UNIVERSITEIT GENT

FACULTY OF SCIENCES

Cabling non-involutive set-theoretic solutions of YBE

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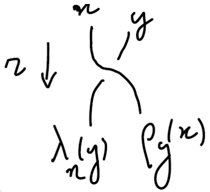
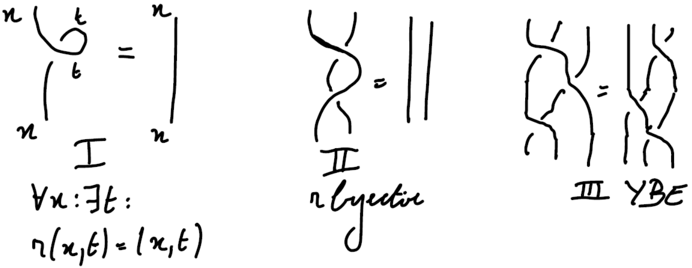
The Yang-Baxter equation: a picture

Definition

A biquandle is a colouring via a non-empty set X, i.e. map $r : X \times X \to X \times X$ that satisfies the rules below + bijectivity of λ_x and ρ_y.





Examples

- Twist solution: $r(x,y) = (y,x)$,
- Lyubashenko, where $f,g : X \to X$ are permutations with $fg = gf$: $r(x,y) = (f(y), g(x))$.
- Quandles: self-distributive structures $(S, \triangleright)$ such that left multiplication is bijective and $x \triangleright x = x$. Then set $r(x,y) = (y, y \triangleright x)$.

"Building blocks"of solutions give

Definition

A biquandle (X, r) is said to be indecomposable, if there does not exist a partition $X = X_1 \cup X_2$, where $(X_i, r|_{X_i \times X_i})$ are subbiquandles.

Denote $T : X \to X$ the map $x \mapsto \lambda_x^{-1}(x)$, the diagonal map. Some results for involutive solutions:

- Rump: If $T = \mathrm{id}_X$, then (X, r) is decomposable,
- Ramirez and Vendramin: Numerical results,
- Camp-Mora and Sastriques: If $\gcd(|T|, |X|) = 1$, then (X, r) is decomposable.

The Yang–Baxter monoid

Definition

Let (X, r) be a solution of the Yang–Baxter equation. Then the monoid

$$M(X, r) = \langle x \in X \mid xy = \lambda_x(y)\rho_y(x)\rangle ,$$

is called the Yang–Baxter monoid of (X, r).

Theorem (CJKvAV)

Let (X, r) be a solution. Then, the monoid $M = M(X, r)$ has a Yang–Baxter semitruss structure with r_M the associated solution. One has the tuple $(M, +, \circ)$ with $a \circ b = a + \lambda_a(b)$.

This is reminiscent of a 'skew brace' but on a tuple of monoids.

New solutions up

Theorem (Gateva-Ivanova, Majid)

Let (X, r) be a solution of the Yang–Baxter equation, then there exists a unique solution r_M on the monoid $M(X, r)$ such that

$$r_M|_{X\times X} = r.$$

Hence, several approaches to construct novel solutions related to a solution (X, r) popped up.

- The k-Veronese solution as the subsolution of $M(X, r)$ of words of length (a multiple of) k. (Gateva-Ivanova)
- For an involutive solution (X, r): $\left\{ kx = \underbrace{x + \dots + x}_{k \text{ times}} \mid x \in X \right\}$ is of size $|X|$ and a subsolution of $M(X, r)$. This is the k-cabled solution. (Lebed, Ramirez and Vendramin)

For non-degenerate solutions one usually has that $(M(X,r),+)$ is not free abelian, and the words kx may be related/equal.

Observation

Let (X,r) be a biquandle. Then, $x \triangleleft x = x$ for all $x \in X$. Thus, the words $kx = x + \ldots + x$ are unrewritable for all positive integers k, hence are in 1-to-1 with X.

The set $\{kx \mid x \in X\}$ is a subsolution of (M, r_M). Transporting this up towards X one obtains the k-cabled solution $(X, r^{(k)})$.

Extending decomposability results

The following results were extended from involutive solutions, i.e. $r^2 = \mathrm{id}_{X \times X}$ to bijective solutions.

Theorem (Rump; Colazzo, vA)

Let (X, r) be a finite solution. If (X, r) is square-free and the derived solution (X, r') is multipermutation, then (X, r) is decomposable.

Theorem (Camp-Mora, Sastriques; Colazzo, vA)

Let (X, r) be a finite solution. If the derived solution (X, r') is multipermutation and the order of the diagonal map T and $|X|$ are coprime, then (X, r) is decomposable.

Simple solutions

Definition

A solution (X, r) is said to be simple, if for every map of solutions $f : (X, r) \to (Y, s)$ is an isomorphism or $|Y| = 1$.

Theorem (CvA)

Let (X, r) be a biquandle and k a positive integer. If (X, r) is a simple solution and k is coprime to the Dehornoy class of X, then $(X, r^{(k)})$ is simple.

Cabling preserves morphisms

Underlying idea to the previous results.

Theorem (Colazzo, vA)

Let k be a positive integer. Then, k-cabling is an endo-functor on the category of biquandles.

- Decomposability of a solution is having an epimorphism to the twist solution on $\{0, 1\}$.
- Simplicity is not having a non-trivial epimorphism.

Future research directions

- Controlling invariants of the solution under cabling,
- solutions of maximal Dehornoy class,
- graph of solutions on a given set X,
- endocabling $\rightarrow$ C.Dietzel.

References

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4. V. Lebed, S. Ramirez and L. Vendramin, *Involutive Yang-Baxter: cabling, decomposability and Dehornoy class*, Rev. Mat. Ib. 40 no.2 (2023): 625-635.

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