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GROUPS IN WHICH EVERY NON-ABELIAN SUBGROUP IS PERMUTABLE

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A subgroup H of a group G is said to be *permutable* if $HX = XH$ for every subgroup X of G. In this paper the structure of groups in which every subgroup either is abelian or permutable is investigated.

1. Introduction.

A subgroup H of a group G is said to be *permutable* (or *quasinormal*) if $HX = XH$ for every subgroup X of G. This concept was introduced by Ore [5], and the behaviour of permutable subgroups was later investigated by several authors. Obviously, normal subgroups are always permutable, and it is easy to show that every maximal permutable subgroup of a group is normal, so that in particular permutable subgroups of finite groups must be subnormal. A group is called *quasihamiltonian* if all its subgroups are permutable. It has been proved by Stonehewer [12] that permutable subgroups of arbitrary groups are ascendant, so that any quasihamiltonian group is locally nilpotent, and the structure of quasihamiltonian groups was described by Iwasawa [3]. In particular, it turns out that quasihamiltonian non-abelian groups have torsion-free rank at most 1, and that primary groups with such property contain a bounded abelian subgroup of finite index. For a detailed account of results concerning permutable subgroups we refer to [11].

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A group G is called *metahamiltonian* if all its non-abelian subgroups are normal. The structure of metahamiltonian groups has been investigated by G. M. Romalis and N. F. Sesekin in a series of papers ([8], [9],[10]), where they proved in particular that if G is a soluble metahamiltonian group, then the commutator subgroup G' of G is finite. The aim of this article is to obtain a corresponding theorem for *metaquasihamiltonian* groups, i.e. groups in which every subgroup is either abelian or permutable. In fact, we shall prove the following result.

THEOREM. *Let G be a locally graded metaquasihamiltonian group. Then G contains a finite normal subgroup N such that the factor group G/N is quasihamiltonian. Moreover, G is soluble with derived length at most* 4.

It is not clear whether the bound for the derived length obtained in the above theorem is best possible.

Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. In the above statement the assumption that the group is locally graded cannot be omitted, since all Tarski groups are metahamiltonian (and so also metaquasihamiltonian).

Most of our notation is standard and can be found in [7].

2. Proof of the Theorem.

Our first lemma follows directly from the definition of a metaquasihamiltonian group.

LEMMA 2.1. *Let G be a metaquasihamiltonian group, and let N be a non-abelian normal subgroup of G. Then the factor group G/N is quasihamiltonian.*

A lattice $\mathcal{L}$ with 0, 1 is said to be *metamodular* if for each $a \in \mathcal{L}$ either the interval $[a/0]$ is a modular lattice or a is a modular element of $\mathcal{L}$. Clearly every metaquasihamiltonian group has metamodular subgroup lattice, and the structure of periodic groups with this property has recently been described in [1].

LEMMA 2.2. *Let G be a locally graded metaquasihamiltonian group. Then G is soluble with derived length at most* 4.

Proof. As the subgroup lattice $\mathcal{L}(G)$ of G is metamodular, the group G is soluble (see [1], Theorem 2.4). We may suppose that G is not metabelian, so that the factor group $G/G^{(3)}$ is not quasihamiltonian, and hence $G^{(3)}$ is abelian by Lemma 2.1. Therefore G has derived length at most 4. □

The following result allows to reduce our consideration to the case of non-periodic groups.

LEMMA 2.3. *Let G be a periodic locally graded metaquasihamiltonian group. Then G contains a finite normal subgroup N such that the factor group G/N is quasihamiltonian.*

Proof. We may obviously suppose that the group G is not abelian. Since G is locally finite by Lemma 2.2, it contains a finite non-abelian subgroup E, and each subgroup of G containing E is permutable; in particular, if X is any subgroup of G, the product XE is a permutable subgroup of G, and the index $|XE : X|$ is finite. Therefore every subgroup of G has finite index in a permutable subgroup, and hence G contains a finite normal subgroup N such that G/N is quasihamiltonian (see [2]). □

Our next two lemmas prove in particular that the commutator subgroup of any locally graded metaquasihamiltonian group is periodic.

