branches. Also any two immediate successors of an arbitrary point do not generate a p-group. These branches are just the required chains. □

Theorem 3.2 enables us to prove very easily the "Asar-Hartley theorem" which characterises locally finite groups satisfying the strong Sylow Theorem for the prime p by a cardinality result without the need to endeavour the continuum hypothesis (for a proof closer to the original one of Asar, the reader can consult [10], pp. 8–9).

Theorem 3.3 (see Asar [1], Hartley [6],[8][*]) *Let* G *be a locally finite group and* p *be a prime. Suppose that for every countable subgroup* H *of* G *we have* $|\mathrm{Syl}_p H| < 2^{\aleph_0}$. *Then* G *satisfies the strong Sylow* p*-Theorem.*

Proof — Suppose G does not satisfy the strong Sylow Theorem for the prime p. Then there is a subgroup U of G which does not satisfy the Sylow Theorem for the prime p. Thus according to Theorems 3.1 and 3.2 there are $2^{\aleph_0}$ many infinite ascending chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

of finite p-subgroups of U with the properties from point a) of Theorem 3.2. Let $\mathcal{M}$ be the set of all p-subgroups of U which are an ascending union of one of these chains. Then it follows $|\mathcal{M}| = 2^{\aleph_0}$ and that any two $\mathcal{M}$-groups cannot generate a p-group. Now let

$$H_n := \langle X_{i_1 i_2 \ldots i_n} \mid i_k \in \{0, 1\},\ 1 \leqslant k \leqslant n \rangle \quad (n \in \mathbb{N})$$

and

$$H := \bigcup_{n \in \mathbb{N}} H_n.$$

Then H is a countable subgroup of U and so of G. Since H contains every $\mathcal{M}$-group it follows that $|\mathrm{Syl}_p H| = 2^{\aleph_0}$. This contradicts the assumption on the countable subgroups of G. □

[*] The result for countable locally finite groups was obtained independently by Brian Hartley using a quite different method which allowed him to generalise it from the prime p to a set of primes π when the finite groups of a nested local system have each a nilpotent Hall π-subgroup (see [6]). However, Hartley has extended his proof in [8] to uncountable locally finite groups by another beautiful method.

The cardinality statement of Theorem 3.3 has an immediate first corollary for countable locally finite groups.

Theorem 3.4 *Let* G *be a countable locally finite group. The following properties are equivalent:*

1) *For every (countable) subgroup* H *of* G *we have* $|\mathrm{Syl}_p H| < 2^{\aleph_0}$.

2) G *satisfies the strong Sylow Theorem for the prime* p.

3) G *satisfies the Sylow Theorem for the prime* p.

4) $|\mathrm{Syl}_p G| < 2^{\aleph_0}$.

5) *Every (countable) subset of* G *is contained in a subgroup* U *of* G *with* $|\mathrm{Syl}_p U| < 2^{\aleph_0}$.

The second corollary of Theorem 3.3 would certainly as a conjugacy assertion be very difficult to be proved but is as a cardinality statement trivial. Recall first that a class of groups $\mathfrak{X}$ is *countably recognisable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group (see Baer [2]).

Theorem 3.5 *The locally finite group* G *satisfies the strong Sylow Theorem for the prime* p *if and only if every countable subgroup of* G *satisfies the strong Sylow Theorem for the prime* p. *In particular, the class* Syl-p *of all locally finite groups satisfying the strong Sylow Theorem for the prime* p *is countably recognisable.*

We now can prove our key discovery whenever the Sylow Theorem for the prime p is not valid in a countable locally finite group which shows a symmetry between not conjugate Sylow p-subgroups.

Theorem 3.6 *Let* G *be a countable locally finite group and* p *be a prime. If two Sylow* p*-subgroups of* G *are not conjugate, then neither is singular.*

Proof — Let S and T be Sylow p-subgroups of G which are not conjugate. We saw in Theorem 3.1 that one of S or T is not singular. Without loss of generality (w.l.o.g.) we may suppose that S is not singular. To prove the result we must show that T is not singular either. If T is not good, it cannot be singular, since by Theorem 2.8 singular Sylow p-subgroups are very good. So let T be good w.r.t. the nested local system $\{G_n \mid n \in \mathbb{N}\}$ for G and let F be an arbitrary finite subgroup of T. We show that F cannot be a p-uniqueness subgroup for T and so T is not singular since F is chosen arbitrarily. Since S and T are not conjugate, we have $S \neq T$.

