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Groups Satisfying the Double Chain Condition on Subnormal Non-Normal Subgroups

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Abstract

If θ is a subgroup property, a group G is said to satisfy the double chain condition on θ-subgroups if it admits no infinite double sequences

$$\ldots < X_{-n} < \ldots < X_{-1} < X_0 < X_1 < \ldots < X_n < \ldots$$

consisting of θ-subgroups. This paper investigates the structure of subsoluble groups satisfying the double chain condition on subnormal non-normal subgroups and a complete description of these groups is given in the periodic case. It follows that for periodic subsoluble groups the double chain condition and the minimal condition on subnormal non-normal subgroups are equivalent.

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1 Introduction

A group G is called a T-*group* if all its subnormal subgroups are normal, i.e. if normality is a transitive relation in G. The structure of soluble T-groups was described by W. Gaschütz [14] in the finite case and by D.J.S. Robinson [16] for arbitrary groups. It turns out that all soluble groups with the T-property are metabelian and hypercyclic, while any finitely generated soluble T-group either is finite or abelian. Relevant classes of generalized T-groups can be introduced by imposing that the set of all subnormal non-normal subgroups of the group is small in some sense (see for instance [2],[11],[12]).

It is well-known that chain conditions play a relevant role in the study of infinite soluble groups. In a very early stage, a complete description of soluble groups satisfying the minimal condition on all subgroups was obtained by S.N. Černikov [9], while it is clear that a soluble group satisfies the maximal condition on subgroups if and only if it is polycyclic. Afterwards, the effect of the minimal and the maximal conditions on several systems of relevant subgroups was investigated, and this point of view was also adopted in [11],[12], where groups satisfying either the minimal or the maximal condition on subnormal non-normal subgroups were considered. Even the imposition of weaker forms of the classical chain conditions in many cases produces remarkable effects. In particular, T.S. Shores [19] and D.I. Zaicev [20] independently proved that if G is a generalized soluble group whose subgroup lattice admits no chains with the same order type of the set of integers, then G is soluble-by-finite and satisfies either the minimal or the maximal condition on subgroups. If θ is a subgroup property, we shall say that a group G satisfies the *double chain condition* on θ-subgroups if for each *double chain*

$$\dots \leqslant X_{-n} \leqslant \dots \leqslant X_{-1} \leqslant X_0 \leqslant X_1 \leqslant \dots \leqslant X_n \leqslant \dots$$

of θ-subgroups of G there exists an integer k such that either $X_n = X_k$ for all $n \leqslant k$ or $X_n = X_k$ for all $n \geqslant k$. Obviously, both the minimal and the maximal condition on θ-subgroups imply the double chain condition on θ-subgroups. Groups satisfying the double chain condition on some relevant systems of subgroups have been recently investigated (see [1],[3],[4],[6],[7]).

The aim of this paper is to give a further contribution to the theory of generalized T-groups in this context, by studying soluble groups satisfying the double chain condition on subnormal non-normal subgroups (DC_{snn}-*groups*). A complete description of DC_{snn}-groups is obtained in the periodic case, and it follows actually from our main result that for periodic soluble groups the double chain condition on the system of subnormal non-normal subgroups is equivalent to the minimal condition on the same system. The structure of soluble non-periodic groups with the DC_{snn}-property will be described in the forthcoming paper [5].

Our notation is mostly standard and can be found in [18]; in particular, for any group G, we denote by $\mathrm{Fit}(G)$ the *Fitting subgroup* of G, i.e. the subgroup generated by all nilpotent normal subgroups of G.

2 General properties

It is useful to point out that any infinite direct decomposition of a group G gives rise to a double chain of subgroups which is unbounded on both sides. In fact, if

$$G_1, G_2, \dots, G_n, \dots \tag{*}$$

is a countably infinite collection of non-trivial subgroups of G such that

$$\langle G_n \mid n \in \mathbb{N} \rangle = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, G_n,$$

then G admits the double chain of subgroups

$$\dots < U_{-k} < \dots < U_{-1} < U_0 < U_1 < \dots < U_k < \dots \tag{**}$$

where

$$U_k = \Big(\underset{n\in\mathbb{N}}{\mathrm{Dr}}\, G_{2n-1}\Big) \times \Big(\underset{1\leqslant n\leqslant k}{\mathrm{Dr}}\, G_{2n}\Big) \quad \text{and} \quad U_{-k} = \underset{n>k}{\mathrm{Dr}}\, G_{2n-1}$$

for each non-negative integer k. We say that (**) is the *double chain associated* to the collection of subgroups (*).

Lemma 2.1 *Let G be a DC_{snn}-group admitting a subnormal section X/Y that can be decomposed into the direct product of infinitely many cyclic non-trivial subgroups. Then X and Y are normal in G, and all cyclic subnormal subgroups of G/Y are G-invariant.*

Proof — Clearly, X/Y contains a subgroup X^*/Y which is the direct product of a countably infinite collection $(X_n/Y)_{n\in\mathbb{N}}$ of cyclic non-trivial subgroups, and

$$X^*/Y = V/Y \times W/Y,$$

where

$$V/Y = \operatorname*{Dr}_{n\in\mathbb{N}}(X_{2n}/Y) \quad \text{and} \quad W/Y = \operatorname*{Dr}_{n\in\mathbb{N}}(X_{2n-1}/Y).$$

As G satisfies the double chain condition on subnormal non-normal subgroups, it follows that there exist normal subgroups V^* and W^* of G such that $Y \leqslant V^* \leqslant V$ and $Y \leqslant W^* \leqslant W$. Thus also the intersection $Y = V^* \cap W^*$ is normal in G. For each positive integer n, the group X^*/X_n is likewise the direct product of a countably infinite collection of cyclic non-trivial subgroups, and hence each X_n is normal in G.

