PRONORMALITY IN INFINITE GROUPS

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Abstract

A subgroup H of a group G is said to be *pronormal* if H and H^x are conjugate in $\langle H, H^x \rangle$ for each element x of G. In this paper properties of pronormal subgroups of infinite groups are investigated, and the connection between pronormal subgroups and groups in which normality is a transitive relation is studied. Moreover, we consider the *pronorm* of a group G, i.e. the set of all elements x of G such that H and H^x are conjugate in $\langle H, H^x \rangle$ for every subgroup H of G; although the pronorm is not in general a subgroup, we prove that this property holds for certain natural classes of (locally soluble) groups.

1. Introduction

A subgroup H of a group G is said to be *pronormal* if for every element x of G the subgroups H and H^x are conjugate in $\langle H, H^x \rangle$. Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups; moreover, Sylow subgroups of finite groups and Hall subgroups of finite soluble groups are always pronormal. The concept of a pronormal subgroup was introduced by P. Hall, and the first results about pronormality appeared in a paper by Rose [13]. More recently, several authors have investigated the behaviour of pronormal subgroups, mostly dealing with properties of pronormal subgroups of finite soluble groups and with groups which are rich in pronormal subgroups. In the first part of this paper some information about pronormal subgroups of infinite groups will be obtained. In particular, we will prove that, if G is an FC-group (i.e. a group with finite conjugacy classes of elements) and K is a locally soluble subgroup of G whose cyclic subgroups are pronormal in G, then also K is pronormal in G; an example shows that such a result is no longer true for subgroups of arbitrary soluble groups. Moreover, it will be shown that in a polycyclic-by-finite group G the pronormality of a subgroup can be detected from its behaviour in the finite homomorphic images of G.

A group G is said to be a *T-group* if every subnormal subgroup of G is normal, i.e. if normality is a transitive relation in G. It is well known that pronormal subgroups are strictly related to T-groups, and in Section 3 some further evidence of this connection will be given. In particular, we will prove that an FC-group is a soluble T-group if and only if all its subgroups are pronormal. A similar result for finite groups was obtained by Peng [10], and here Peng's theorem will also be extended to locally finite groups satisfying the minimal condition on primary subgroups.

In [1] Baer introduced the *norm* $N(G)$ of a group G as the intersection of all the normalisers of subgroups of G, i.e. $N(G)$ is the set of all elements x of G such that $H^x = H$ for every subgroup H of G. Clearly $N(G)$ is a characteristic subgroup of G, and its elements induce by conjugation power automorphisms of G, so that in particular $N(G)$ is contained in the second term $Z_2(G)$ of the upper central series of G (see for instance [3]). In order to define an analogue to the norm for pronormality, we shall say that an element x of a group G *pronormalises* a subgroup H of G if the subgroups H and H^x are conjugate in $\langle H, H^x\rangle$. Thus a subgroup H of a group G is pronormal in G if and only if it is pronormalised by all elements of G. The *pronorm* of a group G is the set $P(G)$ of all elements of G pronormalising every subgroup of G. In particular, $P(G) = G$ if and only if all subgroups of G are pronormal. The consideration of the alternating group A_5 shows that for an arbitrary finite group G the pronorm $P(G)$ need not be a subgroup. The last section of this article is devoted to the study of the pronorm. Among other results it will be proved that the pronorm of a locally soluble FC-group is a subgroup, and that in a polycyclic group with nilpotent commutator subgroup the pronorm coincides with the Wielandt subgroup.

Most of our notation is standard and can be found in [12]. Moreover, we refer to [11] for results concerning the structure of soluble T-groups.

2. Pronormal subgroups

Our first two lemmas provide local versions of some elementary results about pronormal subgroups that can be found in [13].

Lemma 2.1. *Let G be a group, and let H and K be subgroups of G such that $H^K = H$. If x is an element of G normalising H and pronormalising K, then x pronormalises HK.*

Proof. Since x pronormalises K, there exists an element $z \in \langle K, K^x\rangle$ such that $K^{xz} = K$. On the other hand,

$$\langle K, K^x\rangle \leqslant \langle K, x\rangle \leqslant N_G(H),$$

so that $(HK)^{xz} = H^{xz}K^{xz} = HK$, and x pronormalises HK. ■

Lemma 2.2. *Let G be a group, and let H and K be subgroups of G such that $H^K = H$. If N is a normal subgroup of G such that K is pronormal in KN and x is an element of N pronormalising H, then x also pronormalises HK.*

Proof. Since x pronormalises H and

$$\langle H, H^x\rangle = \langle H, H^x\rangle \cap NH = H(N \cap \langle H, H^x\rangle),$$

there exists an element y of $N \cap \langle H, H^x\rangle$ such that $H^{xy} = H$. On the other hand, K is pronormal in KN and so there is $z \in \langle K, K^{xy}\rangle$ such that $K^{xyz} = K$. Put $J = HK$. Then $H = H^{xy}$ is a normal subgroup of $\langle J, J^{xy}\rangle$, and hence $H^{xyz} = H^z = H$. Thus $J^{xyz} = J$, where y is an element of $\langle J, J^x\rangle$ and z belongs to $\langle K, K^{xy}\rangle \leqslant \langle J, J^x\rangle$. Therefore yz is an element of $\langle J, J^x\rangle$, and hence x pronormalises J. ■

Lemma 2.3. *Let G be a group, and let K be a subgroup of G and Ω a chain of subgroups of G such that $K = \bigcup_{H\in\Omega} H$. If x is an element of G pronormalising every element of Ω and the subgroup $[K, x]$ is finite, then x also pronormalises K.*