There exists an $m = m(F) \in \mathbb{N}$ with $F \leqslant G_m$. After the renumeration $\{n \mapsto n+m-1 \mid n \in \mathbb{N}\}$, it is possible to assume $F \leqslant G_1$. Then $F \leqslant T \cap G_1 \in \mathrm{Syl}_p G_1$. If $T \cap G_n$ is the unique Sylow p-subgroup of G_n for all $n \in \mathbb{N}$ then T is the unique Sylow p-subgroup of G and we obtain the contradiction that $S = T$. Hence there is an $n \in \mathbb{N}$ such that G_n has a Sylow p-subgroup R with $R \neq T \cap G_n$. Renumbering again if needed we may assume that $R \in \mathrm{Syl}_p G_1$ with $R \neq T \cap G_1$. Choose $y \in R \backslash (T \cap G_1)$, so in particular $y \notin T$. By the Sylow p-Theorem for finite groups there is an $x \in G_1$ such that $(T \cap G_1)^x = R$ and so $F^x \leqslant R$ since $F \leqslant T \cap G_1$. From $\langle F^x, y\rangle \leqslant R$ follows that $\langle F^x, y\rangle$ is a finite p-group. Let $Y := \langle F^x, y\rangle \cap T$. Then $Y \neq \langle F^x, y\rangle$ since $y \notin T$.

But Y satisfies, as is well-known, the normaliser condition and so we can choose $z \in N_{\langle F^x, y\rangle}(Y) \backslash Y$. Then $z \notin T$ since otherwise z belongs to $T \cap \langle F^x, y\rangle = Y$. But z is a p-element outside of T and $T \in \mathrm{Syl}_p G$, and so $z \notin N_G(T)$. Therefore $\langle T, T^z\rangle$ is not a p-group. In particular, $T \neq T^z$ and $F \leqslant T \cap T^z$. Therefore the arbitrarily chosen F is not a p-uniqueness subgroup for T. □

Whenever a countable locally finite group contains a singular Sylow p-subgroup then all good Sylow p-subgroups will be conjugate by Theorem 2.4. Whenever every countable subgroup of a (countable) locally finite group contains a singular Sylow p-subgroup then all Sylow p-subgroups are conjugate. This core insight is spelled out by the following theorem.

Theorem 3.7 (see [4]) *Let* G *be a locally finite group and let* p *be a prime. Suppose that every countable subgroup of* G *contains a singular Sylow* p*-subgroup. Then* G *satisfies the strong Sylow Theorem for the prime* p.

Proof — According to Theorem 3.5 we can assume that G is countable, and according to Theorem 3.4 it suffices to show that G satisfies the Sylow Theorem for the prime p. However, this is now immediate since by assumption G has a singular Sylow p-subgroup S. Let T be any Sylow p-subgroup of G. If S and T are not conjugate, then by Theorem 3.6 neither is singular. With this contradiction S and T are conjugate and the result follows. □

Since the above result is very significant, we provide an alternative proof by proving the contrapositive.

Proof — Suppose G does not satisfy the Sylow Theorem for the prime p. Then, according to Theorem 3.1, Theorem 3.2 b), and Theorem 3.2 a), there are $2^{\aleph_0}$ infinite ascending chains

$$X_{i_1} < X_{i_1 i_2} < \ldots < X_{i_1 i_2 \ldots i_n} < \ldots$$

of finite p-subgroups of G with the properties from Theorem 3.2 a). Let

$$U_n := \langle X_{i_1 i_2 \ldots i_n} \mid i_k \in \{0,1\},\ 1 \leqslant k \leqslant n \rangle \quad (n \in \mathbb{N})$$

and

$$U := \bigcup_{n \in \mathbb{N}} U_n = \langle X_{i_1 i_2 \ldots i_n} \mid i_k \in \{0,1\},\ 1 \leqslant k \leqslant n \in \mathbb{N} \rangle.$$