Let g be any element of G such that the subgroup $\langle g, Y\rangle$ is subnormal in G. Of course, there is a positive integer m such that

$$\langle g, Y\rangle \cap X^* \leqslant \langle X_1, \dots, X_m\rangle,$$

and hence

$$\langle g, Y\rangle \cap \langle X_n \mid n > m\rangle = Y.$$

Consider in G the double chains

$$\dots < V_{-n} < \dots < V_{-1} < V_0 < V_1 < \dots < V_n < \dots$$

and

$$\dots < W_{-n} < \dots < W_{-1} < W_0 < W_1 < \dots < W_n < \dots$$

respectively associated to the collections of normal subgroups

$$\{X_{2m}, X_{2m+2}, X_{2m+4}, \dots\} \quad \text{and} \quad \{X_{2m+1}, X_{2m+3}, X_{2m+5}, \dots\}.$$

Then

$$\dots < \langle g, Y\rangle V_{-n} < \dots < \langle g, Y\rangle V_{-1} < \langle g, Y\rangle V_0$$
$$< \langle g, Y\rangle V_1 < \dots < \langle g, Y\rangle V_n < \dots$$

and

$$\dots < \langle g, Y\rangle W_{-n} < \dots < \langle g, Y\rangle W_{-1} < \langle g, Y\rangle W_0$$
$$< \langle g, Y\rangle W_1 < \dots < \langle g, Y\rangle W_n < \dots$$

are double chains consisting of subnormal subgroups of G, and so there exist integers r and s such that $\langle g, Y\rangle V_r$ and $\langle g, Y\rangle W_s$ are normal in G. On the other hand, $\langle g, Y\rangle = \langle g, Y\rangle V_r \cap \langle g, Y\rangle W_s$ and hence $\langle g, Y\rangle$ is normal in G. Finally, X is normal in G, as X/Y is generated by cyclic subnormal subgroups of G/Y. □

Lemma 2.2 *Let* G *be a* DC_{snn}*-group, and let* A *be a torsion-free abelian subnormal subgroup of* G *which is not finitely generated. Then all subgroups of* A *are normal in* G.

Proof — Assume for a contradiction that the statement is false, so that A contains a cyclic subgroup $\langle u\rangle$ which is not normal in G. Clearly, there exists an infinite sequence

$$k_1, k_2, \dots, k_n, \dots$$

of positive integers such that

$$\langle u^{k_1}\rangle > \langle u^{k_2}\rangle > \dots > \langle u^{k_n}\rangle > \dots$$

and each $\langle u^{k_n}\rangle$ is not normal in G. Then it follows from the double chain condition that the set $\mathcal{H}$ of all finitely generated subgroups of A which contain $\langle u\rangle$ and are not normal in G satisfies the maximal condition. Let M be a maximal element of $\mathcal{H}$, and let a be an arbitrary element of $A \setminus M$. Then $\langle M, a\rangle$ is normal in G, and so all subgroups of A properly containing M are G-invariant. Furthermore, the intersection

$$\bigcap_{n>0} \langle M, a^n\rangle$$

is likewise normal in G, so that it must properly contain M and hence $M \cap \langle a\rangle \neq \{1\}$. It follows that the infinite group A/M is periodic, whence A and M have the same Prüfer rank r, and $r > 1$ since M cannot be properly contained in a cyclic subgroup of A. Of course, any two non-trivial subgroups of A/M have a non-trivial intersection, and so A/M is a group of type p^∞ for some prime number p. Write

$$M = \langle a_1\rangle \times \dots \times \langle a_r\rangle,$$

where $\langle a_1\rangle$ is not normal in G, and for each prime number $q \neq p$ put

$$M_q = \langle a_1\rangle \times \langle a_2^q\rangle \times \dots \times \langle a_r^q\rangle.$$

Then

$$A/M_q = M/M_q \times P_q/M_q,$$

where P_q/M_q is a group of type p^∞. As $\langle a_1 \rangle$ admits an infinite descending chain of subgroups which are not G-invariant, we have that P_q is a normal subgroup of G for all $q \neq p$, and so

$$P = \bigcap_{q \neq p} P_q$$

is likewise normal in G. Moreover,

$$P \cap M = \bigcap_{q \neq p} (P_q \cap M) = \bigcap_{q \neq p} M_q = \langle a_1 \rangle$$

and hence the subgroup P has rank 1. It follows that $\langle a_1 \rangle$ lies in a cyclic G-invariant subgroup of P, whence it is normal in G. This contradiction completes the proof. □

Let G be a group. Recall that the *Baer radical* of G is the subgroup generated by all abelian subnormal subgroups of G, and G is called a *Baer group* if it coincides with the Baer radical, or equivalently if every cyclic subgroup of G is subnormal. Of course, Baer groups are locally nilpotent. For our purposes, we also need to recall also that a group G is *minimax* if it has a series of finite length each of whose factors satisfies either the minimal or the maximal condition; in particular, an abelian group A is minimax if and only if it contains a finitely generated subgroup E such that A/E satisfies the minimal condition.

