GROUPS WITH FINITELY MANY NORMALIZERS OF NON-NILPOTENT SUBGROUPS

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Abstract

It is known that (generalized) soluble groups in which every non-normal subgroup is locally nilpotent either are locally nilpotent or have a finite commutator subgroup. Here the structure of (generalized) soluble groups with finitely many normalizers of (infinite) non-(locally nilpotent) subgroups is investigated, and the above result is extended to this more general situation.

1. Introduction

A group G is called *metahamiltonian* if every non-abelian subgroup of G is normal. Metahamiltonian groups were introduced and investigated by G.M. Romalis and N.F. Sesekin [15; 16; 17], who proved in particular that locally soluble metahamiltonian groups have a finite commutator subgroup. More recently, B. Bruno and R.E. Phillips [1] considered groups in which every subgroup is either locally nilpotent or normal, and also in this case they obtained that locally soluble groups with such property either are locally nilpotent or have finite commutator subgroups. The consideration of Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) shows that in these situations some (generalised) solubility condition must be required. Actually, Bruno and Phillips proved their result within the universe of $\mathcal{W}$-groups: a group G is called a $\mathcal{W}$-group if every finitely generated non-nilpotent subgroup of G has a finite, non-nilpotent, homomorphic image. It is well known that all locally (soluble-by-finite) groups and all linear groups have the property $\mathcal{W}$ (see [13, part 2, theorem 10.51] and [19]). The aim of this paper is to extend this investigation to groups containing in some sense only a few subgroups, which are neither locally nilpotent nor normal.

In 1980 Y.D. Polovickiĭ [12] showed that a group G has finitely many normalizers of abelian subgroups if and only if the centre $Z(G)$ has finite index. Much

earlier, in a famous paper from 1955, B.H. Neumann [10] proved that a group G has finite conjugacy classes of subgroups if and only if $G/Z(G)$ is finite, and the same conclusion holds if the restriction is imposed only to conjugacy classes of abelian subgroups (see [6]). Thus, central-by-finite groups are precisely those groups in which the normalizers of (abelian) subgroups have finite index, and these results suggested that the behaviour of normalizers has a strong influence on the group. The structure of groups with finitely many normalizers of subgroups with a given property χ has recently been studied for several different choices of χ (see [2; 3; 4; 5]). Here, we consider groups with finitely many normalizers of non-(locally nilpotent) subgroups, and our main result on this subject is the following:

Theorem A   *Let G be a $\mathcal{W}$-group with finitely many normalizers of non-(locally nilpotent) subgroups. Then either G is locally nilpotent or its commutator subgroup G' is finite.*

Of course, all finite groups satisfy the hypotheses of the above theorem, and hence there is no bound for the derived length of soluble groups with finitely many normalizers of non-nilpotent subgroups, Moreover, it follows from Theorem A that if G is a $\mathcal{W}$-group with finitely many normalizers of non-(locally nilpotent) subgroups, then either G is locally nilpotent or all its locally nilpotent subgroups are nilpotent. Thus we have:

Corollary   *Let G be a $\mathcal{W}$-group with finitely many normalizers of non-(locally nilpotent) subgroups. If G is not locally nilpotent, then it has finitely many normalizers of non-nilpotent subgroups.*

Note here that a result corresponding to Theorem A for groups with finitely many normalizers of non-nilpotent subgroups cannot be proved, and it seems difficult to deal with the case of locally nilpotent groups with this property; in fact, there exist (soluble) locally nilpotent groups with trivial centre in which all proper subgroups are nilpotent and subnormal (see [7]), and such groups can have arbitrarily high derived length (see [8]).

Next, we consider the more general situation of groups with finitely many normalizers of infinite non-(locally nilpotent) subgroups. In this case, one cannot expect to obtain the finiteness of the commutator subgroup, as it is shown by the non-abelian extension of a group of type p^∞ (where p is an odd prime) by a group of order 2. In this context we have the following result:

Theorem B   *Let G be a $\mathcal{W}$-group with finitely many normalizers of infinite non-(locally nilpotent) subgroups. If G is not locally nilpotent, then either it is a Černikov group or its commutator subgroup G' is finite.*

Most of our notation is standard and can, for instance, be found in [13].

2. Preliminary results

The following result plays a central role in the study of groups with finitely many normalizers of subgroups with a given property. It was proved by B.H. Neumann [9] in the more general case of groups covered by finitely many cosets.