Proof. Put

$$[K, x] \cap \langle K, K^x\rangle = \{y_1, \ldots, y_t\},$$

and for each $i = 1, \ldots, t$ let Ω_i be the subset of Ω consisting of all subgroups $H \in \Omega$ such that $H^x = H^{y_i}$. For every $H \in \Omega$ the subgroup $\langle H, H^x\rangle$ is contained in the product $H([H, x] \cap \langle K, K^x\rangle)$, and so

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_t.$$

For each $i \leqslant t$ put

$$K_i = \bigcup_{H\in\Omega_i} H,$$

so that $K_i^x = K_i^{y_i}$. If i, j are indices such that K_i is not contained in K_j, there exists an element H of Ω_i which is not contained in K_j, so that every element of Ω_j is contained in H, and hence also in K_i. Thus K_j is contained in K_i, and the finite set $\{K_1, \ldots, K_t\}$ is a chain. On the other hand,

$$K = \bigcup_{H\in\Omega} H = \langle K_1, \ldots, K_t\rangle,$$

so that $K = K_i$ for some $i \leqslant t$, and x pronormalises K. ■

It is now possible to prove that the join of any chain of pronormal subgroups of an FC-group is likewise pronormal.

Corollary 2.4. *Let G be an FC-group, and let Ω be a chain of pronormal subgroups of G. Then also $\bigcup_{H\in\Omega} H$ is a pronormal subgroup of G.*

Proof. Put

$$K = \bigcup_{H\in\Omega} H.$$

Since G is an FC-group, the subgroup $[K, x]$ is finite for each element x of G. Then it follows from Lemma 2.3 that K is pronormalised by every element of G, and hence it is a pronormal subgroup of G. ■

Lemma 2.5. *Let G be an FC-group, and let K be a locally soluble subgroup of G. If N is a normal subgroup of G such that X is pronormal in XN for every cyclic subgroup X of K, then K is a pronormal subgroup of KN.*

Proof. Let

$$\{1\} = K_0 \leqslant K_1 \leqslant \ldots \leqslant K_n \leqslant K_{n+1} \leqslant \ldots \leqslant K_\omega = K$$

be an ascending normal series with abelian factors of length at most ω of the locally soluble FC-group K (see [15, corollary 1.15]). Assume that the lemma is false, so that it follows from Corollary 2.4 that K_m is not pronormal in KN for some positive integer m, and m can be chosen to be the smallest with respect to this condition, so that K_{m-1} is a pronormal subgroup of KN. Since K_m is normal in K, we have that K_m is not pronormal in $K_m N$. Let x be an element of N which does not pronormalise K_m. As G is an FC-group, the subgroup $[K_m, x]$ is finite, so that it follows from Lemma 2.3 that the ordered set $\mathfrak{L}$, consisting of all subgroups of K_m containing K_{m-1} and pronormalised by x, is inductive. Thus by Zorn's Lemma $\mathfrak{L}$ contains a maximal element M. If a belongs to $K_m \backslash M$, the subgroup $\langle a \rangle$ is pronormal in $\langle a \rangle N$, and application of Lemma 2.2 yields that x also pronormalises $\langle a \rangle M$. This contradiction proves the lemma. ∎

The following theorem is a special case of Lemma 2.5.

Theorem 2.6. *Let G be an FC-group, and let K be a locally soluble subgroup of G. If every cyclic subgroup of K is pronormal in G, then also K is pronormal in G.*

In order to show that in the above results the assumption that the group has finite conjugacy classes is essential, we need the following lemma.

Lemma 2.7. *Let π be a set of prime numbers, and let G be a periodic group and P a Sylow π-subgroup of G. If P is pronormal in G, then G/P^G is a π'-group.*

PROOF. As the subgroup P is pronormal in G, we have $G = P^G N_G(P)$, and hence the groups G/P^G and $N_G(P)/(P^G \cap N_G(P))$ are isomorphic. On the other hand, P is the unique Sylow π-subgroup of $N_G(P)$, and so G/P^G is a π'-group. ∎

It follows from a result of Kovacs *et al.* [7] that there exists an uncountable periodic metabelian group G, which is a non-split extension of an abelian normal subgroup A without elements of order 2 by a group of exponent 2; moreover, the Sylow subgroups of G are countable and all subgroups of A are normal in G. Then every subgroup of G is a T-group (see [11, lemma 5.2.2]), and hence all finite subgroups of G are pronormal (see [4, lemma 9]). Let P be a Sylow 2-subgroup of G. If P is pronormal in G, it follows from Lemma 2.7 that G/P^G is a $2'$-group, so that also G/PA is a $2'$-group, and hence $G = PA$. This contradiction proves that the subgroup P is not pronormal in G. On the other hand, all cyclic subgroups of G are pronormal, and P is the join of a chain of finite subgroups. Therefore in the statements of Theorem 2.6 and Corollary 2.4 the assumption that G is an FC-group cannot be omitted.

It was proved by Kegel [5] that a subgroup H of a polycyclic-by-finite group G is subnormal in G if and only if H^σ is subnormal in G^σ for every finite homomorphic image G^σ of G, and a similar result for quasinormality was later obtained by Lennox and Wilson [9]. We prove here the corresponding statement for pronormal subgroups of polycyclic-by-finite groups.