Lemma 2.3 *Let* G *be a* DC_{snn}*-group and let* B *be the Baer radical of* G*. If* G *contains an abelian subnormal subgroup* A *which is not minimax, then all subgroups of* B *are normal in* G*.*

Proof — Assume for a contradiction that the statement is false, and let U be a free abelian subgroup of A such that A/U is periodic. It follows from Lemma 2.1 that U is finitely generated, and so A/U cannot satisfy the minimal condition on subgroups. Thus the socle of A/U^{2^k} is infinite for each positive integer k, and hence a further application of Lemma 2.1 yields that U^{2^k} is normal in G and all subgroups of B/U^{2^k} are G-invariant. In particular, B/U^{2^k} is abelian for each $k > 2$, and so B itself is abelian because

$$\bigcap_{k>2} U^{2^k} = \{1\}.$$

Let T be the subgroup consisting of all elements of finite order of B. Again Lemma 2.1 shows that T satisfies the minimal condition on subgroups, so that $B = T \times V$, where V is a torsion-free subgroup. Of course, V is not minimax, and hence all its subgroups are normal in G by Lemma 2.2. By assumption, there exists an element b of B such that $\langle b \rangle$ is not normal in G. As B/V is periodic, we have $\langle b \rangle \cap V = \langle b^m \rangle$ for some $m > 0$, and $\langle b^m \rangle$ is a normal subgroup of G. Let $W/\langle b^m \rangle$ be a free abelian subgroup of $V/\langle b^m \rangle$ such that V/W is periodic. Clearly, W is finitely generated by Lemma 2.1, so that V/W does not satisfy the minimal condition on subgroups and hence its socle S/W is infinite. Again by Lemma 2.1 all subgroups of B/W are G-invariant, and in particular the subgroup $\langle b, W \rangle$ is normal in G. Since

$$\langle b, W \rangle / \langle b^m \rangle = \langle b \rangle / \langle b^m \rangle \times W / \langle b^m \rangle,$$

we have that the subgroup $\langle b \rangle / \langle b^m \rangle$ is characteristic in $\langle b, W \rangle / \langle b^m \rangle$ and so normal in $G/\langle b^m \rangle$. It follows that $\langle b \rangle$ is normal in G, and this contradiction completes the proof. □

A group G is called *subsoluble* if it has an ascending series of subnormal subgroups whose factors are abelian. Of course, soluble groups are subsoluble, while subsolubility is equivalent to solubility within the universe of finite groups. Moreover, it is easy to see that a group is subsoluble if and only if all its non-trivial homomorphic images have a non-trivial Baer radical and that in any subsoluble group the Baer radical contains its centralizer. It is also clear that subsoluble T-groups are soluble, and it is even true that any subsoluble group satisfying the maximal condition on subnormal non-normal subgroups is soluble (see [11]). At the end of this section, we will prove that a similar conclusion holds more in general for subsoluble groups with the DC_{snn}-property.

Corollary 2.4 *Let G be a periodic subsoluble group with the DC_{snn}-property and let B be the Baer radical of G. Then either G is a Černikov group or all subgroups of B are normal in G. In particular, any periodic Baer group with the DC_{snn}-property is either Černikov or Dedekind.*

Proof — Assume that B contains a subgroup which is not normal in G. Then it follows from Lemma 2.3 that B satisfies the minimal condition on abelian subnormal subgroups, and hence it is a Černikov group by a result of Robinson (see [17], Theorem E). Moreover, the factor group $G/C_G(B)$ is likewise Černikov (see [18] Part 1, Theorem 3.29) and $C_G(B) \leqslant B$, so that G is a Černikov group. □

Lemma 2.5 *Let G be a DC_{snn}-group, and let X be a subnormal non-normal subgroup of G. If X is contained in the Baer radical of G, then its normal closure X^G is either Černikov or polycyclic.*

Proof — Assume first that the Baer group X^G is periodic, but not a Černikov group. Then it follows from Corollary 2.4 that X^G is a Dedekind group, so that X^G contains a subgroup that can be decomposed into the direct product of infinitely many cyclic non-trivial subgroups and hence all its subgroups are normal in G by Lemma 2.1. This contradiction shows that the statement holds when X^G is periodic.

Suppose now that X^G is not periodic, so that also X is not periodic. Thus X is generated by elements of infinite order, and hence it contains an infinite cyclic subgroup $\langle x\rangle$ which is not normal in G. Then the DC_{snn}-property yields that G satisfies the maximal condition on subnormal non-normal subgroups containing $\langle x\rangle$. As every finitely generated subgroup of X is subnormal in G, it follows that the set of all finitely generated subgroups of X containing $\langle x\rangle$ which are not normal in G has a maximal element M. If $M \neq X$, we have that $\langle M, y\rangle$ is a normal subgroup of G for each element y of $X \setminus M$, and so also

$$X = \langle \langle M, y\rangle \mid y \in X \setminus M\rangle$$

is normal in G. This contradiction shows that $X = M$ is finitely generated. Since G satisfies the maximal condition on subnormal non-normal subgroups containing $\langle x\rangle$, the group X^G satisfies the maximal condition on subnormal subgroups containing X. If

$$X = X_0 < X_1 < \ldots < X_t = X^G$$

is a finite series between X and X^G, it follows that each X_{i+1}/X_i satisfies the maximal condition on subnormal subgroups and hence it is polycyclic (see [18] Part 1, Theorem 5.37). Therefore also X^G is polycyclic. □

We need also the following elementary property of locally nilpotent groups.