Lemma 2.1. *Let the group $G = X_1 \cup \ldots \cup X_t$ be the union of finitely many subgroups $X_1, \ldots, X_t$. Then any X_i of infinite index can be omitted from this decomposition; in particular, at least one of the subgroups $X_1, \ldots, X_t$ has finite index in G.*

Our next lemma shows that any group with finitely many normalizers of non-nilpotent subgroups contains a subgroup of finite index in which all non-nilpotent subgroups are subnormal, with defect of at most 2.

Lemma 2.2. *Let G be a group with finitely many normalizers of non-nilpotent subgroups. Then G contains a characteristic subgroup M of finite index, such that $N_M(X)$ is normal in M for each non-nilpotent subgroup X of M.*

PROOF. Let $\mathcal{H}$ be the set of all non-nilpotent subgroups of G. If X is any element of $\mathcal{H}$, its normalizer $N_G(X)$ has obviously finitely many images under automorphisms of G; in particular, the subgroup $N_G(X)$ has finitely many conjugates in G and so the index $|G : N_G(N_G(X))|$ is finite. It follows that the characteristic subgroup

$$M(X) = \bigcap_{\alpha \in Aut\, G} N_G(N_G(X))^\alpha$$

also has finite index in G. If X and Y are elements of $\mathcal{H}$ such that $N_G(X) = N_G(Y)$, then $M(X) = M(Y)$, and hence also

$$M = \bigcap_{X \in \mathcal{H}} M(X)$$

is a characteristic subgroup of finite index of G. Let X be any non-nilpotent subgroup of M. Then

$$M \leq M(X) \leq N_G(N_G(X)),$$

and so the normalizer $N_M(X) = N_G(X) \cap M$ is a normal subgroup of M. ∎

Recall that a group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index; of course, all $\mathcal{W}$-groups are locally graded.

Corollary 2.3. *Let G be a locally graded group with finitely many normalizers of non-nilpotent subgroups. Then G is soluble-by-finite.*

PROOF. By Lemma 2.2 there exists in G a normal subgroup M of finite index, such that all non-nilpotent subgroups of M are subnormal with defect of at most 2 (in M). Then M is soluble (see [18, theorem 2]), and hence G is soluble-by-finite. ∎

The same argument used in the proof of Lemma 2.2 shows that any group with finitely many normalizers of (infinite) non-(locally nilpotent) subgroups contains a

subgroup of finite index, in which every (infinite) subgroup is either locally nilpotent or subnormal with defect of at most 2. In fact, we have:

Lemma 2.4. *Let G be a group with finitely many normalizers of (infinite) subgroups that are not locally nilpotent. Then G contains a characteristic subgroup M of finite index such that $N_M(X)$ is normal in M for each (infinite) subgroup X of M that is not locally nilpotent.*

It follows from a theorem of Phillips and Wilson (see [11, theorem A]) that any $\mathcal{W}$-group whose infinite non-subnormal subgroups are locally nilpotent is a finite extension of a locally nilpotent group. Thus, the above lemma can be applied to obtain the following result:

Lemma 2.5. *Let G be a $\mathcal{W}$-group with finitely many normalizers of infinite non-(locally nilpotent) subgroups. Then the Hirsch–Plotkin radical of G has finite index; in particular, G locally satisfies the maximal condition on subgroups.*

PROOF. By Lemma 2.4 there exists a normal subgroup M of G of finite index, such that every infinite subgroup of M is either subnormal or locally nilpotent. Then M contains a locally nilpotent subgroup of finite index, and hence the Hirsch–Plotkin radical of G has finite index. ∎

In the last part of this section, we consider groups in which all infinite non-normal subgroups are locally nilpotent; it turns out that such groups are essentially the same as those considered by Bruno and Phillips.

Lemma 2.6. *Let G be a non-periodic $\mathcal{W}$-group whose infinite non-normal subgroups are locally nilpotent. Then, either G is locally nilpotent or all non-normal subgroups of G are nilpotent.*

PROOF. Suppose that G is not locally nilpotent, and let X be any finite non-nilpotent subgroup of G. Consider an element $g \in G$ of infinite order. Since $E = \langle X, g \rangle$ is a finitely generated non-nilpotent subgroup of G, it contains a normal subgroup K such that E/K is a finite non-nilpotent group. Obviously, there exists a non-normal subgroup L of E such that $K < L < E$. Thus, K is a finitely generated infinite nilpotent group; and so, replacing K by a suitable subgroup, it can be assumed that K is torsion-free. For each positive integer n, the infinite subgroup XK^n is normal in G, and hence also

$$X = \bigcap_n XK^n$$

is a normal subgroup of G. Therefore, all non-normal subgroups of G are locally nilpotent, and so even nilpotent (see [1, theorem B]). ∎

Corollary 2.7. *Let G be a non-periodic $\mathcal{W}$-group whose infinite non-normal sub-*

groups are locally nilpotent. Then either G is locally nilpotent or its commutator subgroup G' is finite.