Theorem 2.8. *Let G be a polycyclic-by-finite group, and let H be a subgroup of G. Then an element x of G pronormalises H if and only if x^σ pronormalises H^σ for every finite* *homomorphic image G^σ of G. In particular, H is pronormal in G if and only if H^σ is pronormal in G^σ for every finite homomorphic image G^σ of G.*

PROOF. The condition of the statement is obviously necessary. Suppose conversely that H^σ is pronormalised by x^σ for every finite homomorphic image G^σ of G, and let K be any normal subgroup of finite index of $\langle H, H^x\rangle$. As G is polycyclic-by-finite, both $\langle H, H^x\rangle$ and K are intersections of subgroups of finite index of G (see [14, p. 18]), and hence there exists a subgroup of finite index L of G such that $\langle H, H^x\rangle \cap L = K$. Let N be the core of L in G. By hypothesis the coset xN pronormalises HN/N, and so HN/N and H^xN/N are conjugate in $\langle H, H^x\rangle N/N$. Using the natural isomorphism between the groups $\langle H, H^x\rangle N/N$ and $\langle H, H^x\rangle/(\langle H, H^x\rangle \cap N)$, it follows that the subgroups

$$H(\langle H, H^x\rangle \cap N)/(\langle H, H^x\rangle \cap N)$$

and

$$H^x(\langle H, H^x\rangle \cap N)/(\langle H, H^x\rangle \cap N)$$

are conjugate in $\langle H, H^x\rangle/(\langle H, H^x\rangle \cap N)$. On the other hand, $\langle H, H^x\rangle \cap N$ is contained in $\langle H, H^x\rangle \cap L = K$, and hence HK/K and H^xK/K are conjugate in $\langle H, H^x\rangle/K$. Therefore H and H^x are conjugate in every finite homomorphic image of $\langle H, H^x\rangle$, so that they are conjugate in $\langle H, H^x\rangle$ (see [14, p. 69]) and H is pronormalised by x. ■

Corollary 2.9. *Let G be a polycyclic-by-finite group, and let K be a soluble subgroup of G. If every cyclic subgroup of K is pronormal in G, then also K is pronormal in G.*

PROOF. If G^σ is a finite homomorphic image of G, it follows from Theorem 2.6 that K^σ is pronormal in G^σ, so that K is a pronormal subgroup of G by Theorem 2.8. ■

3. Pronormal subgroups and *T*-groups

Let G be a group. The *Wielandt subgroup* $\omega(G)$ of G is the intersection of all the normalisers of subnormal subgroups of G. Thus a group G is a T-group if and only if $\omega(G) = G$. Recall that a subgroup H of a group G is said to be *ascendant* if there is an ascending series between H and G. Here we will denote by $\tau(G)$ the intersection of all the normalisers of ascendant subgroups of G. Clearly every subnormal subgroup is also ascendant, so that for any group G the subgroup $\tau(G)$ is contained in $\omega(G)$. Moreover, if G is a polycyclic-by-finite group, ascendant and subnormal subgroups of G coincide, and hence $\tau(G) = \omega(G)$. The subgroup $\tau(G)$ will play a relevant role in our considerations.

The first lemma of this section is almost obvious.

Lemma 3.1. *Let G be a group whose Fitting subgroup F is nilpotent. Then F/F' is the Fitting subgroup of G/F'.*

PROOF. Let N/F' be a nilpotent normal subgroup of G/F'. Then NF/F' is nilpotent, and hence also NF is nilpotent by P. Hall's nilpotency criterion (see [12, part 1, theorem 2.27]). Therefore N is contained in F, and F/F' is the Fitting subgroup of G/F'. ■

Let G be a group. An automorphism θ of G is called a *power automorphism* if $H^\theta = H$ for every subgroup H of G, and θ is said to be *locally universal* if for every finitely generated subgroup E of G there exists an integer n such that $x^\theta = x^n$ for every element x of E.

Lemma 3.2. *Let G be a hyperabelian group with nilpotent Fitting subgroup, and let N be a normal subgroup of G. If $N \leqslant \omega(G)$, then $[N, G]$ is contained in the Fitting subgroup of N.*

Proof. Let x be any element of N. Since N is contained in $\omega(G)$, x induces on F/F' a power automorphism, which is locally universal (see [3, theorem 3.4.1]). Thus $[x, g]$ acts trivially on F/F' for every $g \in G$. On the other hand, F/F' is the Fitting subgroup of G/F' by Lemma 3.1, so that $C_{G/F'}(F/F') = F/F'$ and $[x, g]$ belongs to F. Therefore, $[N, G]$ is contained in $F \cap N$, and so also in the Fitting subgroup of N. ∎

We will often use the following result, that has already been proved by the second author (see [16, lemma 3.3]).

Lemma 3.3. *Let G be a periodic soluble group, and let N be a normal π-subgroup of G and H a π'-subgroup of G (where π is a set of primes), and suppose that either N or H is finite. If x is an element of G such that the subgroups HN and H^xN are conjugate in $\langle H, H^x \rangle N$, then H and H^x are conjugate in $\langle H, H^x \rangle$. In particular, if HN is a pronormal subgroup of G, then also H is pronormal in G.*

Lemma 3.4. *Let G be a periodic soluble group, and let E be a finite p-subgroup of G (p prime). If N is a normal subgroup of G such that $N \leqslant \tau(EN)$, then E is pronormal in EN.*

Proof. Put $K = EN$. Then N is contained in $\tau(K)$ and hence $K = E\tau(K)$. As the Fitting subgroup V of $\tau(K)$ is obviously nilpotent, we have $V = F \cap \tau(K)$, where F is the Fitting subgroup of K. Then F is nilpotent-by-finite, and so even nilpotent. Application of Lemma 3.2 yields that $[\tau(K), K]$ is contained in V, so that K/V is a nilpotent group. In particular, the subgroup EV is subnormal in K. Consider now the normal subgroup $L = O_{p'}(V)$ of K. Clearly EV/L is a nilpotent-by-finite p-group, so that it is hypercentral and hence EL is ascendant in EV, and so also in K. Thus EL is a normal subgroup of $K = E\tau(K)$, and it follows from Lemma 3.3 that E is pronormal in K. ∎

A group G is called a *$\bar{T}$-group* if each subgroup of G is a T-group. It is well known that every soluble non-periodic $\bar{T}$-group is abelian. We will prove that a soluble group G is a $\bar{T}$-group if and only if $\tau(G) = G$, i.e. if and only if all its ascendant subgroups are normal.

