A Lazard correspondence between post-Lie rings and skew braces

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Lie rings

Definition

A Lie ring is a triple $(\mathfrak{g}, +, [-,-])$, henceforth simply denoted by $\mathfrak{g}$, such that $(\mathfrak{g}, +)$ is an abelian group and

$$\mathfrak{g}^2 \to \mathfrak{g} : (x, y) \mapsto [x, y],$$

is a biadditive skew symmetric operation on $\mathfrak{g}$ satisfying the Jacobi identity.

The BCH formula

Originates in Lie theory, where it expresses (locally) the multiplication of a Lie group in terms of its associated Lie algebra. The first few terms of the Baker–Campbell–Hausdorff formula are given by

$$\mathrm{BCH}(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \frac{1}{24}(\dots) + \dots$$

Fact

The prime factors appearing in the denominator of the coefficient of degree n in $\mathrm{BCH}(x,y)$ never exceed n.

Lazard correspondence

Let p^n be a prime power.

Definition

A Lie ring $\mathfrak{g}$ of size p^n is Lazard if it is endowed with a descending chain of ideals $\mathfrak{g} = \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \ldots \supseteq \mathfrak{g}_p = \{0\}$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all $i, j \geq 1$.

Definition

A group G of size p^n is Lazard if it is endowed with a descending chain of normal subgroups $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_p = \{1\}$ such that $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \geq 1$.

Theorem (Lazard, 1954)

Let $\mathfrak{g}$ be a Lazard Lie ring. Then $\mathrm{BCH}(x, y)$ is a well-defined element in $\mathfrak{g}$ for all $x, y \in \mathfrak{g}$ and moreover $(\mathfrak{g}, \mathrm{BCH})$ is a Lazard group. The map

$$\mathfrak{g} \mapsto \mathbf{Laz}(\mathfrak{g}) := (\mathfrak{g}, \mathrm{BCH})$$

yields a functorial correspondence between Lazard Lie rings and Lazard groups.

Lazard correspondence for finite groups

More well-known version:

Theorem (Lazard, 1954)

Let p^n be a prime power. Then the Lazard correspondence restricts to:

$$\left\{ \begin{array}{c} \text{Lie rings of size } p^n \text{ and} \\ \text{nilpotency class} < p \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{groups of size } p^n \text{ and} \\ \text{nilpotency class} < p \end{array} \right\}$$

Here we consider all structures with their filtration given by their lower central series.

Skew braces and the holomorph

Definition

A skew brace is a triple $(A, \cdot, \circ)$ with A a set, $(A, \cdot)$ and $(A, \circ)$ groups, and for all $a, b, c \in A$,

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c),$$

with a^{-1} the inverse in $(A, \cdot)$. If $(A, \cdot)$ is abelian, then A is a brace.

Proposition (Bachiller, 2016; Guarnieri, Vendramin, 2017)

Let $(A, \cdot)$ be a group. There is a bijective correspondence between operations $\circ$ making $(A, \cdot, \circ)$ into a skew brace and subgroups G of the holomorph

$$\mathrm{Hol}(A, \cdot) := (A, \cdot) \rtimes \mathrm{Aut}(A, \cdot),$$

such that the projection $G \to A : (a, \lambda) \mapsto a$ is bijective.

Post-Lie rings and the affine Lie ring

Definition

A post-Lie ring is a pair $(\mathfrak{a}, \triangleright)$ with $\mathfrak{a}$ a Lie ring and $\triangleright$ a biadditive operation s.t.

$$x \triangleright [y, z] = [x \triangleright y, z] + [y, x \triangleright z],$$
$$[x, y] \triangleright z = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z - y \triangleright (x \triangleright z) + (y \triangleright x) \triangleright z.$$

for all $x, y, z \in \mathfrak{a}$. If the Lie bracket on $\mathfrak{a}$ is a trivial, then $(\mathfrak{a}, \triangleright)$ is a pre-Lie ring.

