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TRIPLE FACTORIZATIONS BY LOCALLY SUPERSOLUBLE GROUPS

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1. Introduction

It was shown by Kegel [1] that every finite group $G = AB = AC = BC$, factorized by two nilpotent subgroups A and B and a supersoluble subgroup C, is supersoluble too. The authors extended this result in [2] to the case of a trifactorized soluble-by-finite group $G = AB = AC = BC$ with finite abelian section rank, proving that if A and B are nilpotent and C is locally supersoluble then G itself is locally supersoluble. Here a group G is said to have *finite abelian section rank* if it lacks infinite abelian sections of prime exponent. Even in the case of finite groups, it is clearly not enough to assume that the subgroups A, B, and C all are supersoluble. In fact, there exists a finite nonsupersoluble group $G = AB = AC = BC$ written as the product of two supersoluble normal subgroups A and B and a nilpotent subgroup C (see, for instance, [13, p. 152]). On the other hand, Baer [4] proved that, if G is a finite group with nilpotent commutator subgroup and H and K are supersoluble normal subgroups of G, then the product HK is supersoluble. This result suggests that the behavior of the commutator subgroup is the main obstacle in studying the groups factorized by supersoluble subgroups. In fact, we prove in this article that, if $G = AB = AC = BC$ is a group with finite abelian section rank factorized by three locally supersoluble subgroups A, B, and C and the commutator subgroup G' of G is locally nilpotent, then G is locally supersoluble. An analogous result holds for groups with finite abelian section rank having a triple factorization by locally nilpotent subgroups. These results are proved in Section 2, where an extension of Baer's theorem to infinite groups can also be found. Finally, in Section 3 we consider groups with a triple factorization by subgroups having (generalized) nilpotent commutator subgroups.

Most of our notation is standard and can be found in [3]. We refer to [5] for the main properties of factorized groups.

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2. Groups with a Supersoluble Triple Factorization

Our first result generalizes Baer's theorem on products of supersoluble normal subgroups to the case of infinite groups. Recall that a group G is *FC-hypercentral* if it has an ascending normal series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_\tau = G$$

such that every element of $G_{\alpha+1}/G_\alpha$ has finitely many conjugates in G/G_α for each ordinal $\alpha < \tau$.

Lemma 2.1. *Let G be a group with locally nilpotent commutator subgroup and let H and K be normal subgroups of G.*

(a) *If H and K are supersoluble then HK is supersoluble.*

(b) *If H and K are locally supersoluble then HK is locally supersoluble.*

(c) *If H and K are hypercyclic then HK is hypercyclic.*

PROOF. (a) Clearly, the group $L = HK$ is polycyclic and all its finite homomorphic images are supersoluble (see [4, p. 186]). It follows that L itself is supersoluble by a result of Baer (see [6]).

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(b) Let E be a finitely generated subgroup of $L = HK$. Then there exist finitely generated subgroups H_0 of H and K_0 of K such that E is contained in $M = \langle H_0, K_0 \rangle$, and clearly $M = (H \cap M)(K \cap M)$. Since L is locally polycyclic (see [7, Part 1, Theorem 2.31]), the group M is polycyclic, so that $H \cap M$ and $K \cap M$ are supersoluble, and so M is supersoluble by (a). Therefore, L is locally supersoluble.

(c) Let $L = HK$. As the hypotheses are inherited by homomorphic images, it is enough to show that, if $L \neq 1$, then L contains a cyclic nontrivial normal subgroup. The subgroups H and K are FC-hypercentral, so that L is FC-hypercentral too (see [7, Part 1, p. 130). Let $x \neq 1$ be an element of L having only finitely many conjugates. Then $\langle x \rangle^L$ is finitely generated, and $C = C_L(\langle x \rangle^L)$ is a normal subgroup of finite index in L. Therefore, there exists a finitely generated subgroup E of L, containing $\langle x \rangle^L$, such that $L = CE$. Since L is locally supersoluble by (b), the subgroup E is supersoluble, and so it contains a cyclic nontrivial normal subgroup N such that $N \leq \langle x \rangle^L$. Clearly, $C \leq C_L(N)$, and hence N is normal in L. □

Lemma 2.2. *Let G be a group with locally nilpotent commutator subgroup and let H be a locally supersoluble ascendant subgroup of G. Then the normal closure H^G is locally supersoluble.*