Lemma 3.5. *Let G be a group whose cyclic subgroups are pronormal. Then G is a $\bar{T}$-group.*

Proof. Let x be any element of G. Since $\langle x \rangle$ is a pronormal subgroup of G, we have

$$G = N_G(\langle x \rangle)\langle x \rangle^G,$$

and hence

$$\langle x \rangle^G = \langle x \rangle^{N_G(\langle x \rangle)\langle x \rangle^G} = \langle x \rangle^{\langle x \rangle^G}.$$

Then G is a T-group (see [11, lemma 2.1.1]). As the hypotheses are inherited by every subgroup of G, it follows that G is even a $\bar{T}$-group. ■

Following the notation introduced in [11] we will say that a soluble non-abelian T-group G is of *type I* if the centraliser $C_G(G')$ is not periodic, and that G is of *type II* if it is not periodic and $C_G(G')$ is periodic. Note also that for a T-group the properties of being soluble or hyperabelian are obviously equivalent.

Theorem 3.6. *Let G be a soluble group. Then G is a $\bar{T}$-group if and only if all its ascendant subgroups are normal.*

Proof. Suppose first that G is a $\bar{T}$-group, and assume by contradiction that G contains an ascendant non-normal subgroup H. Let

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_\tau = G$$

be an ascending series, and let $\alpha \leqslant \tau$ be the least ordinal such that H is not normal in H_α. Clearly α is not a limit ordinal, and H is a normal subgroup of $H_{\alpha-1}$. On the other hand, $H_{\alpha-1}$ is normal in the T-group H_α, so that H is also normal in H_α. This contradiction shows that every ascendant subgroup of G is normal.

Conversely, suppose that all ascendant subgroups of G are normal, so that $\tau(G) = G$. If G is periodic, it follows from Lemma 3.4 and Lemma 2.2 that every cyclic subgroup of G is pronormal, so that G is a $\bar{T}$-group by Lemma 3.5. Assume now that the soluble T-group G is not periodic, and let F be the Fitting subgroup of G. If G is of type I, and T is the largest periodic normal subgroup of G, the factor group G/T is not abelian. Therefore without loss of generality it can be assumed that G has no periodic non-trivial normal subgroups, and in particular F is a torsion-free abelian group. Let z be an element of G such that $a^z = a^{-1}$ for all $a \in F$ (see [11, theorem 3.1.1]). Then $z^4 = 1$ and z^2 belongs to F, so that $z^2 = 1$ and $F^2 = F^4$. It follows that F contains a subgroup L such that F/L is a group of type 2^∞. Then G/L is a hypercentral non-abelian group, contradicting the hypothesis. Assume now that G is of type II, so that F is periodic. Clearly there exists a prime p such that $G/O_{p'}(G)$ is not abelian, and the above argument proves that the Fitting subgroup of $G/O_{p'}(G)$ is periodic, so that it is a p-group. Thus we may suppose that the Fitting subgroup of G is a p-group for some prime p. For each positive integer n let S_n be the nth term of the upper socle series of F. If x is any element of infinite order of G, we have that $y = x^{p-1}$ acts trivially on S_{n+1}/S_n for all n, and hence $\langle y, S_n \rangle$ is normal in $\langle y, S_{n+1} \rangle$. Then $\langle y \rangle$ is an ascendant subgroup of G, so that $\langle y \rangle$ is normal in G, and y belongs to F. This last contradiction completes the proof of the theorem. ■

It follows in particular from Theorem 3.6 that the class of soluble groups whose ascendant subgroups are normal is subgroup-closed.

Lemma 3.7. *Let G be an FC-group. Then $\omega(G) = \tau(G)$.*

Proof. Let x be an element of $\omega(G)$, and let H be any ascendant subgroup of G and h an element of H. As the normal closure $N = \langle h, x\rangle^G$ is a finitely generated FC-group, the factor group $N/Z(N)$ is finite, so that in particular $H \cap N$ is a subnormal subgroup of N. Thus $H \cap N$ is subnormal in G, and hence $(H \cap N)^x = H \cap N$. It follows that h^x belongs to H, and so $H^x \leqslant H$. Therefore $\omega(G)$ is contained in the normaliser of every ascendant subgroup of G, and hence $\omega(G) = \tau(G)$. ■

In the above lemma the assumption that G is an FC-group cannot be omitted. In fact, if D_{2^∞} is the locally dihedral 2-group, the subgroup $\tau(D_{2^\infty})$ has order 2 and $\omega(D_{2^\infty}) = D_{2^\infty}$.

Corollary 3.8. *Let G be a locally soluble T-group. If G is an FC-group, then it is also a $\bar{T}$-group.*

Proof. By Lemma 3.7 we have $G = \omega(G) = \tau(G)$, so that every ascendant subgroup of G is normal. Moreover, the group G is hyperabelian (see [15, corollary 1.15]), and so even soluble, so that G is a $\bar{T}$-group by Theorem 3.6. ■

It was proved by Peng [10] that a finite group G is a soluble T-group if and only if all its primary subgroups are pronormal, and this is also equivalent to the property that all subgroups of G are pronormal (see [4, lemma 9]).