LEMMA 2.4. *Let G be a locally graded metaquasihamiltonian group. Then G is either locally nilpotent or finite-by-abelian.*

Proof. Suppose that G is not locally nilpotent. Since G is soluble by Lemma 2.2 and all its non-abelian subgroups are ascendant, it follows from a result of Phillips and Wilson that either G is a Černikov group or $G/Z(G)$ is finite (see [6], Theorem C(i)). Thus without loss of generality it can be assumed that G is a Černikov group, so that by Lemma 2.3 it contains a finite normal subgroup N such that G/N is quasihamiltonian. In particular, G/N is the direct product of finitely many primary groups. As every quasihamiltonian p-group of infinite exponent is abelian (see [11], Theorem 2.4.14), it follows that G is finite-by-abelian. □

LEMMA 2.5. *Let G be a torsion-free locally graded metaquasihamiltonian group. Then G is abelian.*

Proof. The group G is locally nilpotent by Lemma 2.4. Assume that G is not abelian, and let E be a finitely generated non-abelian subgroup of G. Since

E is a torsion-free nilpotent group, it is not abelian-by-finite and so all its finite homomorphic images are quasihamiltonian by Lemma 2.1; it follows that E itself is quasihamiltonian (see [4]). Thus E is abelian, and this contradiction proves the lemma. □

Let G be a group, and let H be a subgroup of G. If x is any element of G, the *order* of x modulo H is the order of the group $\langle x\rangle/\langle x\rangle \cap H$. In particular, x has infinite order modulo H if and only if x has infinite order and $\langle x\rangle \cap H = \{1\}$.

LEMMA 2.6. *Let G be a group, and let H be a subgroup of G such that all subgroups of G containing H are permutable. If there exists an element of G having infinite order modulo H, then H is normal in G.*

Proof. Let x and y be elements of G having finite order modulo H. As the subgroups H and $\langle H, x\rangle$ are permutable in G, the index

$$|\langle H, x, y\rangle : H| = |\langle H, x, y\rangle : \langle H, x\rangle| \cdot |\langle H, x\rangle : H|$$

is finite, and hence also the product xy has finite order modulo H. It follows that the set T of all elements of G having finite order modulo H is a proper subgroup of G, and G is generated by $G \setminus T$. On the other hand, it is well known that every element of $G \setminus T$ normalizes H (see [11], Lemma 5.2.7), and so H is normal in G. □

LEMMA 2.7. *Let G be a non-periodic metaquasihamiltonian group. Then every periodic non-abelian subgroup of G is normal. In particular, all periodic subgroups of G are metahamiltonian groups.*

Proof. Let H be a periodic non-abelian subgroup of G, so that all subgroups of G containing H are permutable. Since G is not periodic, there exist elements of G having infinite order modulo H, and hence H is normal in G by Lemma 2.6. □

It has been mentioned in the introduction that every quasihamiltonian group with torsion-free rank at least 2 is abelian; in the case of metaquasihamiltonian groups the following result can be proved.

LEMMA 2.8. *Let G be a locally graded metaquasihamiltonian group with torsion-free rank at least* 2. *Then G is finite-by-abelian.*