Lemma 2.6 *Let G be a locally nilpotent group whose commutator subgroup G' is finitely generated. Then G is nilpotent.*

Proof — As G' is a finitely generated nilpotent group, the factor group $G/C_G(G')$ is likewise finitely generated (see for instance [18] Part 1, Theorem 3.27), and hence there exists a finitely generated subgroup E of G such that $G' \leqslant E$ and $G = EC_G(G')$. Thus G is the product of two nilpotent normal subgroups, and hence it is likewise nilpotent. □

Theorem 2.7 *Let G be a Baer group with the DC_{snn}-property. Then G is nilpotent.*

Proof — Of course, it can be assumed that G is not a Dedekind group or, equivalently, that it does not have the T-property. If G is periodic, then it is a Černikov group by Corollary 2.4, and hence G is nilpotent. Suppose now that G is not periodic, so that it is generated by elements of infinite order, whence it must contain an infinite cyclic non-normal subgroup $\langle x \rangle$. Clearly, $\langle x \rangle$ admits an infinite descending chain of subgroups which are not normal in G and so the DC_{snn}-property yields that the set of all subnormal non-normal subgroups of G containing $\langle x \rangle$ has a maximal element M. Then G/M^G is a Dedekind group and in particular its commutator subgroup is finite. Moreover, it follows from Lemma 2.5 that the normal closure M^G satisfies the maximal condition on subgroups, so that G' is finitely generated and hence G is nilpotent by Lemma 2.6. □

Corollary 2.8 *Let G be a DC_{snn}-group. Then the Baer radical of G is nilpotent and coincides with the Fitting subgroup of G.*

Corollary 2.9 *Let G be a finitely generated soluble DC_{snn}-group. Then G is polycyclic.*

Proof — Let F be the Fitting subgroup of G. By induction on the derived length of G, we may suppose that the factor group G/F is polycyclic. Then there exists a finitely generated subgroup E of F such that $F = E^G$. Since F is nilpotent by Corollary 2.8, the subgroup E is subnormal in G and so it follows from Lemma 2.5 that F satisfies either the minimal or the maximal condition on subgroups. If F satisfies the minimal condition on subgroups, then it is covered by finite characteristic subgroups, and so it is finite. Therefore F satisfies the maximal condition on subgroups, so that it is polycyclic, and hence G itself is polycyclic. □

Theorem 2.10 *Let G be a subsoluble group with the DC_{snn}-property. Then G is soluble.*

Proof — Assume for a contradiction that the statement is false, and let B be the Baer radical of G. Since B is nilpotent by Corollary 2.8, the factor group G/B cannot be soluble, and so it does not satisfy the maximal condition on subnormal non-normal subgroups. It follows that B satisfies the minimal condition on subgroups which are not G-invariant, and so in particular all infinite cyclic subgroups of B are normal in G.

If B is not periodic, it is generated by its elements of infinite order and hence the group $G/C_G(B)$ is abelian, which is impossible because $C_G(B) \leqslant B$. Thus B is periodic and of course it must contain a subgroup which is not normal in G. Application of Lemma 2.3 yields now that all abelian subgroups of B satisfy the minimal condition and hence B is a Černikov group. Let T be the largest periodic normal subgroup of G. Then $B \leqslant T$ and $T/C_T(B)$ is Černikov (see [18] Part 1, Theorem 3.29), so that T itself is a soluble Černikov group. On the other hand, the Baer radical of G/T is torsion-free and hence G/T is soluble by the first part of the proof. This contradiction completes the proof. □

It is known that there exist locally nilpotent T-groups which are insoluble (see for instance [15]), so that in particular Theorem 2.10 cannot be extended to wider group classes, like for instance the class of radical groups, i.e. those groups admitting an ascending (normal) series with locally nilpotent factors.

3 Periodic DC_{snn}-groups

The aim of this section is to provide a complete description of periodic soluble groups satisfying the double chain condition on subnormal non-normal subgroups. As in many problems concerning soluble T-groups and their generalizations, an important role is played by power automorphisms. Recall here that an automorphism of a group G is a *power automorphism* if it maps every subgroup of G onto itself. The set of all power automorphisms of G is an abelian residually finite normal subgroup of the full automorphism group of G (see [10] for this and other basic information on power automorphisms of groups). Of course, if N is a normal subgroup of a group G, all subgroups of N are normal in G if and only if each element of G induces by conjugation a power automorphism on N, so that in particular $G/C_G(N)$ embeds into the group of power automorphisms of N and hence it is abelian and residually finite.

The first lemma of this section shows that the behaviour of periodic soluble groups with the double chain condition on subnormal non-normal subgroups is not too far from that of soluble T-groups.

Lemma 3.1 *Let G be a periodic soluble group with the DC_{snn}-property.*

(a) *If G is not Černikov, then it is metabelian and hypercyclic.*

(b) *If G is not a T-group, then it is abelian-by-finite and G/G' satisfies the minimal condition on subgroups.*

Proof — Of course, it can be assumed that G neither is Černikov nor a T-group. If F is the Fitting subgroup of G, it follows from Corollary 2.4 that all subgroups of F are normal in G, so that $G/C_G(F)$ is abelian and hence $G' \leqslant C_G(F) = \zeta(F)$. Therefore G is metabelian and it is also hypercyclic because all subgroups of G' are normal in G.