PROOF. Let

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_\tau = G$$

be an ascending series with smallest ordinal type τ. By induction on τ we may suppose that the subgroup H^{H_α} is locally supersoluble for each ordinal $\alpha < \tau$. If τ is a limit ordinal, then

$$G = \bigcup_{\alpha<\tau} H_\alpha$$

and hence

$$H^G = \bigcup_{\alpha<\tau} H^{H_\alpha}$$

is obviously locally supersoluble. Suppose that τ is not a limit ordinal, and put $K = H_{\tau-1}$. As H^K is locally supersoluble, on replacing H by H^K we can assume without loss of generality that H is a normal subgroup of K. Then H^x is a locally supersoluble normal subgroup of K for every element x of G. From Lemma 2.1 it follows now that H^G is locally supersoluble. □

Theorem 2.3. *Let G be a group with locally nilpotent commutator subgroup and let H and K be ascendant subgroups of G.*

(a) *If H and K are supersoluble then $\langle H, k \rangle$ is supersoluble.*

(b) *If H and K are locally supersoluble then $\langle H, k \rangle$ is locally supersoluble.*

PROOF. Suppose first that H and K of G are locally supersoluble ascendant subgroups of G. Then the normal subgroups H^G and K^G are locally supersoluble by Lemma 2.2, and Lemma 2.1 implies that $H^G K^K$ is locally supersoluble. In particular, $\langle H, K \rangle$ is a locally supersoluble subgroup of G. Assume now that H and K are supersoluble. Then the locally supersoluble subgroup $\langle H, K \rangle$ is finitely generated and hence supersoluble. □

Note that a result similar to statements (a) and (b) of Theorem 2.3 does not hold for the join of hypercyclic ascendant subgroups of a group. In fact, there exists a locally finite 2-group not hypercentral but generated by two abelian subnormal subgroups (see [8, p. 22]).

A group G is called *parasoluble* if it has a normal series of finite length

$$1 = G_0 \leq G_1 \leq \cdots \leq G_t = G$$

such that for every $i \leq t-1$ the group G_{i+1}/G_i is abelian and all its subgroups are normal in G/G_i. Clearly, a group is supersoluble if and only if it is parasoluble and finitely generated. Moreover, the commutator subgroup of a parasoluble group is nilpotent. The following result on products of parasoluble normal subgroups in particular provides an alternative proof of Baer's theorem for finite groups.

Proposition 2.4. *Let G be a group with nilpotent commutator subgroup and let H and K be parasoluble normal subgroups of G. Then HK is parasoluble.*

PROOF. Clearly, we may suppose that $G = HK$. If the factor group G/G'' is parasoluble, then also G is parasoluble [9, Lemma 2.1], so that without loss of generality G' can be assumed abelian. Then there exist finite G-invariant series with abelian factors

$$1 = H_0 \leq H_1 \leq \cdots \leq H_s = H$$

and

$$1 = K_0 \leq K_1 \leq \cdots \leq K_t = K$$

such that all subgroups of H_{i+1}/H_i are normal in H/H_i if $i \leq s-1$ and all subgroups of K_{j+1}/K_j are normal in K/K_j if $j \leq t-1$ [9, Lemma 2.4). By induction on $s+t$ we infer that the factor group G/H_1 is parasoluble. Moreover, every subgroup of $(H_1 \cap K_{j+1})/(H_1 \cap K_j)$ is normal in $K/(H_1 \cap K_j)$ and so in $G/(H_1 \cap K_j)$ for each $j \leq t-1$. On the other hand, $H_1/(H_1 \cap K)$ is centralized by K, so that all its subgroups are normal in $G/(H_1/K)$. It follows that the group G is parasoluble. □

Our next two lemmas deal with the behavior of self-centralizing radicable abelian normal subgroups of a group.

Lemma 2.5. *Let G be a group and let J be a radicable abelian normal subgroup of G. If N is a normal subgroup of G such that $N \cap J$ is finite, then $N \leq C_G(J)$ and $C_{G/N}(JN/N) = C_G(J)/N$. In particular, if $C_G(J) = J$, then $C_{G/N}(J/N) = J/N$.*

PROOF. Let $C/N = C_{G/N}(JN/N)$. Then $[J, C]$ is contained in $N \cap J$, and so it is finite. On the other hand, $[J, C]$ is a radicable subgroup of G, so that $[J, C] = 1$ and $C = C_G(J)$. □

Lemma 2.6. *Let G be a group, K a locally nilpotent normal subgroup of G, and J a periodic radicable abelian normal subgroup of G contained in K and whose proper G-invariant subgroups are finite. If $C_G(J) = J$, then $J = Z_\omega(K)$.*

PROOF. Clearly, J satisfies the minimal condition on subgroups, and hence it is contained in $Z_\omega(K)$. If n is any positive integer then the subgroup $JZ_n(K)$ is nilpotent, so that $[J, Z_n(K)]$ is a proper G-invariant subgroup of J. Thus, $[J, Z_n(K)]$ is finite. On the other hand, $[J, Z_n(K)]$ is a radicable subgroup of G, and hence $[J, Z_n(K)] = 1$. Therefore, $Z_n(K) \leq C_G(J) = J$, and $J = Z_\omega(K)$. □