Then U is a (countable) subgroup of G and $\{U_n \mid n \in \mathbb{N}\}$ is a nested local system for U. We show that U does not contain any singular Sylow p-subgroup. Let F^* be a finite p-subgroup of U. There exists an $m = m(F^*) \in \mathbb{N}$ with $F^* \leqslant U_m$. By definition of U_m there are indices $j_1, j_2, \ldots, j_m, \ldots, k_1, k_2, \ldots, k_m, \ldots, l_1, l_2, \ldots, l_m$ with

$$F^* \leqslant \langle X_{j_1 j_2 \ldots j_m}, X_{k_1 k_2, \ldots, k_m}, \ldots, X_{l_1 l_2 \ldots l_m} \rangle.$$

Then

$$P_1 := \langle X_{j_1 j_2 \ldots j_m 0}, X_{k_1 k_2 \ldots k_m 0}, \ldots, X_{l_1 l_2 \ldots l_m 0} \rangle$$

and

$$P_2 := \langle X_{j_1 j_2 \ldots j_m 1}, X_{k_1 k_2 \ldots k_m 1}, \ldots, X_{l_1 l_2 \ldots l_m 1} \rangle$$

are finite p-subgroups of U with $F^* \leqslant P_1 \cap P_2$ such that $\langle P_1, P_2 \rangle$ is not a p-group. We now choose $Q_{1,0}, Q_{2,0} \in \mathrm{Syl}_p U_m$ with $P_1 \leqslant Q_{1,0}$ and $P_2 \leqslant Q_{2,0}$. If

$$Q_{1,0} \leqslant Q_{1,1,} \leqslant \ldots \leqslant Q_{1,n} \quad \text{and} \quad Q_{2,0} \leqslant Q_{2,1} \leqslant \ldots \leqslant Q_{2,n}$$

are already p-subgroups of U with $Q_{1,i}, Q_{2,i} \in \mathrm{Syl}_p U_{m+i}$ $(0 \leqslant i \leqslant n)$, let $Q_{1,n+1}, Q_{2,n+1} \in \mathrm{Syl}_p U_{m+n+1}$ such that $Q_{1,n} \leqslant Q_{1,n+1}$ and $Q_{2,n} \leqslant Q_{2,n+1}$ $(n \in \mathbb{N}_0)$. Let

$$Q_1 := \bigcup_{n \in \mathbb{N}_0} Q_{1,n} \quad \text{and} \quad Q_2 := \bigcup_{n \in \mathbb{N}_0} Q_{2,n}.$$

Then Q_1 and Q_2 are both p-subgroups of U with $F^* \leqslant Q_1 \cap Q_2$ such that $\langle Q_1, Q_2\rangle$ is not a p-group. Per construction, Q_1 and Q_2 reduce into the nested local system $\{U_{m+n} \mid n \in \mathbb{N}_0\}$ for U. By Lemma 2.1, the groups Q_1 and Q_2 are two good Sylow p-subgroups of U containing F^*, that is, F^* is not a p-uniqueness subgroup of U. Thus U does not contain any p-uniqueness subgroup. □

Third, we supplement Theorem 3.7 with an example of a countable locally finite group H without the (strong) Sylow Theorem for the prime p but with a (countable) subgroup U without singular Sylow p-subgroups. Let $H := DG^{(\mathbb{N})}$ be the group from p. 21, $V := G^{(\mathbb{N})}$ and F be a finite subgroup of the good Sylow p-subgroup P^0 of V from p. 21. We show that F cannot be a p-uniqueness subgroup of V. Since F is finite, there is an $m = m(F) \in \mathbb{N}$ with $F \leqslant U_m$. Because of $|\mathrm{Syl}_p G| \geqslant 2$ there is a $Q_{m+1} \in \mathrm{Syl}_p G_{m+1}$ with $Q_{m+1} \neq P_{m+1}$. Then

$$Q^0 := \big(P_1 \times P_2 \times \ldots \times P_m \times Q_{m+1} \times P_{m+2} \times \ldots\big) \cap G^{(\mathbb{N})}$$

contains the group F and we have $Q^0 \neq P^0$. So V has the distinguished good Sylow p-subgroup P^0 which is not singular (notice that by Theorem 3.1 there must be such a Sylow subgroup since V does not satisfy the Sylow p-Theorem). By the second part of the proof of Theorem 3.7, there is a (countable) subgroup U of V which does not contain any singular Sylow p-subgroup.