Proof. By Lemma 2.4 we may clearly suppose that G is a locally nilpotent non-abelian group. Let T be the subgroup consisting of all elements of finite order of G, and suppose first that T is not abelian, so that G contains a finite non-abelian subgroup E. Then E is normal in G by Lemma 2.7 and G/E is a quasihamiltonian group with torsion-free rank at least 2, so that G/E is abelian and G is finite-by-abelian. Suppose now that T is abelian, and assume that T is not contained in the centre $Z(G)$, so that there exist elements x of T and y of $G \setminus T$ such that $[x, y] \neq 1$. As $\langle x, y\rangle$ is not abelian, all subgroups of G containing $\langle x, y\rangle$ are permutable and it follows from Lemma 2.6 that $\langle x, y\rangle$ is normal in G. Thus $T_0 = T \cap \langle x, y\rangle$ is a finite normal subgroup of G, and the centralizer $C = C_G(T_0)$ is properly contained in G. Let u be any element of $G \setminus C$; then u has infinite order and the non-abelian subgroup $\langle u, T_0\rangle$ is normal in G by Lemma 2.6, so that uT_0 belongs to $Z(G/T_0)$. Therefore G/T_0 is abelian, and G is finite-by-abelian. Suppose finally that T is a subgroup of $Z(G)$, so that in particular G is nilpotent of class at most 2 by Lemma 2.5. Let a be a non-central element of G, and let b be an element of G such that $c = [a, b] \neq 1$. Clearly a and b have infinite order, and $\langle a\rangle \cap \langle b\rangle = \{1\}$, since otherwise $\langle a, b\rangle$ would have torsion-free rank 1 and $\langle a, b\rangle/T \cap \langle a, b\rangle$ would be cyclic, contrary to the assumption $ab \neq ba$. Assume that c has not prime order, so that there exists a prime number q such that $\{1\} < \langle c^q\rangle < \langle c\rangle$. Then $[a, b^q] = c^q \neq 1$, so that the subgroup $H = \langle a, b^q\rangle$ is not abelian, and so it is permutable in G. Clearly $\langle a, b\rangle = H\langle b\rangle$, so that $|\langle a, b\rangle : H| = q$ and H is normal in $\langle a, b\rangle$. It follows that c belongs to H, a contradiction since $T \cap H = \langle c^q\rangle$. Therefore c has prime order p, and the prime number p clearly does not depend on the choice of the element b, so that the abelian group $[a, G]$ has exponent p. Let u be any element of $G \setminus C_G(a)$. The above argument shows that $[u, G]$ has prime exponent, and since $[a, u] \in [u, G]$, we have that $[u, G]$ has exponent p, so that also $G' = [G \setminus C_G(a), G]$ has exponent p. Thus the factor group $G/Z(G)$ has exponent p. By Schur's theorem it can be assumed that $G/Z(G)$ is infinite, so that the torsion-free abelian group G/T has infinite rank. Let X be a finitely generated non-abelian subgroup of G; then all subgroups of G containing X are permutable, and it follows from Lemma 2.6 that X is normal in G. The factor group G/X is quasihamiltonian and has infinite torsion-free rank, so that it is abelian and G' is contained in X. Therefore G' is finitely generated, and so finite also in this last case. □

We can now prove the main result of the paper.

Proof of the Theorem. By Lemma 2.2 the group G is soluble with derived length at most 4. Moreover, it follows from Lemma 2.3, Lemma 2.4

and Lemma 2.8 that it is enough to prove the theorem when G is a locally nilpotent group with torsion-free rank 1. Let T be the subgroup consisting of all elements of finite order of G. If T is not abelian, it contains a finite non-abelian subgroup E; then E is normal in G by Lemma 2.7 and G/E is a quasihamiltonian group, so that G is finite-by-quasihamiltonian. Therefore it can be assumed that T is abelian. As G/T is a torsion-free abelian group of rank 1, we may also suppose that T is not contained in $Z(G)$, so that there exist elements $x \in T$ and $y \in G \setminus T$ such that $xy \neq yx$. Moreover,

$$G = \bigcup_{n \in \mathbb{N}} \langle g_n \rangle T,$$

where $g_1 = y$ and $\langle g_n \rangle T \leq \langle g_{n+1} \rangle T$ for all n. Put $N = \langle x, y \rangle \cap T$, so that $\langle x, y \rangle = \langle y \rangle \ltimes N$. If n is any positive integer, we have $y = g_n^{k_n} u_n$, where k_n is an integer and $u_n \in T$, so that $g_n^{k_n}$ lies in $N_G(N) \setminus C_G(N)$ and hence the subgroup $\langle g_n^{k_n}, N \rangle$ is permutable in G. It follows that the subset $\langle g_n \rangle N$ is a subgroup, and so the finite subgroup N is normal in G. Since the subgroup $\langle g_n, N \rangle$ is not abelian, it is permutable in G, and so all subgroups of T containing N are normalized by g_n. In particular, all elements of prime order and all elements of order 4 of T/N belong to the centre of $\langle g_n, T \rangle / N$, and hence $\langle g_n, T \rangle / N$ is a quasihamiltonian group (see [11], Theorem 2.4.11). Therefore G/N is a quasihamiltonian group, and the theorem is proved. □

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