Let X be a subnormal non-normal subgroup of G. Then $X' \leqslant G' \leqslant F$ is normal in G and X/X' is an abelian subnormal non-normal subgroup of G/X', so that it follows again from Corollary 2.4 that G/X' is Černikov. Therefore G/G' satisfies the minimal condition. In particular, $G/\zeta(F)$ is a Černikov group which is also residually finite, so that G is abelian-by-finite. □

Corollary 3.2 *Let G be a periodic nilpotent group with the DC_{snn}-property. Then G is either Černikov or a Dedekind group.*

Proof — Assume that G is not Dedekind. Then G is not a T-group and so G/G' satisfies the minimal condition on subgroups by Lemma 3.1. Since G is nilpotent, it follows that in this case G is Černikov. □

Our next result shows in particular that, like Černikov groups, in any periodic soluble DC_{snn}-group the conjugacy classes of subnormal subgroups are finite. The structure of soluble groups in which every subnormal subgroup has only finitely many conjugates has been described by Casolo in [8].

Lemma 3.3 *Let G be a periodic soluble DC_{snn}-group and let X be a subnormal non-normal subgroup of G. Then X has only finitely many conjugates in G and G/X_G is Černikov. Moreover, if G is not Černikov, then X^G/X_G is finite.*

Proof — It is well known that each subnormal subgroup of a Černikov group has finitely many conjugates (see for instance [18] Part 1, Theorem 5.49), so that we may suppose without loss of generality that G is not a Černikov group. It follows from Lemma 3.1 and Corollary 2.4 that G is metabelian and all subgroups of G' are normal in G. In particular, X' is normal in G and X/X' is an abelian subnormal non-normal subgroup of G/X'. Then Corollary 2.4 again yields that the factor group G/X' is Černikov, so that G/X_G is Černikov too and hence X has finitely many conjugates in G. Finally, since the Fitting subgroup of G has finite index by Lemma 3.1 and all its subgroups are normal in G, we have that the index $|X : X_G|$ is finite and so X^G/X_G is finite. □

Our next resut describes the behaviour of the lower central series of a locally finite DC_{snn}-group, without any solubility assumption. Combined with Lemma 3.1, it shows in particular that any locally finite hypocentral DC_{snn}-group is nilpotent and has a finite commutator subgroup.

Proposition 3.4 *Let G be a locally finite DC_{snn}-group. Then the lower central series of G stops after finitely many steps and its last term $\overline{\gamma}(G)$ has finite index in G'.*

Proof — Assume, for sake of contradiction, that the statement is false. Clearly, the factor group $G/\gamma_\omega(G)$ is also a counterexample, so that we may suppose without loss of generality $\gamma_\omega(G) = \{1\}$. Then G is locally nilpotent and $\gamma_{n+1}(G) < \gamma_n(G)$ for each positive integer n. Since any Dedekind group has class at most 2, it follows from Corollary 3.2 that for each n the nilpotent group $G/\gamma_n(G)$ is Černikov, so that $G'/\gamma_n(G)$ is finite and G' is residually finite.

Assume now that $G^{(n+1)} < G^{(n)}$ for every non-negative integer n. If $G^{(n)}/G^{(n+1)}$ is finite for some n, then $G^{(n)} = EG^{(n+1)}$ for a suitable finite subgroup E and, for any normal subgroup K of finite index of $G^{(n)}$, we have that $G^{(n+1)}K/K$ is contained in the Frattini subgroup of $G^{(n)}/K$. It follows that $G^{(n)} = EK$ and hence $|G^{(n)} : K| \leqslant |E|$, which is impossible because G' is residually finite. Thus the group $G^{(n)}/G^{(n+1)}$ is infinite for all n. It follows that the soluble group $G/G^{(4)}$ cannot satisfy the maximal condition on subnormal non-normal subgroups (see [11], Corollary 2.11), so that $G^{(4)}$ must satisfy the minimal condition on subnormal non-normal subgroups and hence $G^{(4)}$ is soluble (see [12], Theorem 2.9). This contradiction shows that the derived series of G stops after finitely many steps and so G is soluble.

Let D be the largest divisible abelian normal subgroup of G. Then the subgroup $[D, G]$ is likewise divisible and so $D \leqslant \zeta(G)$ because G' is residually finite. On the other hand, each subnormal subgroup of G has finite index in its normal closure by Lemma 3.3, so all Sylow subgroups of the factor group G/D are nilpotent (see [8], Theorem 3.2). It follows that G is a Baer group and so it is nilpotent by Theorem 2.7. This contradiction shows that the lower central series of G terminates after finitely many steps. Then $G/\overline{\gamma}(G)$ is nilpotent, so that it is either a Černikov or a Dedekind group and hence its commutator subgroup $G'/\overline{\gamma}(G)$ is finite. □

Lemma 3.5 *Let* G *be a periodic soluble* DC_{snn}*-group, and let* $L = \overline{\gamma}(G)$ *be the last term of its lower central series. If* G *is not a Černikov group, then* $L^2 = L$.

