Enumeration of quandles: Part 1

Petr Vojtěchovský

UNIVERSITY OF DENVER

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Coauthors

H = Alexander Hulpke

K = Louis Kauffman

M = Sujoy Mukherjee

P = Jesse Parrish

S = David Stanovský

Y = Seung-Yeop Yang

Outline

Coloring rules

Color a diagram of an oriented knot K by an algebra $(X, \triangleleft, \triangleright)$ according to these rules:

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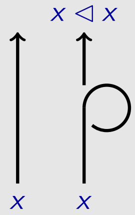
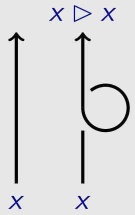
Which properties must $\triangleleft$, $\triangleright$ satisfy for the coloring to be invariant under Reidemeister moves? There are many oriented Reidemeister moves, but all are consequences of the following five:

Reidemeister I





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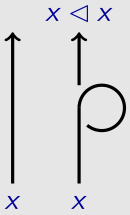
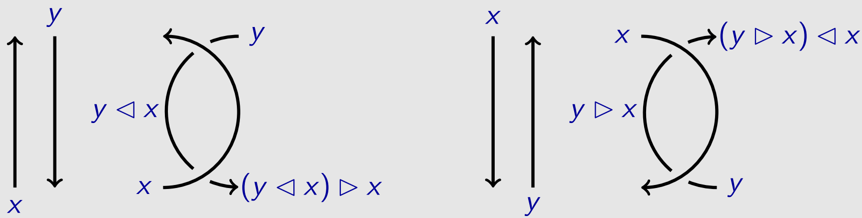


So far we have $x \lhd x = x = x \rhd x$.

Reidemeister II



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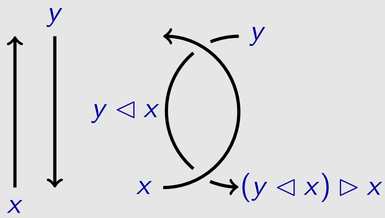
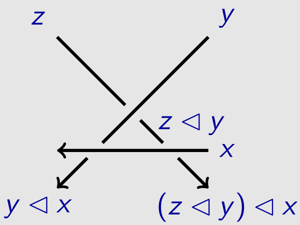


So far we have $x \triangleleft x = x = x \triangleright x$, $(y \triangleleft x) \triangleright x = y$ and $(y \triangleright x) \triangleleft x = y$.

Hence $R_x^{\triangleleft} = (R_x^{\triangleright})^{-1}$ and we don't need to keep track of $\triangleright$ anymore.

Reidemeister III



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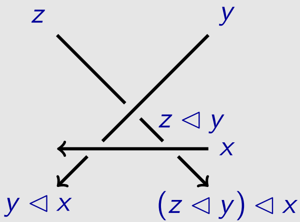

Altogether, we have: $(X, \triangleleft)$ such that $x \triangleleft x = x$ and $R_x \in \mathrm{Aut}(X, \triangleleft)$.

Racks and quandles

Definition

A magma $(Q, \cdot)$ is a **rack** if

- R_x is a permutation of Q for every $x \in Q$,
- $(yx)(zx) = (yz)x$ for every $x, y, z \in Q$.

A rack $(Q, \cdot)$ is a **quandle** if

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Definition

For a rack Q, let

$$\mathrm{Mlt_r(Q)} = \langle R_x : x \in Q\rangle \leq \mathrm{Aut}(Q)$$

be the **right multiplication group** of Q.

Examples of quandles

Projection quandle

$x * y = x$

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$Hx * Hy = H\varphi(xy^{-1})y$

for a group $(G, \cdot)$, subgroup $H \leq G$ and $\varphi \in \mathrm{Aut}(G)$ centralizing H

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Key properties of racks and quandles

Denote by g^G the conjugacy class of g in G.

If G acts on X, let $O(x)$ be the orbit of x and X/G a complete set of orbit representatives.

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Let Q be a rack and $G = \mathrm{Mlt_r(Q)}$. Then:

- $R_x \in C_G(G_x)$,
- $R_x^G = \{R_{\varphi(x)} : \varphi \in G\} = \{R_y : y \in O(x)\}$,
- $G = \langle \bigcup_{x \in Q/G} R_x^G \rangle$,
- if Q is a quandle, then $R_x \in Z(G_x)$ since $R_x(x) = xx = x$.