Let N be a normal subgroup of a group G. We say that N is *hypercyclically embedded* in G if N has an ascending G-invariant series with cyclic factors. Clearly, every group G contains a largest hypercyclically embedded normal subgroup N, and the factor group G/N has no cyclic nontrivial normal subgroups. Recall also that, if $G = AB$ is a factorized group and N is a normal subgroup of G, the *factorizer* $X(N)$ of N in G is the subgroup $AN \cap BN$. Sesekin [10] proved that this subgroup has the triple factorization

$$X(N) = A^*B^* = A^*N = B^*N,$$

where $A^* = A \cap BN$ and $B = B \cap AN$.

Theorem 2.7. *Let the group $G = AB = AC = BC$ be the product of three locally supersoluble subgroups A, B, and C. If C has finite abelian section rank and the commutator subgroup G' of G is locally nilpotent then G is locally supersoluble.*

PROOF. Suppose first that the commutator subgroup G' of G is nilpotent. Then it is enough to prove that the factor group G/G'' is locally supersoluble. Hence, G' can be assumed abelian without loss of generality. Let U be the largest hypercyclically embedded normal subgroup of AG', and put $\overline{G} = G/U$. Clearly $\overline{A} \cap \overline{G}'$ is a hypercyclically embedded normal subgroup of $\overline{AG}'$, so that $\overline{A} \cap \overline{G}' = 1$, and in particular the subgroup $\overline{A}$ is abelian. Consider now the largest hypercyclically embedded normal subgroup $\overline{V}$ of $\overline{BG}'$ and the largest hypercyclically embedded normal subgroup $\overline{W}$ of $\overline{CG}'$, and put $\widetilde{G} = \overline{G}/\overline{V}$ and $\widehat{G} = \overline{G}/\overline{W}$. Then $\widetilde{B} \cap \widetilde{G}' = 1$ and $\widehat{C} \cap \widehat{G}' = 1$, so that the groups $\widetilde{A}$, $\widetilde{B}$, $\widehat{A}$, and $\widehat{C}$ are abelian. It follows that the factorized groups $\widetilde{G}$ and $\widehat{G}$ are locally supersoluble (see [5, Theorem 6.6.11]). Therefore, the normal subgroups $\bar{B}\bar{G}'$ and $\bar{C}\bar{G}'$ of $\bar{G}$ are locally supersoluble as well, and hence $\bar{G} = (\bar{B}\bar{G}')(\bar{C}\bar{G}')$ is locally supersoluble by Lemma 2.1. Then AG' is a locally supersoluble normal subgroup of G. The same argument shows that BG' is a locally supersoluble normal subgroup of G, so that $G = (AG')(BG')$ is locally supersoluble by Lemma 2.1.

Suppose now that G' is locally nilpotent and let T be the subgroup consisting of all elements of finite order of G'. Then G'/T is nilpotent (see [7, Part 2, Theorem 6.36]), and the first part of the proof implies that the factor group G/T is locally supersoluble. It is clearly enough to show that all Sylow subgroups of T are hypercyclically embedded in G, so that on replacing G by $G/T_{p'}$ we can assume that T is a p-group for some prime p, and in particular T is a Černikov group. Let J be the finite residual of T. Then T/J is finite, so that G'/J is nilpotent, and G/J is locally supersoluble by the first part of the proof. Assume that the theorem is false, and let G be a counterexample such that J has minimal rank. Consider an infinite minimal G-invariant subgroup J_0 of J. Then G/J_0 is locally supersoluble, and J_0 is a radicable subgroup of G whose proper G-invariant subgroups are finite. Clearly, we may suppose that G has no cyclic nontrivial normal subgroups. Then there exists a subgroup L of G such that $G = L \ltimes J_0$ (see [5, Theorem 5.3.14), and L is locally supersoluble. The subgroup $C_L(J_0)$ is normal in G and the factor group $G/C_L(J_0)$ is a counterexample with $J_0 \simeq J_0C_L(J_0)/C_L(J_0)$. Since