Theorem 3.9. *Let G be an FC-group. The following statements are equivalent:*

(i) *G is a soluble T-group;*
(ii) *every subgroup of G is pronormal;*
(iii) *every cyclic subgroup of G is pronormal.*

Proof. If every cyclic subgroup of G is pronormal, it follows from Lemma 3.5 that G is a $\bar{T}$-group. In particular, the locally finite group $G/Z(G)$ is metabelian, and so G is soluble. Thus it is enough to prove that if G is a soluble T-group, then all its subgroups are pronormal. Let $\langle a\rangle$ be a cyclic subgroup of G, and let x be any element of G. As G is a $\bar{T}$-group by Corollary 3.8, the subgroup $\langle a, x\rangle$ is a finitely generated soluble T-group, so that it is either finite or abelian (see [11, theorem 3.3.1]), and in particular all its cyclic subgroups are pronormal. Therefore $\langle a\rangle$ and $\langle a\rangle^x$ are conjugate in $\langle a, a^x\rangle$, and $\langle a\rangle$ is a pronormal subgroup of G. It follows now from Theorem 2.6 that every subgroup of G is pronormal. ■

A group G is called *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Since finite $\bar{T}$-groups are metabelian, it follows easily that every periodic locally graded $\bar{T}$-group is locally finite, and so also metabelian.

Lemma 3.10. *Let G be a periodic locally graded $\bar{T}$-group, and let P be a Sylow p-subgroup of G (p prime). If P is pronormal in G, then all subgroups of P are pronormal in G.*

PROOF. Let A be the last term of the lower central series of G. Then G/A is a Dedekind group, A is an abelian Hall subgroup of G and all subgroups of A are normal in G (see [11, theorem 6.1.1]), so that without loss of generality it can be assumed that $P \cap A = \{1\}$. The subgroup $N = PA$ is normal in G, and the factor group G/N has no elements of order p by Lemma 2.7, so that N is a Hall subgroup of G. Let π be the set of all prime numbers which are orders of elements of G/N. Consider any element x of G, and let E be the π-component of $\langle x \rangle$. Then x belongs to the subgroup $K = EN$. Let $L = N_K(P)$ be the normaliser of P in K. Since $L \cap N$ is a Hall normal subgroup of L, and $L/L \cap N$ is finite, it is well known that L contains a finite π-subgroup V such that $L = V(L \cap N)$ and $V \cap N = \{1\}$. Put $B = VP$, so that $B \cap A = \{1\}$ and B is a Dedekind group. As P is a pronormal subgroup of K, we have

$$K = LN = VN = BA,$$

and so $x = ba$, where $b \in B$ and $a \in A$. Let H be any subgroup of P. Clearly H is a Sylow p-subgroup of $H\langle a \rangle$, and the index $|H\langle a \rangle : H|$ is finite, so that the subgroups H and H^a are conjugate in $\langle H, H^a \rangle$. On the other hand, $H^x = H^{ba} = H^a$ since B is a Dedekind group, and so H and H^x are conjugate in $\langle H, H^x \rangle$. Therefore H is a pronormal subgroup of G. ■

The last result of this section is another extension of Peng's theorem.

Theorem 3.11. *Let G be a periodic locally graded group satisfying the minimal condition on primary subgroups. Then G is a $\bar{T}$-group if and only if all its primary subgroups are pronormal.*

PROOF. Suppose first that G is a $\bar{T}$-group, so that in particular it is soluble. As in any soluble locally finite group satisfying the minimal condition on primary subgroups there is a unique conjugacy class of Sylow p-subgroups for each prime p (see for instance [6, corollary 3.18]), it follows that every Sylow subgroup of G is pronormal, and hence all primary subgroups of G are pronormal by Lemma 3.10. Conversely, if every primary subgroup of G is pronormal, then all cyclic subgroups of G are pronormal, and G is a $\bar{T}$-group by Lemma 3.5. ■

In the situation of Theorem 3.11 it cannot be proved that all subgroups of G are pronormal; this can be seen from the following example due to Kuzennyi and Subbotin [8]. Let

$$\{p_n \mid n \in \mathbb{N}\}$$

and

$$\{q_n \mid n \in \mathbb{N}\}$$

be infinite disjoint sets of odd prime numbers such that $p_n = 1 + k_n q_n$ for all n (where k_n is a suitable integer), and let $\langle a_n \rangle$ and $\langle b_n \rangle$ be groups of order p_n and q_n respectively. Put

$$A = \underset{n \in \mathbb{N}}{\mathrm{Dr}} \langle a_n \rangle$$

and

$$B = \langle b \rangle \times \underset{n \in \mathbb{N}}{\mathrm{Dr}} \langle b_n \rangle,$$

where b has order 2. Consider now the semidirect product $G = B \rtimes A$, where $[a_n, b_m] = 1$ if $n \neq m$ and

$$a_n^b = a_n^{-1}, \quad a_n^{b_n} = a_n^{-k_n}$$

for all n. Clearly all Sylow subgroups of G have prime order, and hence all its primary subgroups are pronormal. On the other hand, it is easy to prove that the direct product

$$H = \underset{n \in \mathbb{N}}{\mathrm{Dr}} \langle a_n b_n \rangle$$

is a Sylow π-subgroup of G, where $\pi = \{q_n \mid n \in \mathbb{N}\} \cup \{2\}$, and it follows from Lemma 2.7 that H is not pronormal in G, since G/H^G contains elements of order 2.

4. The pronorm of a group

Let χ be a property pertaining to subgroups. The *χ-pronorm* of a group G is the set $P_\chi(G)$ of all elements of G pronormalising every χ-subgroup of G. In particular, if $\mathfrak{X}$ is a group class, the $\mathfrak{X}$-pronorm $P_{\mathfrak{X}}(G)$ is the subset of G consisting of the elements pronormalising every subgroup of G which belongs to $\mathfrak{X}$. If $\mathfrak{U}$ is the class of all groups, $P_{\mathfrak{U}}(G)$ is just the *pronorm* $P(G)$ of G. Here we will also be interested in the pronorms relative to the classes $\mathfrak{C}$ of all cyclic groups and $\mathfrak{F}$ of all finite groups. The subset $P_{\mathfrak{C}}(G)$ will be called the *cyclic pronorm* of G. The first lemma of this section shows in particular that, if *sn* and *asc* denote the properties of being a subnormal subgroup and an ascendant subgroup respectively, then $P_{sn}(G) = \omega(G)$ and $P_{asc}(G) = \tau(G)$ for any group G.