Joyce-Blackburn representation for racks

Theorem (Blackburn, VY)

Let G be a group acting on X. Then there is a one to one correspondence between:

- *racks Q defined on X with $\mathrm{Mlt_r(Q)} = G$, and*
- **rack envelopes** *for G, that is, tuples $(\rho_x : x \in X/G)$ such that $\rho_x \in C_G(G_x)$ and $\langle \bigcup_{x \in X/G} \rho_x^G \rangle = G$.*

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From a quandle envelope to a quandle: An example

Let $G = \langle (2,3,4),(2,3) \rangle \cong S_3$ act on $\{1,2,3,4\}$. The orbits are $\{1\},\{2,3,4\}$ and we can take $X/G = \{1,2\}$.

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We obtain the quandle

Q	1	2	3	4
1	1	1	1	1
2	2	2	4	3
3	3	4	3	2
4	4	3	2	4

Outline

Isomorphisms and conjugation

Proposition (folklore for right quasigroups)

Let X be a set.

- *If $(X,*)$, $(X,\circ)$ are isomorphic racks then $\mathrm{Mlt_r}(X,*)$, $\mathrm{Mlt_r}(X,\circ)$ are conjugate subgroups of S_X.*
- *Let G, H be conjugate subgroups of S_X. Then the set of racks on X with right multiplication group equal to G contains the same isomorphism types as the set of racks on X with right multiplication group equal to H.*

(iii) *Let $(X,*)$, $(X,\circ)$ be two racks with $\mathrm{Mlt_r}(X,*) = G = \mathrm{Mlt_r}(X,\circ)$. Then $(X,*)$, $(X,\circ)$ are isomorphic if and only if there is an isomorphism $f:(X,*)\to(X,\circ)$ satisfying $f\in N_{S_X}(G)$.*

Action on parameter spaces

For a group $G \leq S_X$ let

$$\mathrm{Par}_r(G) = \prod_{x \in X/G} C_G(G_x), \quad \mathrm{Par}_q(G) = \prod_{x \in X/G} Z(G_x).$$

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Difficulties:

- $\mathrm{Par}_r(G)$ can be large, especially if G is an elementary abelian 2-group. There is a nonabelian $G \le S_{13}$ for which $\mathrm{Par}_r(G)$ has over 2 billion elements.
- Not every $(\rho_x^G : x \in X/G) \in \mathrm{Par}_r(G)$ generates G. This must be explicitly tested.
- Not clear how to use Burnside's Lemma efficiently for envelopes.

Conjugacy classes of subgroups of symmetric groups

It is a nontrivial problem to calculate subgroups of S_n up to conjugation. The following takes several hours in GAP:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$s(n)$	1	2	4	11	19	56	96	296	554	1593	3094	10723	20832

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State of the art:

Theorem (Holt)

There are 7598016157515302757 subgroups of S_{18}, partitioned into 7274651 conjugacy classes.

The enumeration algorithm

Let $r(n)$ (resp. $q(n)$) denote the number of racks (resp. quandles) of order n up to isomorphism.

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- takes about a day to find isomorphism types for $r(11)$ and $q(12)$,
- crashes on $r(12)$, $r(13)$ and $q(13)$,
- takes 3 weeks to determine isomorphism types of racks of order 13 with **nonabelian** right multiplication groups.

Racks with commuting right translations

Theorem

The following conditions are equivalent for a rack Q:

- $\mathrm{Mlt_r(Q)}$ *is abelian,*
- $(xy)z = (xz)y$,
- $x(yz) = xy$,
- Q *is* 2-**reductive** *(that is, $x(yu) = x(yv)$),*
- Q *is* **medial** *(that is, $(xu)(vy) = (xv)(uy)$) and* **paragraphic** *(that is, $x(yx) = xy$).*

Jedlička, Pilitowska, Stanovský and Zamojska-Dzienio used **affine meshes** to construct all 2-reductive racks, in principle. They used Burnside's Lemma efficiently to count 2-reductive racks up to $n \leq 14$. Using their counts for the abelian case, we determined $r(12)$, $r(13)$ and $q(13)$.

Enumeration results: Racks and quandles

n	$q(n)$	$r(n)$	comments
1	1	1	
2	1	2	
3	3	6	
4	7	19	
5	22	74	
6	73	353	
7	298	2080	easy; add column, test, backtrack

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12	21101335	836395102	BUT WAIT, THERE IS MORE
13	469250886	25794670618	VY

Asymptotic growth

Theorem (Blackburn 2013)

For all sufficiently large orders n, we have

$$2^{n^2/4-o(n\log(n))} \le q(n) \le r(n) \le 2^{cn^2},$$

where c is a constant approximately equal to 1.5566.