Fourth, let $G = \underline{S}^{(\mathbb{N})}$ be the countable locally finite group of finitary permutations on a countably infinite set (that is, which move only finitely many elements), p a prime, and $\{n_i \mid i \in \mathbb{N}\}$ a sequence in $\mathbb{N}$ with $n_i + 2p \leqslant n_i + 1$ $(i \in \mathbb{N})$. Then $\Sigma := \{\underline{S}^{n_i} \mid i \in \mathbb{N}\}$ is a nested local system for G. By Lemma 2.2 b) there exists an $S \in \mathrm{Syl}_p G$ which is good w.r.t. Σ. We know that $|\mathrm{Syl}_p \underline{S}^{2p}| \geqslant 2p-2 \geqslant 2$. Let $T_1, T_2 \in \mathrm{Syl}_p \underline{S}^{2p}$ with $T_1 \neq T_2$. Let $i \in \mathbb{N}$. Then

$$\underline{S}^{n_i} \leqslant \underline{S}^{n_i} \times \underline{S}^{2p} \leqslant \underline{S}^{n_{i+1}}.$$

We put $F_i := \underline{S}^{n_i} \times T_2$, if $S \cap \underline{S}^{2p} = T_1$, and $F_i := \underline{S}^{n_i} \times T_1$ otherwise. Then we have $S \cap F_i \notin \mathrm{Syl}_p F_i$ and $\underline{S}^{n_i} \leqslant F_i \leqslant \underline{S}^{n_{i+1}}$. Hence $\{F_i \mid i \in \mathbb{N}\}$ is a (nested) local system for G containing no local subsystem of Σ into which S reduces. Thus S is a good Sylow p-subgroup of G which is not very good.

Fifth, the good Sylow p-subgroup P^0 of $V := G^{(\mathbb{N})}$ provides an example of a Sylow p-subgroup which is very good but not singular.

Let Σ^* be a local system for V; by Lemma 2.2 a) there exists a nested local subsystem $\Sigma_1 = \{V_n \mid n \in \mathbb{N}\}$ of Σ and by Lemma 2.2 b) there is a Sylow p-subgroup Q of V which is good w.r.t Σ_1. Since P^0 is good w.r.t. $\Sigma = \{U_i \mid i \in \mathbb{N}\}$, it will contain a conjugate of every finite p-subgroup P of V: there is a Σ-group $U = U(P)$ with $P \leqslant U$; let $R \in \mathrm{Syl}_p U$ with $P \leqslant R$; by Sylow Theorem there is a $y \in U$ with $R^y = P^0 \cap U$; hence $P^y \leqslant P^0$. Therefore

$$(Q \cap V_n)^{x_n} \leqslant V_n^{x_n} \cap P^0$$

for some $x_n \in V$ ($n \in \mathbb{N}$). Thus $V_n^{x_n} \cap P^0$ is a Sylow p-subgroup of V_n and therefore $|P^0 \cap V_n| = |Q \cap V_n|$. It follows that $P^0 \cap V_n$ has the size of a Sylow p-subgroup of V_n ($n \in \mathbb{N}$), and consequently P^0 reduces into the subsystem Σ_1 of the given local system Σ^*.

The following core result may be very well-known but we can present a novel and shorter proof.

Theorem 3.8 (see [4]) *Let G be a locally finite group and let p be a prime. To any finite p-subgroup P of G shall pertain two finite p-subgroups P_1 and P_2 of G with $P \leqslant P_1 \cap P_2$ such that $\langle P_1, P_2 \rangle$ is not a p-group. Then there will exist a countable subgroup H of G with $|\mathrm{Syl}_p H| = 2^{\aleph_0}$.*

Proof — We construct recursively an infinite ascending chain

$$F_0 < F_1 < \ldots < F_n < \ldots$$

of finite subgroups of G and for every $n \in \mathbb{N}_0$ a set Σ_n of p-subgroups of F_n such that for every $n \in \mathbb{N}_0$ we have: (i) $|\Sigma_n| = 2^n$; (ii) every two Σ_n-groups do not generate a p-group; (iii) for $n \geqslant 1$ every Σ_{n-1}-group lies in at least two Σ_n-groups.