Lemma 4.1. *Let G be a group, and let H be an ascendant subgroup of G. If x is an element of G pronormalising H, then $H^x = H$.*

Proof. Assume that $H^x \neq H$, and let

$$H = H_0 < H_1 < \ldots < H_\alpha < H_{\alpha+1} < \ldots < H_\tau = G$$

be an ascending series. By hypothesis there exists an element y of $\langle H, H^x \rangle$ such that $H^x = H^y$, so that y belongs to $\langle H, H^y \rangle$. Consider now the least ordinal μ such that $y \in H_\mu$. Clearly $\mu = \alpha + 1$ for some ordinal α, and $H \leqslant H_\alpha \lhd H_\mu$ so that $y \in \langle H, H^y \rangle \leqslant H_\alpha$. This contradiction shows that $H^x = H$. ■

The above lemma shows in particular that in a hypercentral group the pronorm and the norm coincide. This property also holds in the case of locally nilpotent groups. In fact, if G is a locally nilpotent group, and x is an element of G pronormalising a subgroup H, then $H^x = H$ (see [16, lemma 2.5]), and hence $P(G) = N(G)$.

Lemma 4.2. *Let G be a hyperabelian group, and let x be an element of G pronormalising all indecomposable cyclic subgroups of G. Then x normalises every ascendant subgroup of G.*

PROOF. Assume by contradiction that the lemma is false, and consider a counterexample G containing an ascendant subgroup K such that $K^x \neq K$ and K has an ascending normal series

$$\{1\} = K_0 < K_1 < \ldots < K_\tau = K$$

with abelian factors for which the ordinal τ is minimal. Clearly τ cannot be a limit ordinal, and hence the subgroup $H = K_{\tau-1}$ is normal in $L = \langle K, x\rangle$. Let C/H be any indecomposable cyclic subgroup of L/H. Then $C = \langle c\rangle H$, where the cyclic subgroup $\langle c\rangle$ of G is also indecomposable, so that x pronormalises $\langle c\rangle$. Application of Lemma 2.1 yields that C is pronormalised by x, and hence the coset xH pronormalises all indecomposable cyclic subgroups of L/H. It follows now from Lemma 4.1 that xH normalises every indecomposable cyclic subgroup of the abelian group K/H, so that $K^x = K$. This contradiction proves the lemma. ■

In the alternating group A_5 all non-cyclic subgroups are pronormal, so that $P(A_5) = P_{\mathfrak{C}}(A_5)$. Moreover, every subgroup of order 2 of A_5 is not pronormal, and it is easy to show that $P(A_5)$ has order 40. In particular, the pronorm of A_5 is not a subgroup.

In order to prove that in certain relevant cases the pronorm and the cyclic pronorm of a group are subgroups, we introduce for any group G two descending normal series related to the subgroups $\omega(G)$ and $\tau(G)$.

Let G be a group. The *lower Wielandt series* of G is the descending normal series whose terms $\omega_\alpha(G)$ are defined inductively by the positions

$$\omega_0(G) = G, \quad \omega_{\alpha+1}(G) = \bigcap_{K \in \Omega_\alpha(G)} \omega(K),$$

where $\Omega_\alpha(G)$ is the set of all subgroups of G containing $\omega_\alpha(G)$, and

$$\omega_\lambda(G) = \bigcap_{\beta < \lambda} \omega_\beta(G)$$

if λ is a limit ordinal. The last term of the lower Wielandt series of G will be denoted by $\bar{\omega}(G)$. The *lower τ-series* of G is the descending normal series obtained by replacing in the above definition the Wielandt subgroup $\omega(X)$ by the subgroup $\tau(X)$ for each group X. The last term of the lower τ-series of G will be denoted by $\bar{\tau}(G)$. Clearly $\omega_1(G) = \omega(G)$ and $\tau_1(G) = \tau(G)$. Our next two lemmas describe the behaviour of $\bar{\tau}(G)$ and $\bar{\omega}(G)$ with respect to subgroups and homomorphic images.

Lemma 4.3. *Let G be a group, and let H be a subgroup of G. Then $\bar{\tau}(G)$ is contained in $\bar{\tau}(H\bar{\tau}(G))$ and $\bar{\omega}(G)$ is contained in $\bar{\omega}(H\bar{\omega}(G))$.*

PROOF. Let α be an ordinal such that $\bar{\tau}(G) = \tau_\alpha(G)$, and suppose that $\bar{\tau}(G)$ is contained in $\tau_\beta(H\bar{\tau}(G))$ for some ordinal β. If K is any subgroup of G such that

$$\tau_\beta(H\bar{\tau}(G)) \leqslant K \leqslant H\bar{\tau}(G),$$

then $\tau_\alpha(G)$ is contained in K, and hence

$$\bar{\tau}(G) = \tau_{\alpha+1}(G) \leqslant \tau(K).$$

Thus $\bar{\tau}(G)$ lies in $\tau_{\beta+1}(H\bar{\tau}(G))$, and it follows that $\bar{\tau}(G)$ is contained in $\bar{\tau}(H\bar{\tau}(G))$. A similar argument proves the statement for the subgroup $\bar{\varpi}(G)$. ■

Lemma 4.4. *Let G be a group, and let N be a normal subgroup of G. Then $\bar{\tau}(G)N/N$ is contained in $\bar{\tau}(G/N)$ and $\bar{\varpi}(G)N/N$ is contained in $\bar{\varpi}(G/N)$.*