$$C_{L/C_L(J_0)}(J_0C_L(J_0)/C_L(J_0)) = 1,$$

it can be assumed without loss of generality that $C_L(J_0) = 1$. Hence, $C_G(J_0) = J_0$. In particular $J_0 = J$, and $J = Z_\omega(G') \leq G'$ by Lemma 2.6. Since G is not locally supersoluble, it follows from Lemma 2.1 that at most one of the normal subgroups AG', BG', and CG' is locally supersoluble. Suppose that AG' is not locally supersoluble, and let U be the largest G-invariant subgroup of J which is hypercyclically embedded in AG'. Clearly, the factor group $\bar{G} = G/U$ is not locally supersoluble, so that U is finite, and it follows from Lemma 2.5 and Lemma 2.6 that $\bar{J} = C_{\bar{G}}(\bar{J}) = Z_\omega(\bar{G}')$. Assume that $\bar{A} \cap \bar{J} \neq 1$. Then $\bar{A} \cap \bar{J}$ contains a cyclic nontrivial subgroup $\langle \bar{a} \rangle$ which is normal in $\bar{A}$. Let n be the smallest positive integer such that $\bar{a}$ belongs to $Z_n(\bar{G}')$, and consider the normal subgroup $\bar{H} = \langle \bar{a}, Z_{n-1}(\bar{G}')\rangle$ of $\bar{A}\bar{G}'$. Since $G = LJ$, we also have $AG' = L_0J$, where $L_0 = AG' \cap L$. Since $\bar{A}\bar{G}'$ has no cyclic nontrivial normal subgroups which are contained in $\bar{J}$, the subgroup $Z_{n-1}(\bar{G}')$ does not contain cyclic nontrivial L_0-invariant subgroups. On the other hand, $\langle \bar{H}, \bar{L}_0\rangle/Z_{n-1}(\bar{G}')$ is locally supersoluble, and so there exists a subgroup $\bar{M}$ of $\bar{G}$ such that $\langle \bar{H}, \bar{L}_0\rangle = \bar{M} \ltimes Z_{n-1}(\bar{G}')$ (see [5, Theorem 5.3.14). Then

$$\bar{H} = (\bar{H} \cap \bar{M}) \times Z_{n-1}(\bar{G}'),$$

and $\bar{H} \cap \bar{M}$ is a cyclic nontrivial normal subgroup of $\bar{M}\bar{J} = \bar{A}\bar{G}'$. This contradiction shows that $\bar{A} \cap \bar{J} = 1$. Since $\bar{G}$ is not locally supersoluble, it follows from Lemma 2.1 that the normal subgroups $\bar{B}\bar{G}'$ and $\bar{C}\bar{G}'$ cannot be both locally supersoluble. Suppose that $\bar{B}\bar{G}'$ is not locally supersoluble, and let $\bar{V}$ be the largest G-invariant subgroup of $\bar{J}$ that is hypercyclically embedded in $\bar{B}\bar{G}'$. The factor group $\tilde{G} = \bar{G}/\bar{V}$ is not locally supersoluble, so that $\bar{V}$ is finite, and the same argument used above proves that $\tilde{B} \cap \tilde{J} = 1$. Since $\bar{V}$ is contained in $\bar{J}$ and $\bar{A} \cap \bar{J} = 1$, we also have $\tilde{A} \cap \tilde{J} = 1$. Moreover, $C_{\tilde{G}}(\tilde{J}) = \tilde{J}$ by Lemma 2.5. Let $\tilde{X}$ be the factorizer of $\tilde{J}$ in $\tilde{G} = \tilde{A}\tilde{B}$. Then

$$\tilde{X} = \tilde{A}_1\tilde{B}_1 = \tilde{A}_1\tilde{J} = \tilde{B}_1\tilde{J},$$

where $\tilde{A}_1 = \tilde{A} \cap \tilde{B}\tilde{J}$ and $\tilde{B}_1 = \tilde{B} \cap \tilde{A}\tilde{J}$. Since $\tilde{A}_1 \cap \tilde{J} = \tilde{B}_1 \cap \tilde{J} = 1$ and $C_{\tilde{X}}(\tilde{J}) = \tilde{J}$, it follows that $\tilde{J} = \tilde{X}$ (see [5, Lemma 6.5.8]). Then $\tilde{J}$ is a factorized subgroup of $\tilde{G} = \tilde{A}\tilde{B}$, and so $\tilde{J} = (\tilde{A} \cap \tilde{J})(\tilde{B} \cap \tilde{J}) = 1$. This last contradiction completes the proof of the theorem. □

Theorem 2.7 has the following obvious consequence.