Proof — The statement is known if G has the T-property, so we may suppose that G is not a T-group. Then G/G' satisfies the minimal condition on subgroups by Lemma 3.1, so the nilpotent group G/L is Černikov and hence L is not Černikov. Since G is metabelian by Lemma 3.1, the subgroup L is abelian and all its subgroups are normal in G by Corollary 2.4. It follows that L/L^2 is contained in the centre of G/L^2, so G/L^2 is nilpotent and hence $L^2 = L$. □

Next step is to study primary soluble DC_{snn}-groups. To this aim, let

$$P_\infty = \langle c_0, c_1, \dots, c_k, \dots \rangle$$

be a group of type 2^∞, with the usual relations $c_0 = 1$ and $c_k^2 = c_{k-1}$ if $k > 0$, and put $P_k = \langle c_k \rangle$ for each non-negative integer k. Denote by D_k the group generated by P_k and an element g_k such that $c^{g_k} = c^{-1}$ for all $c \in P_k$, $g_0^2 = 1$ and $g_k^2 = c_1$ if $k = 1, 2, \dots, \infty$. Moreover, for any $k \in \mathbb{N} \cup \{\infty\}$, let A_k be a finite abelian 2-group such that $A_k^2 \neq \{1\}$ if $k = 0, 1, 2, \infty$, and let J be a divisible abelian 2-group of infinite rank. Consider the semidirect product

$$G(k, A_k, J) = D_k \ltimes (A_k \times J),$$

where $[P_k, A_k \times J] = \{1\}$ and $x^{g_k} = x^{-1}$ for all $x \in A_k \times J$. We have that $F = P_k \times A_k \times J$ is the Fitting subgroup of $G(k, A_k, J)$ and the last term L of the lower central series of $G(k, A_k, J)$ coincides with J for $k < \infty$ and with $P_\infty J$ when $k = \infty$. Let X be any subnormal non-normal subgroup of $G(k, A_k, J)$; then X cannot be contained in F, so that $[L, X] = L^2 = L$ and hence $L \leqslant X$. Since L has finite index in $G(k, A_k, J)$, it follows that $G(k, A_k, J)$ has only finitely many subnormal non-normal subgroups. These groups occur in the classification of 2-groups with the DC_{snn}-property.

Theorem 3.6 *Let* p *be a prime number and let* G *be a* p*-group with the* DC_{snn}*-property which is not Černikov.*

(a) *If* $p > 2$*, then* G *is abelian;*

(b) *If* $p = 2$*, then either* G *is a* T*-group or it is isomorphic to a group of type* $G(k, A_k, J)$*.*

Proof — By Lemma 3.3 each subnormal subgroup of G has finite index in its normal closure, so that we may apply Theorem 3.2 of [8] obtaining that the Fitting subgroup F of G has index at most 2. If $F = G$, we have that G is nilpotent and so it is even a Dedekind group by Corollary 3.2.

Assume now $|G : F| = 2$, so that in particular G is a 2-group and $G = \langle F, z \rangle$, where z is an element of $G \setminus F$. All subgroups of F are normal in G by Corollary 2.4 and so z induces on F a non-trivial power automorphism. On the other hand, it follows from Lemma 3.5 that the last term L of the lower central series of G is a divisible abelian non-trivial subgroup of F and hence z inverts each element of F. Thus $[F, z] = F^2$ and so $L = F^{2^k}$ for some non-negative integer k. It follows that G/L has finite exponent, so that G/G' is finite by Lemma 3.1 and hence G/L is finite too. Let X be any subnormal non-normal subgroup of G. Then X cannot be contained in F, whence $[L, X] = L^2 = L$ and so $L \leqslant X$. It follows that if G is not a T-group, then G/L has a finite non-normal subgroup and hence G is isomorphic to some $G(k, A_k, J)$. □

The above result has the following consequence that shows in particular that all primary DC_{snn}-groups satisfy the minimal condition on subnormal non-normal subgroups.

Corollary 3.7 *Let* p *be a prime number and let* G *be a* p*-group with the* DC_{snn}*-property. Then either* G *is Černikov or it has only finitely many subnormal non-normal subgroups.*

Lemma 3.8 *Let* G *be a periodic group and let* P *be a Sylow* p*-subgroup of* G*, where* p *is an odd prime number. If all subgroups of* P *are normal in* G*, then either* $[P, G] = \{1\}$ *or* $[P, G] = P$.

Proof — Suppose $[P, G] \neq \{1\}$, so that P is not contained in $\zeta(G)$. Clearly, P is abelan and $G/C_G(P)$ is isomorphic to a periodic non-trivial group of power automorphisms which has no elements of order p. If g is any element of $G \setminus C_G(P)$, we have $[P, g] = P$ and so also $[P, G] = P$. □

It is well known that if G is a periodic soluble T-group and L is the last term of its lower central series, then the set $\pi(L) \cap \pi(G/L)$ does not contain odd prime numbers (see [16], Theorem 4.2.2). In the case of periodic soluble DC_{snn}-groups we have the following weaker result.

Lemma 3.9 *Let* G *be a periodic soluble group with the* DC_{snn}*-property and let* $L = \overline{\gamma}(G)$ *be the last term of the lower central series of* G*. If* p *is a prime number such that the* p*-component* L_p *of* L *is not Černikov, then the set* $\pi(L) \cap \pi(G/L)$ *does not contain any odd prime* $q \geqslant p$.