Proof. Assume by contradiction that $\bar{\tau}(G)N/N$ is not contained in $\bar{\tau}(G/N)$, and let α be the smallest ordinal such that $\bar{\tau}(G)N/N$ is not a subgroup of $\tau_\alpha(G/N)$. Clearly α is not a limit ordinal, and $\bar{\tau}(G)N/N$ lies in $\tau_{\alpha-1}(G/N)$. Let K/N be any subgroup of G/N containing $\tau_{\alpha-1}(G/N)$. Then $\bar{\tau}(G) \leqslant \tau(K)$, and hence $\bar{\tau}(G)N/N \leqslant \tau(K/N)$, so that $\bar{\tau}(G)N/N$ is also contained in $\tau_\alpha(G/N)$. This contradiction proves that $\bar{\tau}(G)N/N$ is a subgroup of $\bar{\tau}(G/N)$. A similar argument can be used to prove the statement concerning the last term of the lower Wielandt series. ■

Lemma 4.5. *Let G be a hyperabelian group. Then the cyclic pronorm $P_{\mathfrak{C}}(G)$ of G is contained in $\bar{\tau}(G)$.*

Proof. Suppose that $P_{\mathfrak{C}}(G)$ is contained in $\tau_\alpha(G)$ for some ordinal α, and let K be any subgroup of G such that $\tau_\alpha(G) \leqslant K$. Then $P_{\mathfrak{C}}(G)$ normalises every ascendant subgroup of K by Lemma 4.2, so that $P_{\mathfrak{C}}(G)$ is contained in $\tau(K)$, and hence also in $\tau_{\alpha+1}(G)$. Therefore $P_{\mathfrak{C}}(G)$ is a subset of $\bar{\tau}(G)$. ■

The example by Kovacs, Neumann and de Vries considered in Section 2 also shows that in a periodic soluble group G the pronorm $P(G)$ can be a proper subset of the cyclic pronorm $P_{\mathfrak{C}}(G)$. On the other hand, it is possible to obtain the following result, showing in particular that in every finite soluble group the pronorm is a subgroup.

Theorem 4.6. *Let G be a periodic hyperabelian group. Then $P_{\mathfrak{F}}(G) = P_{\mathfrak{C}}(G) = \bar{\tau}(G)$.*

Proof. It follows from Lemma 4.5 that

$$P_{\mathfrak{F}}(G) \subseteq P_{\mathfrak{C}}(G) \subseteq \bar{\tau}(G),$$

so that it is enough to prove that every finite subgroup E of G is pronormalised by all elements of $\bar{\tau}(G)$. By induction on the derived length of E it can be assumed that the commutator subgroup E' of E is pronormal in $E'\bar{\tau}(G)$. Write

$$E/E' = \langle u_1 E' \rangle \times \ldots \times \langle u_t E' \rangle,$$

where each u_i is an element of prime-power order. Since $\bar{\tau}(G)$ is a hyperabelian T-group, it is soluble, so that also $\langle u_i \rangle \bar{\tau}(G)$ is a soluble group. Moreover, $\bar{\tau}(G)$ is contained in $\tau(\langle u_i \rangle \bar{\tau}(G))$ by Lemma 4.3, and hence $\langle u_i \rangle$ is a pronormal subgroup of $\langle u_i \rangle \bar{\tau}(G)$ by Lemma 3.4. It follows now from Lemma 2.2 that $\langle u_i, E' \rangle$ is pronormal in $\langle u_i, E' \rangle \bar{\tau}(G)$, and so also that E is a pronormal subgroup of $E\bar{\tau}(G)$. ■

Corollary 4.7. *Let G be a polycyclic group. Then $P(G) = P_{\mathfrak{C}}(G) = \bar{\varpi}(G)$, and in particular the pronorm of G is a subgroup.*

PROOF. By Lemma 4.5 we have

$$P(G) \subseteq P_{\mathfrak{C}}(G) \subseteq \bar{\tau}(G) = \bar{\omega}(G).$$

Let H be any subgroup of G, and let x be an element of $\bar{\omega}(G)$. If N is a normal subgroup of finite index of G, by Theorem 4.6 we have that $P(G/N) = \bar{\omega}(G/N)$. On the other hand, $\bar{\omega}(G)N/N$ is contained in $\bar{\omega}(G/N)$ by Lemma 4.4, and hence xN pronormalises HN/N. It follows now from Theorem 2.8 that x pronormalises H, so that $\bar{\omega}(G)$ is contained in the pronorm of G, and so $P(G) = P_{\mathfrak{C}}(G) = \bar{\omega}(G)$. ■

Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G having only finitely many conjugates. Thus a group is an FC-group if and only if it coincides with its FC-centre.

Lemma 4.8. *Let G be a hyperabelian group such that $\bar{\tau}(G)$ is contained in the FC-centre of G. Then the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a subgroup of G.*

PROOF. By Lemma 4.5 the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a subset of $\bar{\tau}(G)$, and hence it is contained in the FC-centre of G. Let X be a finite subset of $P_{\mathfrak{C}}(G)$, and let a be any element of G. As the normal closure $\langle X \rangle^G$ is a polycyclic group, also $\langle X, a \rangle$ is polycyclic, and hence the cyclic pronorm of L is a subgroup by Corollary 4.7. Thus $\langle a \rangle$ is pronormalised by every element of $\langle X \rangle$, and $\langle X \rangle$ is contained in $P_{\mathfrak{C}}(G)$. It follows that $P_{\mathfrak{C}}(G)$ is a subgroup of G. ■

Theorem 4.9. *Let G be a locally soluble FC-group. Then $P(G) = P_{\mathfrak{C}}(G)$, and in particular the pronorm of G is a subgroup.*