The following lemma is an easy consequence of a result of D.I. Zaičev [20].

Lemma 2.8. *Let G be a periodic (locally soluble)-by-finite group, and let X be a finite subgroup of G. If G is not a Černikov group, there exists a collection $(K_i)_{i \in I}$ of infinite subgroups of G such that $\bigcap_{i \in I} K_i = X$.*

PROOF. As X is finite, G contains an abelian subgroup A such that $A^X = A$ and A does not satisfy the minimal condition on subgroups (see [20]). Clearly, the socle S of A is infinite, and the subgroup XS is residually finite. Then there exists a collection $(L_i)_{i \in I}$ of normal subgroups of finite index of XS such that $X \cap L_i = \{1\}$ for all i, and

$$\bigcap_{i \in I} L_i = \{1\}.$$

It follows that each XL_i is infinite and

$$\bigcap_{i \in I} XL_i = X.$$

Thus, the lemma is proved.   ∎

Corollary 2.9. *Let G be a $\mathcal{W}$-group whose infinite, non-normal subgroups are locally nilpotent. If G is not locally nilpotent, then either G is a Černikov group or all non-normal subgroups of G are nilpotent.*

PROOF. It follows from Lemma 2.5 that G contains a locally nilpotent subgroup of finite index. Moreover, by Lemma 2.6 it can be assumed that G is periodic. Suppose that G is not a Černikov group, and let X be any finite, non-nilpotent subgroup of G. It follows from Lemma 2.8 that X can be obtained as the intersection of a collection of infinite subgroups, and hence it is normal in G. Therefore, all non-normal subgroups of G are locally nilpotent, and hence G contains a finite, normal subgroup N such that G/N is a Dedekind group. In particular, G has a finite commutator subgroup and so its locally nilpotent subgroups are even nilpotent. ∎

3. Proofs of the theorems

The first results of this section deal with the behaviour of torsion-free, normal subgroups in groups with finitely many normalizers of infinite non-(locally nilpotent) subgroups.

Lemma 3.1. *Let G be a $\mathcal{W}$-group with finitely many normalizers of infinite non-*

(locally nilpotent) subgroups, and let A be a torsion-free abelian subgroup of G. If x is an element of finite order of G such that $A^x = A$, then $[A, x] = \{1\}$.

PROOF. Assume for a contradiction that the statement is false, and consider a counterexample G with a minimal number k of normalizers of infinite non-(locally nilpotent) subgroups. Clearly, $\langle x \rangle^A$ is not periodic, and in particular G is not locally nilpotent. Thus $k > 1$, since, by Corollary 2.7, any non-periodic $\mathcal{W}$-group whose infinite, non-normal subgroups are locally nilpotent either is locally nilpotent or has a finite commutator subgroup. Let a be an element of A such that $[a, x] \neq 1$, and put $B = A \cap \langle a, x \rangle$. Then $B^x = B$, and B is finitely generated by Lemma 2.5; moreover, there exists a prime number p such that $[B^{p^n}, x] \neq \{1\}$ for all positive integers n. As

$$\langle x \rangle = \bigcap_{n \in \mathbb{N}} \langle B^{p^n}, x \rangle,$$

the subgroup $\langle B^{p^m}, x \rangle$ is not normal in G for some positive integer m. It follows that $N_G(\langle B^{p^m}, x \rangle)$ has less than k normalizers of infinite non-(locally nilpotent) subgroups, and hence $[B^{p^m}, x] = \{1\}$. This contradiction proves the lemma. ■

Lemma 3.2. *Let G be a $\mathcal{W}$-group with finitely many normalizers of infinite non-(locally nilpotent) subgroups, and let A be a torsion-free abelian normal subgroup of G, such that G/A is periodic. Then A is contained in the centre of G.*