Proof — Let π be the set of all prime numbers strictly larger than p. Then the set N of all π-elements of G is a normal subgroup because G is hypercyclic by Lemma 3.1. Since $L_p \cap N = \{1\}$, the group G/N is not Černikov and it follows from Lemma 3.3 that all subnormal subgroups of N are normal in G. Let q be any prime in the set $\pi \cap \pi(L)$ and let $Q/L_{q'}$ be the unique Sylow q-subgroup of $G/L_{q'}$. Then $Q/L_{q'}$ is abelian by Theorem 3.6; in particular QL/L lies in $\zeta(G/L)$ and so $[Q, G] \leqslant L$. Moreover $Q \leqslant NL_{q'}$ and hence all subgroups of $Q/L_{q'}$ are normal in $G/L_{q'}$. On the other hand, $G/L_{q'}$ cannot be nilpotent because $q \in \pi(L)$ and so $Q = [Q, G]L_{q'}$ by Lemma 3.8. It follows that Q is contained in L and so q does not belong to $\pi(G/L)$. Therefore $\pi \cap \pi(L) \cap \pi(G/L) = \emptyset$ and in particular the statement is proved if $p = 2$.

Suppose finally $p > 2$ and let P/L be the unique Sylow p-subgroup of G/L. Since all subgroups of L are normal in G, we have that P induces on L_p a p-group of power automorphisms. Then either L_p has infinite exponent and is centralized by P or L_p has finite exponent and P acts trivially on $L_p[p^{n+1}]/L_p[p^n]$ for each non-negative integer n. It follows that $P/L_{p'}$ is nilpotent and hence all its subgroups are G-invariant by Corollary 2.4 because L_p is not Černikov. A further application of Lemma 3.8 yields that $[P, G]L_{p'} = P$, whence $P = L$ and $p \notin \pi(G/L)$. □

Our last main result provides a full description of the structure of periodic soluble groups with the DC_{snn}-property.

Theorem 3.10 *Let* G *be a periodic soluble group which is neither Černikov nor a* T*-group. The following statements are equivalent.*

(A) G *satisfies the minimal condition on subnormal non-normal subgroups.*

(B) G *satisfies the double chain condition on subnormal non-normal subgroups.*

(C) *If* F *is the Fitting subgroup of* G, $L = \overline{\gamma}(G)$ *is the last term of the lower central series of* G *and* $\pi = \pi(G/L) \cup \{2\}$*, then all subgroups of* F *are normal in* G, L_2 *is divisible and* $G = K \ltimes L_{\pi'}$ *satisfies one of the following conditions:*

(C1) *If* L_2 *is not Černikov, then* $K = K_1 \times K_2$*, where* K_1 *is a Černikov abelian* $2'$*-subgroup with* $K_1 \cap L = \{1\}$, K_2 *is a 2-group isomorphic to some* $G(k, A_k, J)$ *and* $L_{2'}/[g, L_{2'}]$ *is Černikov for each element* g *of* $K_2 \setminus \mathrm{Fit}(K_2)$.

(C2) *If* L_2 *is Černikov, then also* K *is a Černikov group; moreover,* K *is not a* T*-group and the centralizer* $C_{L_{\pi'}}(Y)$ *is Černikov for each subnormal non-normal subgroup* Y *of* K.

Proof — Obviously, every group satisfying the minimal condition on subnormal non-normal subgroups has the DC_{snn}-property.

Suppose now that G is a DC_{snn}-group. Then G is metabelian and hypercyclic by Lemma 3.1 and all subgroups of F are normal in G by Corollary 2.4; moreover, L_2 is divisible by Lemma 3.5. Since L is contained in F, the factor group $G/C_G(L)$ is isomorphic to a group of power automorphisms of L and hence it is residually finite. Moreover, it follows again from Lemma 3.1 that G/G' satisfies the minimal condition on subgroups, so G/L is Černikov by Proposition 3.4 and hence $G/C_G(L)$ is even finite. Put $N = L_{\pi'}$, so $G/C_G(N)$ is finite and $\pi(N) \cap \pi(G/N) = \emptyset$. It follows from the generalized Schur-Zassenhaus theorem that there exists a π-subgroup K of G such that $G = K \ltimes N$ (see for instance [13], Theorem 2.4.5). Here K cannot have the T-property, because G is not a T-group (see [16], Lemma 5.2.2).

Assume that L_2 is not a Černikov group, so that $\pi(L) \cap \pi(G/L) \subseteq \{2\}$ by Lemma 3.9 and hence L/N is a 2-group. Thus $K \simeq G/N$ contains a unique Sylow 2-subgroup K_2, which is not Černikov because $L_2 \leqslant K_2$. On the other hand, it is well known that the elements of odd order of any locally supersoluble group form a subgroup, so $K = K_1 \times K_2$

where K_1 is a $2'$-subgroup. Of course, $K_1 \cap L = \{1\}$ and hence K_1 is nilpotent and Černikov. But K is a DC_{snn}-group, so that all subgroups of K_1 are normal in K by Corollary 2.4, and in particular K_1 is abelian. Finally, since K is not a T-group, neither K_2 has the T-property and so K_2 is isomorphic to a suitable $G(k, A_k, J)$ by Theorem 3.6. Let now g be any element of $K_2 \setminus \mathrm{Fit}(K_2)$. As L_2 is a divisible subgroup of $K_2 \simeq G(k, A_k, J)$, we have $[g, L_2] = L_2$, so $L = C_{L_{2'}}(g) \times [g, L]$. Moreover, $\langle g\rangle[g, L]$ is a subnormal non-normal subgroup of G, so that $\langle g\rangle[g, L]/[g, L]$ is a non-normal subgroup of the Baer radical of $G/[g, L]$ and hence it follows from Lemma 2.3 that $L/[g, L]$ is Černikov. Thus $L_{2'}/[g, L_{2'}]$ is likewise a Černikov group.