PROOF. Clearly G is hyperabelian, and hence the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a normal subgroup of G by Lemma 4.8. Let K be any subgroup of G. Then X is pronormal in $XP_{\mathfrak{C}}(G)$ for every cyclic subgroup X of K, and it follows from Lemma 2.5 that K is pronormal in $KP_{\mathfrak{C}}(G)$. Therefore $P(G) = P_{\mathfrak{C}}(G)$, and in particular $P(G)$ is a subgroup of G. ■

Note that in the symmetric group S_4 the Wielandt subgroup $\omega(S_4)$ has order 4 and $\omega_2(S_4) = \{1\}$. Therefore the pronorm of a finite soluble group G can be properly contained in the Wielandt subgroup of G. On the other hand, we will prove that such subgroups coincide if G is a polycyclic group with nilpotent commutator subgroup. A corresponding result will also be obtained for periodic FC-groups whose commutator subgroup is locally nilpotent.

Lemma 4.10. *Let G be a periodic FC-group with locally nilpotent commutator subgroup, and let K be a subgroup of G. Then $K \cap \tau(G)$ is contained in $\tau(K)$.*

PROOF. Put $L = K \cap \tau(G)$, and let X be any ascendant subgroup of K. Since G' is a locally nilpotent FC-group, it is hypercentral (see [15, theorem 1.16]), so that $Y = X \cap G'$ is an ascendant subgroup of G and L is contained in the normaliser of Y. For each prime number p, let X_p/Y be the unique Sylow p-subgroup of the abelian group X/Y. Clearly X_p is an ascendant subgroup of K, and in order to prove that L normalises X it is enough to show that L is contained in $N_G(X_p)$ for all p. Therefore without loss of generality it can be assumed that X/Y is a p-group for some prime p. Let $G' = U \times V$, where U is a p-group and V has no elements of order p, and write $\bar{G} = G/V$. Then $\bar{G}'$ is a p-group, and so $\bar{G}$ has a unique Sylow p-subgroup $\bar{P}$. Moreover, $\bar{X}$ is a p-subgroup of $\bar{G}$, so that $\bar{X} \leqslant \bar{P}$ and so $\bar{X}$ is ascendant in $\bar{G}$. It follows that $\bar{X}$ is normalised by $\bar{L}$, and hence $L \leqslant N_G(XV)$. Therefore LY/Y lies in the normaliser of $(XV \cap K)/Y = X(V \cap K)/Y$. On the other hand, the ascendant subgroup X/Y is a Sylow p-subgroup of $X(V \cap K)/Y$, so that X/Y is characteristic in $X(V \cap K)/Y$ and L is contained in $N_G(X)$. ■

Theorem 4.11. *Let G be a periodic FC-group with locally nilpotent commutator subgroup. Then $\bar{\tau}(G) = \tau(G)$. In particular, if G is a finite group with nilpotent commutator subgroup, then $\bar{\omega}(G) = \omega(G)$.*

Proof. Let K be any subgroup of G containing $\tau(G)$. Then $\tau(G)$ is contained in $\tau(K)$ by Lemma 4.10, so that $\tau_2(G) = \tau(G)$, and hence $\bar{\tau}(G) = \tau(G)$. ■

Corollary 4.12. *Let G be a periodic FC-group with locally nilpotent commutator subgroup. Then $P(G) = \tau(G)$.*

Proof. Since G is an FC-group, its locally nilpotent subgroup G' is hypercentral, and so G is hyperabelian. Moreover, $P(G) = P_{\mathfrak{C}}(G)$ by Theorem 4.9, and hence it follows from Theorem 4.6 and Theorem 4.11 that $P(G) = \tau(G)$. ■

If G is a group with finite commutator subgroup, then all ascendant subgroups of G are subnormal, and hence $\tau(G) = \omega(G)$. Therefore it follows from Corollary 4.12 that, if G is a periodic group whose commutator subgroup G' is finite and nilpotent, then $P(G) = \omega(G)$. A similar result also holds for polycyclic groups.

Corollary 4.13. *Let G be a polycyclic group with nilpotent commutator subgroup. Then $P(G) = \omega(G)$.*

Proof. Let x be an element of $\omega(G)$, and let H be any subgroup of G. If G^σ is a finite homomorphic image of G, it follows from Corollary 4.7 and Corollary 4.12 that $P(G^\sigma) = \omega(G^\sigma)$, so that x^σ pronormalises the subgroup H^σ of G^σ. Then x pronormalises H by Theorem 2.8, and hence x belongs to the pronorm $P(G)$ of G. Therefore $P(G) = \omega(G)$ by Lemma 4.1. ■

It was shown in [2] that if G is a finitely generated soluble group with finite Prüfer rank, then the Wielandt subgroup $\omega(G)$ is contained in the FC-centre of G. This property can be used to prove our last result concerning the cyclic pronorm; it turns out in particular that the cyclic pronorm of any locally polycyclic group is a subgroup.

Theorem 4.14. *Let G be a group which is locally (soluble with finite Prüfer rank). Then the cyclic pronorm $P_{\mathfrak{C}}(G)$ is a subgroup of G.*

PROOF. Let X be any finite subset of $P_{\mathfrak{C}}(G)$, and let a be any element of G. Then $L = \langle X, a \rangle$ is a finitely generated soluble with finite Prüfer rank, and hence the Wielandt subgroup $\omega(L)$ of L lies in the FC-centre of L (see [2, theorem A]). Thus the cyclic pronorm of L is a subgroup by Lemma 4.8, and so every element of $\langle X \rangle$ pronormalises $\langle a \rangle$. Therefore $\langle X \rangle$ is contained in $P_{\mathfrak{C}}(G)$, and $P_{\mathfrak{C}}(G)$ is a subgroup of G. ■

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