PROOF. Assume, for a contradiction, that the statement is false, so that $[A, x] \neq \{1\}$ for some element x of G; and it follows from Lemma 3.1 that x must have infinite order. Clearly, $A \cap \langle x \rangle = \langle x^m \rangle$ with $m > 1$, since G/A is periodic. Let $T/\langle x^m \rangle$ be the subgroup consisting of all elements of finite order of $A/\langle x^m \rangle$; then A/T is torsion-free and $[A, x] \leq T$ by Lemma 3.1. On the other hand, x^m belongs to $Z(\langle T, x \rangle)$ and $\langle T, x \rangle/\langle x^m \rangle$ is locally finite, so that $[T, x]$ is locally finite by Schur's theorem, and hence $[T, x] = \{1\}$. Therefore, $[A, x, x] = \{1\}$. It follows that:

$$[A, x]^m = [A, x^m] = \{1\},$$

and so $[A, x] = \{1\}$, which is a contradiction. ■

Corollary 3.3. *Let G be a finitely generated abelian-by-finite group with finitely many normalizers of infinite, non-nilpotent subgroups. Then the factor group $G/Z(G)$ is finite.*

In our proofs we will also need the following elementary characterization of normal subgroups of polycyclic-by-finite groups.

Lemma 3.4. *Let G be a polycyclic-by-finite group, and let X be a subgroup of G such that X^σ is normal in G^σ for each finite homomorphic image G^σ of G. Then X is normal in G.*

PROOF. Let $\mathcal{H}$ be the set of all subgroups of finite index of G containing X. If H is any element of $\mathcal{H}$, the product XH_G is a normal subgroup of G; since every subgroup of a polycyclic-by-finite group is the intersection of subgroups of finite index (see [14, 5.4.16]), we have

$$X = \bigcap_{H \in \mathcal{H}} H = \bigcap_{H \in \mathcal{H}} XH_G,$$

and so X is likewise normal in G.   ∎

The next result provides in particular a complete characterisation of finitely generated, locally graded groups with finitely many normalizers of non-nilpotent subgroups. It shows that for such groups, only the extreme cases actually occur.

Theorem 3.5. *A finitely generated, soluble-by-finite group G has finitely many normalizers of infinite, non-nilpotent subgroups if and only if either G is nilpotent or the factor group $G/Z(G)$ is finite.*

PROOF. The condition of the statement is obviously sufficient. Conversely, suppose that the group G has finitely many normalizers of infinite, non-nilpotent subgroups. It follows from Lemma 2.5 that G is nilpotent-by-finite, and so it contains a torsion-free nilpotent normal subgroup H of finite index. Assume now, for a contradiction, that G neither is nilpotent nor central-by-finite, so that by Corollary 3.3 it is not even abelian-by-finite. Thus, Corollary 2.7 yields that the counterexample G contains infinite subgroups that are neither nilpotent nor normal, and G can be chosen in such a way that the set

$$\{N_G(X_1), \ldots, N_G(X_k)\}$$

of all proper normalizers of infinite, non-nilpotent subgroups has minimal order k. Clearly, each $N_G(X_i)$ is a non-nilpotent group with less than k proper normalizers of infinite, non-nilpotent subgroups, and hence it is central-by-finite; it follows that the index $|G : N_G(X_i)|$ is infinite for all $i = 1, \ldots, k$. Let L be any G-invariant subgroup of finite index of H; then $|L : L \cap N_G(X_i)|$ is infinite for all i, and so L cannot be contained in the set

$$N_G(X_1) \cup \ldots \cup N_G(X_k),$$

because of Lemma 2.1. Therefore, each non-nilpotent subgroup of G/L is normal. Let X be any infinite, non-nilpotent subgroup of G; then X contains a normal subgroup Y such that X/Y is a finite, non-nilpotent group. Moreover, since each subgroup of G is the intersection of subgroups of finite index, there exists a subgroup of finite index U of G such that $X \cap U = Y$. Thus, $V = L \cap U_G$ is a normal subgroup of finite index of G and XV/V is not nilpotent, so that XV is a normal subgroup of G; it also follows that $XL = (XV)L$ is normal in G, and so X itself is a normal subgroup of G by Lemma 3.4. Therefore, all infinite, non-nilpotent subgroups of G

are normal, so that G' is finite by Corollary 2.7. Hence, $H \leq Z(G)$ and $G/Z(G)$ is finite. This contradiction completes the proof of the theorem. ∎

Corollary 3.6. *Let G be a torsion-free $\mathcal{W}$-group with finitely many normalizers of non-(locally nilpotent) subgroups. Then G is locally nilpotent.*

PROOF. Assume that G contains a finitely generated, non-nilpotent subgroup E. Since G locally satisfies the maximal condition on subgroups by Lemma 2.5, it follows from Theorem 3.5 that $E/Z(E)$ is finite, which is a contradiction, as E is a torsion-free, non-abelian group. Therefore, G is locally nilpotent. ∎

We are now in a position to prove our main results.