Let $F_0 := \langle 1 \rangle$ and $\Sigma_0 := \{\langle 1 \rangle\}$. Let $n \in \mathbb{N}$ and suppose

$$F_0 < F_1 < \ldots < F_{n-1} \quad \text{and} \quad \{\Sigma_i \mid i < n\}$$

have already been constructed. We let Σ_n be the set of all finite p-subgroups P_1, P_2 of G such that $\langle P_1, P_2 \rangle$ is not a p-group and there exists exactly one Σ_{n-1}-group P with $P \leqslant P_1 \cap P_2$. From the properties (i)–(iii) of Σ_{n-1} and from the prerequisite on G then follow (i)–(iii) for Σ_n. Let F_n be the span of all Σ_n-groups. Hereafter F_n is a finite subgroup of G with $F_{n-1} < F_n$. Let

$$H := \bigcup_{i \in \mathbb{N}_0} F_i.$$

Then H is a countable subgroup of G. Let $\mathcal{M}$ be the set of all p-subgroups of G which are an ascending union of a chain

$$S_0 < S_1 < \ldots < S_n < \ldots$$

of finite p-subgroups $S_i \in \Sigma_i$ $(i \in \mathbb{N}_0)$. According to (i) and (iii) we have $|\mathcal{M}| = 2^{\aleph_0}$ and according to (ii) any two $\mathcal{M}$-groups cannot generate a p-group. H contains every $\mathcal{M}$-group, so from the properties of $\mathcal{M}$ (and the countability of H) it follows that $|\mathrm{Syl}_p H| = 2^{\aleph_0}$. We have constructed an infinitely high ($\aleph_0$) tree of finite p-subgroups of G which branches properly at each location with proper inclusions and in which any two immediate successors of an arbitrary point do not generate a p-group. This tree has $2^{\aleph_0}$ many infinite branches. □

We are ready to state and prove our *Charakterisierungssatz*.

Theorem 3.9 (see [4]) *Let* G *be a locally finite group and let* p *be a prime. The following properties are equivalent:*

1) G *satisfies the strong Sylow Theorem for the prime* p.

2) *In every subgroup* U *of* G *every Sylow* p*-subgroup of* U *is singular.*

3) *Every countable subgroup* H *of* G *contains a* p*-uniqueness subgroup of* H.

4) *Every countable subgroup* H *of* G *contains a singular Sylow* p*-subgroup of* H.

5) *Every countable subgroup of* G *satisfies the Sylow Theorem for the prime* p.

6) *If* H *is a countable subgroup of* G, *then* $|\mathrm{Syl}_p H| < 2^{\aleph_0}$.

Proof — 2) ⇒ 3) and 3) ⇒ 4) are clear. 4) ⇒ 5) is valid by Theorem 3.7, 5) ⇒ 6) is valid by Theorem 3.4, and 6) ⇒ 1) is valid by Theorem 3.3. It remains to show 1) $\Rightarrow$ 2).* Assume 1) holds and let $U \leqslant G$. Then U satisfies the strong Sylow Theorem for the prime p. By Theorems 3.5 and 3.4 we have that $|\mathrm{Syl}_p H| < 2^{\aleph_0}$ for any countable subgroup H of U. By Theorem 3.8 there is a finite p-subgroup P of U such that for all finite p-subgroups P_1 and P_2 of U with $P \leqslant P_1 \cap P_2$ the group $\langle P_1, P_2 \rangle$ is a p-group. By Proposition 2.3 it follows that P is a p-uniqueness subgroup of U. Let $S \in \mathrm{Syl}_p U$ with $P \leqslant S$. Moreover, let $T \in \mathrm{Syl}_p U$ and $x = x(T) \in U$ with $S = T^{x^{-1}}$. Then P^x is a p-uniqueness subgroup of U with $P^x \leqslant T$, and hence T is singular by means of P^x. □

It would have been easier to show that Theorem 3.9 1) implies that every Sylow p-subgroup S of an arbitrary subgroup U of G is very good. In fact, let Σ be a local system for U. By Lemma 2.2 a) there exists a nested local system Σ_1 of Σ, and by Lemma 2.2 b) there is a $T \in \mathrm{Syl}_p U$ which reduces into Σ_1. Since G satisfies the strong Sylow Theorem for the prime p, we find an $x \in U$ such that $S = T^x$. Let $\Sigma_2 := \{Y \mid Y \in \Sigma_1, x \in Y\}$. Then Σ_2 is a local subsystem of Σ into which S reduces: for $S \cap Y = T^x \cap Y = (T \cap Y)^x \in \mathrm{Syl}_p Y$ when $Y \in \Sigma_2$.