Proposition (Burde, Dekimpe, Vercammen, 2012)

Let $\mathfrak{a}$ be a Lie ring. There is a bijective correspondence between operations $\triangleright$ such that $(\mathfrak{a}, \triangleright)$ is a post-Lie ring and Lie subrings $\mathfrak{g}$ of the affine Lie ring

$$\mathfrak{aff}(\mathfrak{a}) := \mathfrak{a} \rtimes \mathfrak{Der}(\mathfrak{a}),$$

such that the projection $\mathfrak{g} \to \mathfrak{a} : (x, \delta) \mapsto x$ is a bijection.

A known result by Smoktunowicz

Theorem (Smoktunowicz, 2022)

Let p be a prime and $n < p - 1$. Then there are constructions:

$$\left\{ \begin{array}{c} \textit{left nilpotent pre-Lie rings of} \\ \textit{size } p^n \end{array} \right\} \to \left\{ \textit{braces of size } p^n \right\},$$

$$\left\{ \begin{array}{c} \textit{pre-Lie rings of size } p^n \textit{ and} \\ \textit{strong nilpotency class} < p \end{array} \right\} \leftarrow \left\{ \begin{array}{c} \textit{braces of size } p^n \textit{ and strong} \\ \textit{nilpotency class} < p \end{array} \right\}.$$

They restrict to a bijective correspondence:

$$\left\{ \begin{array}{c} \textit{pre-Lie rings of size } p^n \textit{ and} \\ \textit{strong nilpotency class} < p \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{braces of size } p^n \textit{ and strong} \\ \textit{nilpotency class} < p \end{array} \right\}.$$

Does this construction yield a bijective correspondence between left nilpotent pre-Lie rings and braces of size p^n?

Restricting the derivations and automorphisms

- Burde–Dekimpe–Deschamps (2009) proved such a result in a differential geometric setting.
- Key ingredient in their proof: starting from an automorphism of a Lie group, one obtains a derivation of its associated Lie algebra through differentiation.
- To mimic their approach: for a given Lazard Lie ring $\mathfrak{a}$ we would like to relate $\mathfrak{der}(\mathfrak{a})$ and $\mathrm{Aut}(\mathbf{Laz}(\mathfrak{a}))$ through the Lazard correspondence but these are generally not nilpotent, so definitely not Lazard.
- Solution: restrict to a Lie subring $\mathfrak{der}_f(\mathfrak{a}) \leq \mathfrak{der}(\mathfrak{a})$ and a subgroup $\mathrm{Aut}_f(A) \leq \mathrm{Aut}(A)$ that are Lazard with respect to a natural filtration.

Theorem (T., 2024)

Let $\mathfrak{a}$ be a Lazard Lie ring, then

$$\mathbf{Laz}(\mathfrak{der}_f(\mathfrak{a})) \cong \mathrm{Aut}_f(\mathbf{Laz}(\mathfrak{a})),$$
$$\mathbf{Laz}(\mathfrak{aff}_f(\mathfrak{a})) \cong \mathrm{Hol}_f(\mathbf{Laz}(\mathfrak{a})),$$

where $\mathfrak{aff}_f(\mathfrak{a}) := \mathfrak{a} \rtimes \mathfrak{der}_f(\mathfrak{a})$ *and* $\mathrm{Hol}_f(\mathbf{Laz}(\mathfrak{a})) := \mathbf{Laz}(\mathfrak{a}) \rtimes \mathrm{Aut}_f(\mathbf{Laz}(\mathfrak{a}))$.

Main results

Theorem (T., 2024)

For p^n a prime power, there exists a correspondence between:

$$\left\{\begin{array}{c}\text{post-Lie rings of size } p^n \text{ and}\\ L\text{-nilpotency class} < p\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{skew braces of size } p^n \text{ and}\\ L\text{-nilpotency class} < p\end{array}\right\},$$

If moreover $n < p$, then this restricts to a bijective correspondence between:

$$\left\{\begin{array}{c}\text{left nilpotent pre-Lie rings of}\\ \text{size } p^n\end{array}\right\} \leftrightarrow \left\{\text{braces of size } p^n\right\}.$$

In this way we obtain an affirmative answer to Smoktunowicz's question and extend the correspondence to a much larger setting.

Thank you!