Theorem (Ashford and Riordan 2017)

For every $\varepsilon > 0$ and for all sufficiently large orders n we have

$$2^{n^2/4-\varepsilon} \le q(n) \le r(n) \le 2^{n^2/4+\varepsilon}.$$

Outline

Connected quandles

Definition

A quandle Q is **connected** if $\mathrm{Mlt_r(Q)}$ acts transitively on Q.

Theorem

Let G be a group acting transitively on X. Let $x \in X$. There is a one-to-one correspondence between:

- *connected quandles Q on X with $\mathrm{Mlt_r(Q)} = G$, and*
- *the set of all $\rho_x \in C_G(G_x)$ such that $\langle \rho_x^G \rangle = G$.*

Enumeration results: Connected quandles

The enumeration of connected quandles was carried out independently by HSV and by Vendramin.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c(n)$	1	0	1	1	3	2	5	3	8	1	9	10	11	0	7	9

n	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$c(n)$	15	12	17	10	9	0	21	42	34	0	65	13	27	24	29	17

n	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
$c(n)$	11	0	15	73	35	0	13	33	39	26	41	9	45	0	45

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n	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
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(Note $c(2p)$ for primes $p > 5$.)

Commuting translations and connectedness

For a rack Q let $\mathrm{Dis_r(Q)} = \langle R_x^{-1}R_y : x, y \in Q\rangle$ be the **right displacement group**.

Proposition (Joyce)

Let Q be a rack. Then $\mathrm{Dis_r(Q)} \trianglelefteq \mathrm{Mlt_r(Q)}$ and $\mathrm{Mlt_r(Q)}/\mathrm{Dis_r(Q)}$ is cyclic.

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Corollary

Let Q be a connected rack. Then $\mathrm{Mlt_r(Q)}' = \mathrm{Dis_r(Q)}$.

Proof.

For $x, y \in Q$ let $\varphi \in \mathrm{Mlt_r(Q)}$ be such that $\varphi(x) = y$. Then
$R_x^{-1}R_y = R_x^{-1}R_{\varphi(x)} = R_x^{-1}\varphi R_x\varphi^{-1} = [R_x, \varphi^{-1}] \in \mathrm{Mlt_r(Q)}'$. □

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Corollary

Let Q be a connected quandle with $\mathrm{Mlt_r(Q)}$ abelian. Then $Q = 1$.

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Theorem (KMV 2025)

Let $m > 0$ and $e \in \mathbb{Z}_2^m$. Let $Q = (\mathbb{Z}_3 \times \mathbb{Z}_2^m, *)$, where

$$(i,a)*(j,b) = \begin{cases} (-i-j, a), & \textit{if } i-j \equiv 0 \pmod 3, \\ (-i-j, a+b), & \textit{if } i-j \equiv 1 \pmod 3, \\ (-i-j, a+b+e), & \textit{otherwise.} \end{cases}$$

Then Q is a connected quandle in which the 2^m right translations $R_{(0,a)}$ are pairwise distinct and pairwise commute.

Outline

Principal quandles

Recall coset quandles:
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Definition

Let $(G, \cdot, 1)$ be a group and $\varphi \in \mathrm{Aut}(Q)$. Then the **principal quandle** $Q(G, \varphi)$ is defined on G by

$$x * y = \varphi(xy^{-1})y.$$

Basic properties of principal quandles

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- Q is connected iff $S = G$,
- any union of cosets of S is a subquandle of Q.

Essential subquandles

Lemma

Let $Q = Q(G, \varphi)$ *and* $S = S(G, \varphi)$. *For any* $x, y \in G$ *we have* $\theta(Sx) = \theta(Sy)$ *or* $\theta(Sx) \cap \theta(Sy) = \emptyset$.

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An **essential subquandle** of Q is any subquandle of the form

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Lemma (PV)

Any two essential subquandles are isomorphic.

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If φ, ψ are conjugate in $\mathrm{Aut}(G)$ then $Q(G,\varphi) \cong Q(G,\psi)$.

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For which groups G is the converse true? That is, which groups G have the property that whenever $Q(G, \varphi) \cong Q(G, \psi)$ then the automorphisms φ, ψ are conjugate in $\mathrm{Aut}(G)$?