Having proved our *Charakterisierungssatz*, we are now ready to prove the announced main theorem characterising the locally finite groups which satisfy the strong Sylow p-Theorem.

Theorem 3.10 *Let G be a locally finite group and let p be a prime. The following properties are equivalent:*

1) G *satisfies the strong Sylow Theorem for the prime* p.

2) *Every subgroup S of G contains a finite p-subgroup which is singular in S.*

Proof — The result follows from a combination of Proposition 2.3 and Theorem 3.9. □

* In Theorem 1.5 of [10] (*If the locally finite group G satisfies the strong Sylow Theorem for the prime p there exists a finite p-subgroup P which is singular in G*), Kegel ingeniously constructs, by contradiction, an infinite ($\aleph_0$) tower of countable subgroups of G, such that none of the finite p-subgroups of a member can be singular in the upper next, whose union has $2^{\aleph_0}$ maximal p-subgroups and therefore contradicts Theorem 3.4.

4 Novel concepts for Sylow theory in (locally) finite groups

We end this paper with some further thoughts, a result, and some questions that could be quite useful for future researchers into Sylow theory in (locally) finite groups. The status quo of Sylow theory in locally finite groups has been beautifully summarised in [3] and [10]; here, a special place is occupied by the contributions of Brian Hartley (see [6],[7],[8]), who also contributed prodigiously to simple locally finite groups (see [9]). Concerning [9], which appeared posthumously, we notice that it does not cite [10] (not even in its list of 56 references). This is regrettable since Hartley states in his 1990 Mathematical Review of [10] the following: "If the simple locally finite group G satisfies the strong Sylow Theorem for the (even one) prime p, then G is linear. This depends on the classification of finite simple groups and an *assertion* about singular p-subgroups of classical groups. Another proof of this result has since been given by the reviewer (not yet published)." However, due to the tragic death of Brian Hartley on October 8, 1994, aged 55, this certainly very interesting proof was never prepared for publication. With someone of Hartley's stature, there is no question that his word is good enough and that in any case he supplied a new proof with probably quite a number of new insights. It might therefore be worthwhile and even most desirable to inspect Hartley's estate.

In every locally finite group G, for all subgroups U of G, the set $\mathrm{Unique}_p U$ of finite p-subgroups which are p-uniqueness subgroups of U is non-empty if G satisfies the strong Sylow Theorem for the prime p, that is, if G belongs to the class Syl-p of locally finite groups satisfying the strong Sylow Theorem for the prime p, and should this set be non-empty for a countable U then all the good Sylow p-subgroups of U are conjugate. Let U be finite. Then we have already $\mathrm{Unique}_p U \neq \emptyset$ because we have $\mathrm{Syl}_p U \leqslant \mathrm{Unique}_p U$. The Sylow p-subgroups of U are of course the maximal members of $\mathrm{Unique}_p U$, with respect to inclusion and order. It is a very very considerable challenge to try to determine the minimal members of $\mathrm{Unique}_p U$, with respect to either inclusion or order, in case that U and $\mathrm{Syl}_p U$ are sufficiently "known", in particular if U is a "known" finite simple group or a p-soluble group. Note that whenever $P < Q < R$ are p-subgroups of U where Q is a minimal p-uniqueness subgroup, or will be minimal singular in U, then P is contained in at least two, in fact in at least $p+1$, Sylow p-subgroups of U and R will be another p-uniqueness subgroup of U. The author is much hoping that some progress be made to this challenge in the future. For example, the question of whether (resp. when) the minimal p-uniqueness subgroups are conjugate, quite similar to the maximal ones, is surely of some interest, or, whether minimal w.r.t. inclusion implies minimal w.r.t. order, the converse being clearly obvious. We would then also come to better know the p-uniqueness subgroups of locally finite groups, in particular the simple and the locally p-soluble ones, and, many thanks to Kegel's Theorem 4.4, of locally finite groups in general belonging to the lovely class Syl-p. A good starting point would be to study minimal p-uniqueness subgroups of the finite symmetric and alternating groups where a Sylow 2-subgroup of an alternating group is a next to maximal 2-uniqueness subgroup of the symmetric overgroup so that we have to study only the symmetric groups and to show at least that their ranks are "somehow" bounded in terms of a p-uniqueness subgroup and in ideal circumstances to determine all the minimal ones (see what follows).