Assume now that L_2 is Černikov, so that it follows from Lemma 3.9 that L_q is Černikov for any prime $q \in \pi$. As the set π is finite, the subgroup L_π is Černikov and hence $K \simeq G/L_{\pi'}$ is likewise a Černikov group. Assume for a contradiction that K contains a subnormal non-normal subgroup Y such that $C = C_{L_{\pi'}}(Y)$ is not Černikov. Clearly, the subgroup Y acts by conjugation on $L_{\pi'}$ as a group of power automorphisms and $2 \notin \pi'$, so that $L_{\pi'} = C \times M$, where $M = [L_{\pi'}, Y]$. The subgroup YM is subnormal in G, because it is normalized by $L_{\pi'}$ and $L_\pi \leqslant K$; moreover, $[L, Y]$ is contained in M, so that YM/M is nilpotent and hence it lies in the Baer radical of G/M. Application of Corollary 2.4 yields that YM is normal in G, whence $Y = YM \cap K$ is normal in K. This contradiction proves that G has the structure described in (C).

Suppose that statement (C) holds and G satisfies condition (C1). Since K_2 is isomorphic to some $G(k, A_k, J)$, the last term of the lower central series of $K = K_1 \times K_2$ coincides either with J or with $P_\infty J$, according to k being either finite or infinite. Then $L = JL_{\pi'}$ if $k < \infty$ and $L = JL_{\pi'}P_\infty$ if $k = \infty$, and in particuar G/L is a Černikov group. Since $K_2/C_{K_2}(L_{2'})$ is isomorphic to a group of power automorphisms of $L_{2'}$, it is residually finite and so even finite because K_2 is divisible-by-finite. Thus the set

$$\mathfrak{W} = \big\{[g, L_{2'}] \mid g \in K_2 \setminus \mathrm{Fit}(K_2)\big\}$$

is finite and hence $L_{2'}/W$ is a Černikov group, where W is the intersection of all members of $\mathfrak{W}$. Let X be any subnormal non-normal subgroup of G. It follows from Lemma 3.8 that for each prime $q \in \pi'$ we have either $[X, L_q] = \{1\}$ or $L_q = [X, L_q] \leqslant X$, so that $L_{\pi'} \leqslant N_G(X)$ and hence $N_G(X) = L_{\pi'}N_K(X \cap K)$ (replacing eventually X by a suitable conjugate). Thus $X \cap K$ is not normal in K, whence it cannot be contained in $K_1 \times \mathrm{Fit}(K_2)$ and we may consider in X an element g which belongs to $K_2 \setminus \mathrm{Fit}(K_2)$. Of course, g induces the inversion automorphism on the divisible subgroup $L_2 \leqslant K_2$ and so $L_2 = [g, L_2] \leqslant X$; moreover, $L_{2'} = [g, L_{2'}] \times C_{L_{2'}}(g)$ and so X contains also $[g, L_{2'}]$. Therefore $L_2 W \leqslant X$ and G satisfies the minimal condition on subnormal non-normal subgroups because $G/L_2 W$ is a Černikov group.

Assume finally that G is a group satisfying condition (C2). Since all subgroups of F are normal in G, the factor group $G/C_G(F)$ is isomorphic to a group of power automorphisms of F and in particular it is residually finite; moreover, $F/\zeta(F)$ is finite and $\zeta(F) = C_G(F)$, so that also G/F is residually finite and hence even finite. Therefore $G/\zeta(F)$ is finite. Let $\mathfrak{Y}$ be the set of all subnormal non-normal subgroups of K. For each element Y of $\mathfrak{Y}$ and for each prime $q \in \pi'$, the finite group $Y/C_Y(L_q)$ is isomorphic to a group of power automorphisms of L_q and hence it is cyclic because $q > 2$. Thus $L_q = C_{L_q}(Y) \times [L_q, Y]$ for each $q \in \pi'$, so that we have also

$$L_{\pi'} = C_{L_{\pi'}}(Y) \times [L_{\pi'}, Y]$$

and hence $L_{\pi'}/[L_{\pi'}, Y] \simeq C_{L_{\pi'}}(Y)$ is a Černikov group. Put

$$N = \bigcap_{Y \in \mathfrak{Y}} [L_{\pi'}, Y].$$

Since the set

$$\{[L_{\pi'}, Y] \mid Y \in \mathfrak{Y}\}$$

is finite because $\zeta(F)$ has finite index in G, the factor group G/N is Černikov. Let X be any subnormal non-normal subgroup of G. Then

$$X = (X \cap K^g) \ltimes (X \cap L_{\pi'})$$

for a suitable element g of G (see for instance [13], Proposition 2.2.4). Moreover, X is normalized by $L_{\pi'}$ because $X/X \cap L_{\pi'}$ is a subnormal π-subgroup of $G/X \cap L_{\pi'}$, and so $X \cap K^g$ is a subnormal non-normal subgroup of K^g. Thus $Z = X^{g^{-1}} \cap K$ is a subnormal non-normal subgroup of K, so that $L_{\pi'} = C_{L_{\pi'}}(Z) \times [L_{\pi'}, Z]$ and hence

$$N \leqslant [L_{\pi'}, Z] = [L_{\pi'}, Z, Z] \leqslant X^{g^{-1}}.$$

Therefore N is contained in each subnormal non-normal subgroup of G and so G satisfies the minimal condition on subnormal non-normal subgroups. □

Corollary 3.11 *Let* G *be a periodic soluble* DC_{snn}*-group. Then* G *satisfies the minimal condition on subnormal non-normal subgroups.*

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