Corollary 2.8. *Let the group $G = AB = AC = BC$ be the product of three supersoluble subgroups A, B and C. If G has finite abelian section rank and the commutator subgroup G' of G is locally nilpotent, then G is supersoluble.*

In [2] it was also shown that, if $G = AB = AC = BC$ is a soluble-by-finite group with finite abelian section rank factorized by two nilpotent subgroups A and B and a locally nilpotent subgroup C, then G is locally nilpotent. Here we prove the following

Theorem 2.9. *Let the group $G = AB = AC = BC$ be the product of three locally nilpotent subgroups A, B, and C. If G has finite abelian section rank and the commutator subgroup G' of G is locally nilpotent then G is locally nilpotent.*

PROOF. For G' nilpotent, the result can be obtained on using an argument similar to that of the first part of the proof of Theorem 2.7. Suppose now that G' is locally nilpotent and that the theorem is false. In the same way as in the proof of Theorem 2.7 (using Theorem 5.3.7 of [5] instead of Theorem 5.3.14 of [5]) we can reduce the matter to the case of a semidirect product $G = L \ltimes J$, where $J = C_G(J)$ is a radicable abelian normal p-subgroup of finite rank in G whose proper G-invariant subgroups are finite and L is locally nilpotent. Clearly, L is isomorphic with a group of automorphisms of J. Hence, L is linear over the field of p-adic numbers and so it is nilpotent (see [7, Part 2, Theorem 6.32]). In particular G is soluble. Consider the normal subgroup $U = J \cap \overline{Z}(AG')$ of G (where $\overline{Z}(AG')$ denotes the hypercenter of AG') and put $\overline{G} = G/U$. If $\overline{V} = \overline{J} \cap \overline{Z}(\overline{B}\overline{G}')$ and $\widetilde{G} = \overline{G}/\overline{V}$, it can be derived as in the proof of Theorem 2.7 that $\widetilde{A} \cap \widetilde{J} = \widetilde{B} \cap \widetilde{J} = 1$, so that in particular $\widetilde{A}$ and $\widetilde{B}$ are nilpotent subgroups of $\widetilde{G}$. Thus, $\widetilde{G}$ is locally nilpotent (see [5, Theorem 6.6.6]), and so $\overline{B}\overline{G}'$ is locally nilpotent. It follows that

$$\overline{G} = \overline{A}\overline{C} = \overline{A}(\overline{B}\overline{G}') = \overline{C}(\overline{B}\overline{G}')$$

is locally nilpotent (see [5, Theorem 6.3.8]). Then AG' is locally nilpotent, and the same result implies that

$$G = BC = B(AG') = C(AG')$$

is locally nilpotent. This contradiction completes the proof. □

An example of Sysak (see [11] or [5, Theorem 6.1.2]) shows that in the above theorems the hypothesis that the group has finite abelian section rank cannot be omitted. In fact, there exists a group G that is not locally polycyclic but has a triple factorization $G = AB = AK = BK$, where A, B, and K are abelian and K is normal in G. On the other hand, Robinson and Stonehewer [12] proved that, if $G = AB = AC = BC$ is a group with a triple factorization by abelian subgroups then all chief factors of G are central.

3. Groups with a Nilpotent-by-Abelian Triple Factorization

Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and a locally nilpotent normal subgroup K. Assume further that either G has finite abelian section rank or K is a minimax group. If A and B are locally nilpotent or locally supersoluble, then the group G has the same property (see [5, Chapter 6]). Here we obtain similar results when A and B have (generalized) nilpotent commutator subgroups. Note that the example, considered in the introduction, of a finite nonsupersoluble group factorized by three supersoluble subgroups also shows that in such statements it is not enough to assume that the factors A, B, and K all have (generalized) nilpotent commutator subgroups.

A group G is called an $\mathfrak{S}_1$-*group* if it has finite abelian section rank and the set of primes $\pi(G)$ is finite. It is well known that the Fitting subgroup of every $\mathfrak{S}_1$-group is nilpotent. In our proofs we need the following result which has been proved in [13].

Lemma 3.1 *Let the soluble group $G = AB$ be the product of two subgroups A and B and let H and F be the Hirsch–Plotkin radical and the Fitting subgroup of G.*

(a) *If G has finite abelian section rank and A_0 and B_0 are the Hirsch-Plotkin radicals of A and B, then $H = A_0H \cap B_0H$.*

(b) *If G is an $\mathfrak{S}_1$-group and A_1 and B_1 are the Fitting subgroups of A and B, then $F = A_1F \cap B_1F$.*

Theorem 3.2. *Let the group $G = AB = AK = BK$ with finite abelian section rank be the product of two subgroups A and B and a locally nilpotent normal subgroup K. If A' and B' are locally nilpotent then the commutator subgroup G' of G is locally nilpotent.*