Let G be a locally finite group, $S \in \mathrm{Syl}_p G$ and $F \leqslant G$. We call F *minimal* p*-unique w.r.t.* S, if F is a minimal p-uniqueness subgroup of G w.r.t. order such that $F \leqslant S$, that is, F is p-unique with $F \leqslant S$ and each (finite) subgroup P of S with $|P| < |F|$ lies in at least two Sylow p-subgroups of G. If there exists an $S \in \mathrm{Syl}_p G$, such that F is, w.r.t. S, minimal p-unique, then F is called *minimal* p*-unique* (in G). Obviously, G is p-closed if and only if $\langle 1 \rangle$ is minimal p-unique (in G).

Theorem 4.1 (see [4]) *Let G be a locally finite group satisfying the strong Sylow Theorem for the prime p.*

a) *Each Sylow p-subgroup of G contains at least one minimal p-unique subgroup of G.*

b) *Each two minimal p-unique subgroups of G have the same order.*

Proof — a) Let $S \in \mathrm{Syl}_p G$ and let $U(G,S)$ be the set of all p-uniqueness subgroups F of G such that $F \leqslant S$. According to Theorem 3.9 we have $U(G,S) \neq \emptyset$ and of course each $U(G,S)$-group has finite order. Thus $U(G,S)$ contains (w.r.t. S) a minimal p-unique subgroup due to the well ordering of $\mathbb{N}$.

b) Let F_1 and F_2 be two minimal p-unique subgroups of G. For symmetry reasons it suffices to show $|F_1| \leqslant |F_2|$. Let S_1, $S_2 \in \mathrm{Syl}_p G$

with $F_1 \leqslant S_1$ and $F_2 \leqslant S_2$. Since $G \in$ Syl-p there is an $x \in G$ such that $S_1 = S_2^x$. Then F_2^x is a p-uniqueness subgroup of G with $F_2^x \leqslant S_1$. Thus $|F_1| \leqslant |F_2^x| = |F_2|$ since F_1 is minimal p-unique w.r.t. S_1. □

Let G be a locally finite group satisfying the strong Sylow p-Theorem and let $S \in \mathrm{Syl}_p G$. According to Theorem 4.1 a) S contains (w.r.t. S) a minimal p-unique subgroup F. We define $a_p = a_p(G) \in \mathbb{N}_0$ by $|F| =: p^{a_p}$, that is, we let a_p be the composition length of F. According to Theorem 4.1 b) this definition is independent of the special choice of the Sylow p-subgroup S of G. Whereby consequently a_p is a (numeric) Sylow p-invariant of G. We call a_p the p*-uniqueness* of G. This Sylow p-invariant enqueues into the list — even is in the vanguard — of other Sylow p-invariants which play a major role in (locally) finite group theory, e.g. the order p^{b_p} of a Sylow p-subgroup, its nilpotency class c_p, its solubility length d_p, its exponent p^{e_p}, the composition length $i_p - 1$ of a proper maximal (w.r.t. order) Sylow p-intersection and further. The real peculiarity of a_p is that it is not determined by a Sylow p-subgroup as abstract p-group alone but depends on its embedding into the whole group and the respective relationships to the other Sylow p-subgroups. Then (w.r.t. inclusion or order maximal) intersections of two or several Sylow p-subgroups are of interest and deserve further study. For example, two core questions for Sylow theory in (locally) finite groups are how the p-length of a finite p-soluble group and the rank of a (known) finite simple group are bounded in terms of a p-uniqueness subgroup.

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