Uncountable extended residually finite groups

(a joint work with M. De Falco and C. Musella)

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Outline

1. Preliminaries
2. Uncountable groups whose large subgroups are closed
3. Groups whose large proper subgroups satisfy the ERF property
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Closure properties

Definition: A subgroup H of a group G is *closed (with respect to the profinite topology)* if it can be obtained as intersection of a collection of subgroups of finite index in G.

A group G is called *residually finite* if and only if the trivial subgroup is closed. Residual finiteness is not inherited by quotients, and so it is interesting what we can say about *QRF-groups.*

Definition: A group G whose normal subgroups are closed is called QRF-group.
Moreover, if every subgroup of G is closed, then G is said to be *extended residually finite*, or for short, an *ERF-group*.

Some examples of ERF-groups

Clearly:

- every finite group has the ERF-property;
- every polycyclic group is an ERF-group.

Remind that, if $\mathcal{X}$ is any group class, a group G is called *poly-*$\mathcal{X}$ if it has a finite series cointaing the trivial subgroups of G, such that every factor is an $\mathcal{X}$-group.

Theorem (Mal'cev): Let G be a polycylic group. Then every subgroup of G is closed.

Nilpotent groups with the ERF-property

In 1963, Smirov proved the following theorem:

Theorem: Let G be a group. If G is a nilpotent QRF-group, then G is an ERF-group.

D.M. Smirnov: "On the theory of finitely approximable groups", Ukrain. Mat. ˇZ 15 (1963), 453–457.

Nilpotent groups with the ERF-property

Lemma: Let G be a nilpotent group with torsion subgroup $T = T(G)$. Then G is a ERF-group if and only if both T and G/T are ERF-groups.

Theorem: Let G be a nilpotent group with torsion subgroup $T = T(G)$. Then G is a ERF-group if and only if the following hold:

- every primary component of T is abelian-by-finite with finite exponent;
- G/T has no Prüfer sections.

M. Menth, Nilpotent groups with every quotient residually finite, J. Group Theory 5 (2002) 199–217;

FC-group with the ERF-property

Definition: A group G is said to be an FC-group if each element of G has only finitely many conjugates.

A complete characterization of FC-groups with the ERF-property is provided in:

D.J.S. Robinson, A. Russo, G. Vincenzi, On groups whose subgroups are closed in the profinite topology, J. Pure Appl. Algebra 213 (2009) 421–429.

FC-group with the ERF-property

Theorem: Let G be an FC-group. Then G is ERF if and only if the following conditions hold:

- Sylow subgroups of G are abelian-by-finite with finite exponent;
- Sylow subgroups of G' are finite;
- G/T is torsion-free abelian of finite rank[1] and it has no Prüfer sections.

[1] A group G is said to have *finite rank r* if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property.

Groups whose large proper subgroups have a certain property

We can ask what happens if we impose an embedding or an absolute property to *"large"* subgroups in a group, at least for a right choice of the definition of largeness.

Definition: Let $\aleph$ be an uncountable regular cardinal number, and let G be a group of cardinality $\aleph$; a subgroup H of G will be called *large* if it has likewise cardinality $\aleph$ and *small* otherwise.

Groups whose large proper subgroups have a certain property

This point of view was adopted by many authors.

- M. De Falco, F. de Giovanni, H. Heineken, C Musella: "Normality in uncountable groups", Adv. Group Theory Appl. 3 (2017), 13–29.
- F. de Giovanni, M. Trombetti: "Nilpotency in uncountable groups", J. Austral. Math. Soc. 103 (2017), 59–69.
- M. De Falco, F. De Giovanni, C. Musella. "Groups with normality conditions for uncountable subgroups." Journal of the Australian Mathematical Society 111.2 (2021): 268-277.

Uncountable groups whose large subgroups are closed

In the first part of our work, we share the same point of view:

M. De Falco, C. Musella, A. Zaccardo: "Uncountable extended residually finite groups", Rend. Circ. Mat. Palermo, II. Ser 74, 107 (2025).

Proposition (M. De Falco, C. Musella, AZ): Let G be an uncountable group of regular cardinality $\aleph$ whose large subgroups are closed. If G contains an abelian large subgroup, then G is a QRF-group. Moreover, if $Z(G)$ is large, then G is an ERF-group.

Nilpotent-by-finite and FC groups

The following result deals with such property in the class of groups containing a nilpotent subgroup of finite index.

Theorem (M. De Falco, C. Musella, AZ): Let G be an uncountable nilpotent-by-finite group of cardinality $\aleph$ whose large subgroups are closed. If G contains an abelian large subgroup, then G is an ERF-group.

Here we have the case of groups with finite conjugacy classes.

Theorem (M. De Falco, C. Musella, AZ): Let G be an uncountable FC-group of cardinality $\aleph$ whose large subgroups are closed. Then G contains a finite normal subgroup N such that G/N is an ERF-group. In particular, if G is residually finite, it is an ERF-group.

Countably recognizable classes of groups

Definition: A class of groups $\mathcal{X}$ is said to be *countably recognizable* if a group belongs to $\mathcal{X}$ whenever all its countable subgroups lie in $\mathcal{X}$.

Countably recognizable classes of groups were introduced by R. Baer in 1962, who produced many interesting examples of countably recognizable group classes.

R. Baer, Abzählbar erkennbare gruppentheoretische Eigenschaften, Math. Z. 79 (1962), pp. 344–363.

Countably recognizable classes of groups

Theorem: Let G be a group, and let X be a subgroup of G such that $X \cap K$ is closed in K for each countable subgroup K of G. Then X is a closed subgroup of G.

F. de Giovanni, M. Trombetti: "Countable recognizability and residual properties of groups", Rend. Semin. Mat. Univ. Padova 140 (2018), 69–80.

Corollary: The class of ERF-groups is countably recognizable.

Groups whose large proper subgroups satisfy the ERF property

The previous result is a crucial point in the proof of the following one:

Proposition (M. De Falco, C. Musella, AZ): Let G be an uncountable group whose proper large subgroups are ERF-groups. If G is either nilpotent or an FC-group, then G is an ERF-group.

Minimal-not-ERF groups

Let $\mathcal{X}$ be any class of groups. A group G is called a *minimal-not-$\mathcal{X}$* group if G is not an $\mathcal{X}$-group and every proper subgroup of G belongs to the class $\mathcal{X}$.

Proposition (M. De Falco, C. Musella, AZ): Let G be a minimal-not-ERF group. Then either $G' = G$ or G/G' is a Prüfer group.

Thanks to the previuos result, we can obtain another information about uncountable groups in which every proper large subgroup is an ERF-group.

Proposition (M. De Falco, C. Musella, A. Zaccardo): Let G be an uncountable group whose large proper subgroups are ERF-groups. If G is metabelian, then it is an ERF-group.

Definition: A group G is said to be of *Heineken-Mohamed type* if it is a non-nilpotent p-group, for a prime p, and all its proper subgroups are both subnormal in G and nilpotent.

Theorem (M. De Falco, C. Musella, AZ): Let G be an uncountable not-ERF group such that $G'' \neq G'$. Then every large proper subgroup of G is an ERF-group if and only if the following conditions hold:

- G/G' is a Prüfer group;
- G' is an ERF-group;
- G/G'' is a group of Heineken-Mohammed type;
- for every proper large subgroup H of G, HG'' is properly contained in G.

Thanks for your attention!