PROOF. Suppose first that G is soluble and let H be the Hirsch–Plotkin radical of G. Then K is contained in H, so that $AH = BH = G$ and $A'H = B'H$. If A_0 is the Hirsch–Plotkin radical of A and B_0 is the Hirsch–Plotkin radical of B then by Lemma 3.1 we have

$$G' \leq A'H \leq A_0H \cap B_0H = H.$$

Hence, G' is locally nilpotent. In the general case, if T is the largest periodic normal subgroup of G then the factor group G/T is soluble (see [7, Part 2, Lemma 9.34]). The periodic radical group T satisfies the minimal condition on primary subgroups, so that for every prime p the group $T/O_{p'}(T)$ is a finite extension of a p-group (see [14, Theorem 3.17]), and hence is a Černikov group. Thus $G/O_{p'}(T)$ is soluble, and so its commutator subgroup $G'O_{p'}(T)/O_{p'}(T)$ is locally nilpotent. Let K_p be the unique Sylow p-subgroup of K. Since $K_p \cap O_{p'}(T) = 1$, it follows that $K_p \cap G'$ is contained in the hypercenter of G'. As $G'/(G' \cap K \cap T)$ is locally nilpotent, the subgroup G' is also locally nilpotent. □

Theorem 3.3. *Let the $\mathfrak{S}_1$-group $G = AB = AK = BK$ be the product of two subgroups A and B and a nilpotent normal subgroup K. If A' and B' are nilpotent, then the commutator subgroup G' of G is nilpotent too.*

PROOF. Since K is contained in the Fitting subgroup F of G, we have $AF = BF = G$ so that $A'F = B'F$. If A_1 and B_1 are the Fitting subgroups of A and B, then it follows from Lemma 3.1 that

$$G' \leq A'F \leq A_1F \cap B_1F = F.$$

Hence, G' is nilpotent. □

Lemma 3.4 *Let G be a locally nilpotent group and let N be a minimax normal subgroup of G. Then N is contained in the hypercenter of G.*

PROOF. The subgroup T consisting of all elements of finite order of N is a Černikov group, and so it is contained in the hypercenter of the locally nilpotent group G. Moreover, N/T is a torsion-free nilpotent group, and hence it lies in the hypercenter of G/T (see [7, Part 2, Lemma 6.37]). Therefore, $N \leq \bar{Z}(G)$.

Lemma 3.5. *Let G be a group whose finite homomorphic images have nilpotent commutator subgroup and let N be a finite normal subgroup of G such that the commutator subgroup of G/N is locally nilpotent. Then the commutator subgroup of G is locally nilpotent.*

PROOF. Clearly, it can be assumed that N is a minimal normal subgroup of G. Since all finite homomorphic images of G are soluble, the subgroup N is abelian of prime exponent. Assume that G' is not locally nilpotent, so that $[N, G'] = N$. Thus, $H_0(G'/N, N) = 0$, and so $H^2(G/N, N) = 0$ (see [15, Theorem 3.4]). Therefore, G contains a subgroup L such that $G = L \ltimes N$. The centralizer $C_L(N)$ is a normal subgroup of G, and the factor group $G/C_L(N)$ is finite, so that $G'C_L(N)/C_L(N)$ is nilpotent. As G'/N is locally nilpotent, also G' is locally nilpotent, and this contradiction proves the lemma. □

Our next lemma can be proved in the same way as Lemma 6.5.8 of [5].

Lemma 3.6. *Let the group $G = AB = AK = BK$ be the product of two nilpotent-by-finite subgroups A and B and a radicable abelian proper normal p-subgroup of finite rank K. If $A \cap K = B \cap K = 1$ then K is properly contained in its centralizer $C_G(K)$.*

Let G be a soluble minimax group. Then G has a series of finite length whose factors either are finite or infinite cyclic or of type p^∞ for some prime p. The number of infinite factors in such a series is an invariant of G which is called the *minimax rank* of G.

Theorem 3.7. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and a minimax normal subgroup K.*

(a) *If A', B', and K are locally nilpotent then G' is locally nilpotent.*

(b) *If A', B', and K are hypercentral then G' is hypercentral.*

(c) *If A', B', and K are nilpotent then G' is nilpotent.*

PROOF. (a) Assume the result false and choose a counterexample G such that K has minimal minimax rank. Suppose first that K is torsion-free and so nilpotent. Then G/K is not locally nilpotent. Hence, $K' = 1$ by the minimal choice of K, so that K is abelian. The intersection $A' \cap K$ is a normal subgroup of G, and it is contained in the hypercenter of A' by Lemma 3.4. As $G' \leq A'K$, it follows that $A' \cap K$ is contained in the hypercenter of G', and hence $G'/(A' \cap K)$ is not locally nilpotent. Therefore, $A' \cap K = 1$, and so $[A \cap K, A] = 1$. Then $A \cap K$ lies in the center of $G = AK$, and $G'(A \cap K)/(A \cap K)$ is not locally nilpotent, so that $A \cap K = 1$. The centralizer $C_A(K)$ is a normal subgroup of G, and the factor group $G/C_A(K)$ is also a counterexample. As