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- for simple groups [SV]
- for symmetric groups [Higashitani-Kurihara 2023]
- for dihedral groups D_{2p}, p prime [same]

An isomorphism theorem for principal quandles

Theorem (PSV)

For $i \in \{1, 2\}$ *let* $(G_i, \cdot, e_i)$ *be a group,* $\varphi_i \in \mathrm{Aut}(G_i)$, $Q_i = Q(G_i, \varphi_i)$, $S_i = S(G_i, \varphi_i)$ *and* $m_i = m(Q_i)$. *Let* $E_1 = \bigcup_{x \in X_1} S_1 x$ *be an essential subquandle of* Q_1. *Then* $Q_1 \cong Q_2$ *if and only if*

- $m_1 = m_2$,
- *there is a group isomorphism* $\psi : S_1 \to S_2$ *such that* $\psi\varphi_1 = \varphi_2\psi$ *on* S_1,
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This generalizes results of Hou and Holmes for affine quandles.

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Conjecture (Hayashi)

Let Q be a finite connected quandle. Then every right translation of Q has a regular cycle.

Towards disproving Hayashi's conjecture

Proposition

Suppose that G *is a finite group and* $\varphi \in \mathrm{Aut}(G)$ *is such that* $S(G,\varphi) = \langle \varphi(x)^{-1}x : x \in G\rangle = G$ *and* φ *does* **not** *have a regular cycle. Then the principal quandle* $Q = Q(G,\varphi)$ *is a counterexample to Hayashi's conjecture.*

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- There are finite groups possessing an automorphism without a regular cycle [Horoshevskii].
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Problem

Is there a finite group G and an automorphism φ of G such that $\langle \varphi(x)^{-1}x : x \in G \rangle = G$ and φ does not have a regular cycle?

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- whenever φ has at most five cycles [Lages-Lopes],
- for connected quandles of order ≤ 47.

Outline

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- The orbits of this action are precisely the isomorphism types.
- The space it too large.
- Good news: The action restricts to the set of magmas in a given variety V.
- Bad news: The space is typically still too large.

Partial magmas

A **partial magma** on X is a mapping $t : D \to X$, where $D \subseteq X \times X$ is the (partial) domain of t.

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Theorem

The only invariant domains under the action of S_X are the diagonal $\Delta = \{(x, x) : x \in X\}$ and its complement $(X \times X) \setminus \Delta$.

A funny result

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Theorem

$m_2(V) = m_3(V) = \cdots = m_n(V)$. *(But $m_1(V) \neq m_2(V)$ in general.)*

Proof.

For $1 < i < j$, consider $f \in S_X$ such that $f(1) = 1$ and $f(i) = j$. $\square$

Mapping types

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The orbits of S_X on the space of endofuctions are known as **mapping types**. These were first studied by Davis in 1953. In 1972, De Bruijn came up with a recursive (and hard to evaluate) formula for the number a_n of mapping types.

$$\begin{array}{rcrrrrrrrrr} n & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ a_n & = & 1 & 3 & 7 & 19 & 47 & 130 & 343 & 961 & 2615 \end{array}$$

See the sequence OEIS A001372 for a_n with $n \leq 1000$.

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Theorem (Mitchell-M-V 2024)

There is an $O(n^2)$ algorithm that for a given endofunction t on X_n returns the minimal representative of the same mapping type as t.

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- This results in a stratified group action which can be controlled by the orbit-stabilizer theorem.
- We tried this for the variety of right self-distributive magmas, that is, magmas satisfying $(xy)z = (xz)(yz)$.

Results of Ježek

Theorem (Ježek 1997)

The number of right self-distributive magmas of order n in absolute terms and up to isomorphism is

$$
\begin{array}{rcrrrrrr}
n & = & 1 & 2 & 3 & 4 & 5 & 6 \\
a_n & = & 1 & 9 & 224 & 14067 & 3717524 & 25488943921 \\
i_n & = & 1 & 6 & 48 & 720 & 33425 & 35527077^{*}
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- The entry i_6 was corrected by us [MV].
- With the stratified group action, the calculation takes 2 seconds for $n \le 5$, and a few minutes for $n = 6$.

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There are $3,021,268,037,534,480$ *right self-distributive groupoids of order* 7*, counted absolutely.*

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(Disclaimer: We are redoing the calculations to see if we get the same answer with different orbits.)