PROOF OF THEOREM A. The statement is known if every non-normal subgroup of G is locally nilpotent (see [1, theorem B]). Suppose that G is not locally nilpotent and contains subgroups which are neither locally nilpotent nor normal, and let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of such subgroups.

Assume first that

$$G = N_G(X_1) \cup \ldots \cup N_G(X_k).$$

Then, by Lemma 2.1, every subgroup of infinite index can be omitted from this union, and hence

$$G = N_G(X_{i_1}) \cup \ldots \cup N_G(X_{i_t}),$$

where the index $|G : N_G(X_{i_j})|$ is finite for each $j = 1, \ldots, t$. Clearly, $N_G(X_{i_j})$ has less than k proper normalizers of non-(locally nilpotent) subgroups, and so by induction it can be assumed that $N_G(X_{i_j})'$ is finite for $j = 1, \ldots, t$. Since every $N_G(X_{i_j})$ has finitely many conjugates, it follows from Dietzmann's lemma that the normal closure

$$N = \langle N_G(X_{i_1})', \ldots, N_G(X_{i_t})' \rangle^G$$

is likewise finite. Moreover, the factor group G/N has a finite covering consisting of abelian subgroups, and hence it is central-by-finite (see [13, part 1, theorem 4.16]). Therefore G' is finite in this case by Schur's theorem.

Suppose now that $N_G(X_1) \cup \ldots \cup N_G(X_k)$ is properly contained in G, and let g be an element of the set

$$G \setminus (N_G(X_1) \cup \ldots \cup N_G(X_k)).$$

As G is not locally nilpotent, it contains a finitely generated, non-nilpotent subgroup E. Then $\langle g, E \rangle$ is a normal subgroup of G and all subgroups of $G/\langle g, E \rangle$ are normal, so that in particular $|G'\langle g, E \rangle/\langle g, E \rangle| \leq 2$ and G' is finitely generated by Lemma 2.5. It follows that $G' = K'$, where K is a suitable, finitely generated, non-nilpotent subgroup of G. On the other hand, K has finitely many normalizers of non-nilpotent subgroups, so that $K/Z(K)$ is finite by Theorem 3.5 and hence $G' = K'$ is finite. ∎

Proof of Theorem B. The group G contains a locally nilpotent subgroup of finite index by Lemma 2.5. Assume that G is not a Černikov group. If every infinite, non-normal subgroup of G is locally nilpotent, then all non-normal subgroups of G are nilpotent by Corollary 2.9, and so G' is finite. Suppose that G contains infinite subgroups that are neither locally nilpotent nor normal, and let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of such subgroups. By induction on k, it can be assumed that each $N_G(X_i)$ either is a Černikov group or has a finite commutator subgroup. If

$$G = N_G(X_1) \cup \ldots \cup N_G(X_k),$$

by Lemma 2.1 we have also

$$G = N_G(X_{i_1}) \cup \ldots \cup N_G(X_{i_t}),$$

where each $N_G(X_{i_j})$ has finite index in G. In particular, $N_G(X_{i_j})$ is not a Černikov group and hence $N_G(X_{i_j})'$ is finite for every $i = 1, \ldots, t$. In this case, the same argument used in the last part of the proof of Theorem A yields that G' is finite. Therefore, it can be assumed that $N_G(X_1) \cup \ldots \cup N_G(X_k)$ is a proper subset of G.

Let g be an element of

$$G \setminus (N_G(X_1) \cup \ldots \cup N_G(X_k)),$$

and let E be a finitely generated, non-nilpotent subgroup of G. Then $X = \langle g, E \rangle$ is finitely generated and all infinite subgroups of G containing X are normal. Suppose first that G is periodic, so that X is finite and hence it can be obtained as the intersection of infinite subgroups by Lemma 2.8. It follows that X is normal in G and all infinite subgroups of G/X are normal, so that G/X is a Dedekind group by Lemma 2.8 and G' is finite in this case. Assume now that G is not periodic, and let a be an element of infinite order of G. Then the infinite subgroup $\langle a, X \rangle$ is normal in G, and $G/\langle a, X \rangle$ is a Dedekind group. As G locally satisfies the maximal condition on subgroups by Lemma 2.5, it follows that G' is finitely generated. Therefore $G' = K'$, where K is a suitable, finitely generated, non-nilpotent subgroup of G containing X. Since K has finitely many normalizers of non-nilpotent subgroups, $K/Z(K)$ is finite by Theorem 3.5 and hence $G' = K'$ is finite.   ∎

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