$$C_{A/C_A(K)}(KC_A(K)/C_A(K)) = 1,$$

it can be assumed without loss of generality that $C_A(K) = 1$. Then $C_G(K) = K$, and G/K is isomorphic with a group of automorphisms of K. It follows that every abelian subgroup of G/K is minimax (see [7, Part 2, Corollary to Lemma 10.37]), and hence G/K itself is a minimax group (see [7, Part 2, Theorem 10.35]). Therefore, G is minimax, and G' is locally nilpotent by Theorem 3.2. This contradiction shows that K is not torsion-free. Let T be the subgroup consisting of all elements of finite order of K. Then K/T is torsion-free, and $G'T/T$ is locally nilpotent by the first part of the proof. Since G' is not locally nilpotent, there exists a prime number p such that $G'T_{p'}/T_{p'}$ is not locally nilpotent. Hence, without loss of generality it can be assumed that T is a p-group. Let J be the finite residual of T. Then T/J is finite, and so $G'J/J$ is locally nilpotent by Lemma 3.5. Consider an infinite minimal G-invariant subgroup J_0 of J. Then $G'J_0/J_0$ is locally nilpotent, and J_0 is a radicable subgroup of G whose proper G-invariant subgroups are finite. Clearly, $J_0 \cap \overline{Z}(G')$ is a proper subgroup of J_0, so that $J_0 \cap \overline{Z}(G')$ is finite and $G/(J_0 \cap \overline{Z}(G'))$ is a counterexample. Therefore, without loss of generality it can be assumed that $J_0 \cap Z(G') = 1$. Moreover, the locally nilpotent group $G'/C_{G'}(J_0)$ is isomorphic with a group of automorphisms of J_0, and so it is hypercentral (see [7, Part 2, Theorem 6.32]). Then there exists a subgroup L of G such that $G = L \ltimes J_0$ (see [5, Theorem 5.3.7]). The centralizer $C_L(J_0)$ is a normal subgroup of G, and $G'C_L(J_0)/C_L(J_0)$ is not locally nilpotent. Since

$$C_{L/C_L(J_0)}(J_0C_L(J_0)/C_L(J_0)) = 1,$$

we may also suppose that $C_L(J_0) = 1$ and $C_G(J_0) = J_0$. In particular, $J = J_0 = Z_\omega(K)$ by Lemma 2.6. Let A_0 be the Hirsch–Plotkin radical of A, and assume that $A_0 \cap J \neq 1$. Since J satisfies the minimal condition on subgroups, it follows that $Z(A_0) \cap J \neq 1$. Let a be a nontrivial element of $Z(A_0) \cap J$, and let n be the smallest positive integer such that $a \in Z_n(K)$. Put $\overline{G} = G/Z_{n-1}(K)$. Then $\bar{a}$ is a nontrivial element of $Z(\overline{A}_0\overline{K})$. Clearly, $J = [J, G] \leq G'$ and $G' \leq A'K \leq A_0K$, so that $\bar{a} \in \overline{J} \cap Z(\overline{G}')$, a contradiction, since $Z_{n-1}(K)$ is finite and $J \cap Z(G') = 1$ (see [5, Corollary 5.3.8]). It follows that $A_0 \cap J = 1$, so that also $A \cap J = 1$. A similar argument proves that $B \cap J = 1$. Since L is isomorphic with a group of automorphisms of J, the locally nilpotent subgroup L' is nilpotent (see [7, Part 2, Theorem 6.32]), and in particular G is soluble. Moreover, L is abelian-by-finite, since every proper L-invariant subgroup of J is finite (see [5, Lemma 6.6.4]), so that also A and B are abelian-by-finite. The factorizer $X = X(J)$ of J in $G = AB$ has the triple factorization

$$X = A^*B^* = A^*J = B^*J,$$

where $A^* = A \cap BJ$ and $B^* = B \cap AJ$. Since $A^* \cap J = B^* \cap J = 1$ and $C_X(J) = J$, it follows from Lemma 3.6 that $J = X$. Thus J is a factorized subgroup of $G = AB$, and hence $J = (A \cap J)(B \cap J) = 1$. This last contradiction completes the proof of statement (a).

(b) The commutator subgroup G' of G is locally nilpotent by (a), and it follows from Lemma 3.4 that $K \cap G'$ is contained in the hypercenter of G'. Since $G'/(K \cap G') \simeq G'K/K$ is hypercentral, G' is hypercentral.

(c) Assume that this is false, and choose a counterexample G such that K has minimal minimax rank. The subgroup G' is locally nilpotent by (a). Clearly, it is enough to show that the group G'/K' is nilpotent, so that on replacing G by G/K' it can be assumed without loss of generality that K is abelian.

If T is the subgroup of all elements of finite order of K, then $(K \cap G')/(T \cap G')$ is contained in some term with finite ordinal type of the upper central series of $G'/(T \cap G')$ (see [7, Part 2, Lemma 6.37]). As $G'K/K$ is nilpotent, it follows that $G'T/T$ is nilpotent as well. Let J be the finite residual of the Černikov group T. Then T/J is finite, and so $G'J/J$ is nilpotent. By the minimality of the minimax rank of K, we infer that J is a p-group for some prime p. Consider an infinite minimal G-invariant subgroup J_0 of J. Then $G'J_0/J_0$ is nilpotent, and J_0 is a radicable subgroup of G whose proper G-invariant subgroups are finite. Then $J_0 = [J_0, G'] \leq G'$, and there exists a subgroup L of G such that $G = J_0L$ and $J_0 \cap L$ is finite (see [5, Theorem 5.4.4] and [16, Lemma 10]). The subgroup $J_0 \cap L$ is normal in G, and $G'/(J_0 \cap L)$ is not nilpotent, so that we may suppose that $J_0 \cap L = 1$. Then $K = (L \cap K) \times J_0$, where $J_0 \cap L$ is normal in G and $G'(L \cap K)/(L \cap K)$ is not nilpotent. Hence, it can also be assumed that $L \cap K = 1$, and $K = J_0 \leq G'$. Then $G = L \ltimes K$, and as in the proof of (a) we may reduce the matter to the case in which $C_G(K) = K$. Then the soluble group G/K is isomorphic with a group of automorphisms of K, and since all proper G-invariant subgroups of K are finite, we infer that G/K is abelian-by-finite (see [5, Lemma 6.6.4]). Let c be the nilpotency class of A'. Then $A' \cap K$ is contained in $Z_c(A'K)$. Hence,

$$[A \cap K, A'K] = [A \cap K, A'] \leq A' \cap K \leq Z_c(A'K).$$

Therefore, the normal subgroup $A \cap K$ of G is contained in $Z_{c+1}(G')$, and so it is finite. The same argument proves that $B \cap K$ too is finite, so that $E = (A \cap K)(B \cap K)$ is a finite normal subgroup of G. Let $\bar{G} = G/E$. Then $C_{\bar{G}}(\bar{K}) = \bar{K}$ by Lemma 2.5, and on replacing G by $\bar{G}$ it can be assumed without loss of generality that $A \cap K = B \cap K = 1$. In particular, A and B are abelian-by-finite, and so $K = G$ by Lemma 3.6. This last contradiction completes the proof of the theorem. □

Locally finite groups with a triple factorization by locally nilpotent subgroups have been considered in [17]. Here we prove the following result.

Theorem 3.8. *Let the locally finite group $G = AB = AK = BK$ be the product of two subgroups A and B, at least one of which hyperabelian, and a locally nilpotent normal subgroup K. If A' and B' are locally nilpotent then the commutator subgroup G' of G is locally nilpotent.*

PROOF. As locally finite residually locally nilpotent groups are locally nilpotent and $\bigcap_p O_{p'}(G) = 1$, it is enough to show that $G'O_{p'}(G)/O_{p'}(G)$ is locally nilpotent for every prime p. Thus, it can be assumed that $O_{p'}(G) = 1$, so that in particular the locally nilpotent normal subgroup K is a p-group. Suppose that A is hyperabelian, and assume by contradiction that the commutator subgroup A' of A is not a p-group, so that it contains an abelian nontrivial A-invariant p'-subgroup A_0. Let B_1 be the unique Sylow p'-subgroup of B'. Then B_1K is a normal subgroup of G, and the factor group G/B_1K is an extension of a p-group by an abelian group. Since A_0 is contained in G', it follows that $A_0 \leq B_1K$. Consider the normal subgroup $B_0 = A_0K \cap B_1$ of B. Clearly,

$$A_0K = A_0K \cap B_1K = B_0K$$

and

$$\pi(A_0) \cap \pi(K) = \pi(B_0) \cap \pi(K) = \varnothing.$$

Application of Lemma 3.2.3 of [5] yields $N_{A_0}(A_0 \cap B_0) = A_0 \cap B_0$ (see also [18, Lemma 1.59]). Since A_0 is abelian, it follows that $A_0 \leq B_0$, so that $A_0 = B_0$ is a normal p'-subgroup of $G = AB$. This contradiction shows that A' is a p-group, so that $G' \leq A'K$ is a p-group. Hence, it is locally nilpotent.

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