Right Engel conditions for Orderable Groups

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Notation

Given a group G, we use the left-normed simple commutator notation in G:

$$[a_1, a_2, a_3, \ldots, a_r] = [...[[a_1, a_2], a_3], \ldots, a_r].$$

The long commutators $[x, y, \ldots, y]$, where y occurs $i \geq 0$ times, are denoted by $[x,_i y]$ with $[x,_0 y] = x$.

Let $g \in G$ and $n \in \mathbb{N}$. Then, put

$$L_n(g) = \langle [x, {}_n g] \ : \ x \in G \rangle.$$

g left Engel element $\Leftrightarrow \forall x \in G, \exists n = n(g, x) \in \mathbb{N} : [x, {}_n g] = 1$

If n can be chosen independently of x, then g is a *left n-Engel element* of G.

Dually, set

$$R_n(g) = \langle [g, {}_n x] \ : \ x \in G \rangle.$$

g right Engel element $\Leftrightarrow \forall x \in G, \exists n = n(g, x) \in \mathbb{N} : [g, {}_n x] = 1$

If n can be chosen independently of x, then g is a *right n-Engel element* of G.

g is both left n-Engel and right n-Engel
iff $L_n(g) = 1$ and $R_n(g) = 1$.

A group G is called an Engel group if for every $x, g \in G$ the equation

$$[x, g, g, \dots, g] = 1$$

holds, where g is repeated in the commutator sufficiently many times depending on x and g.

A group G is n-Engel if $[x,_n g] = 1$ for all $x, g \in G$.

G is n-Engel iff $L_n(g) = 1$ and $R_n(g) = 1$, for all $g \in G$.

There are several results showing that certain properties of groups with "small" $R_n(g)$ or $L_n(g)$ are close to those of Engel groups.

Well, Engel groups represent a generalization of nilpotent groups.

If G is locally nilpotent then G is Engel.

But Engel groups need not be locally nilpotent.

Wilson and Zelmanov (1992)

Any profinite Engel group is locally nilpotent.

What about profinite groups in which $L_n(g)$ is "small"?
Are they locally nilpotent or close to be locally nilpotent?

Khukhro and Shumyatsky (2016)

If G is a profinite group such that for every $g \in G$ there is $n = n(g)$ such that $L_n(g)$ is finite, then G has a finite normal subgroup N such that G/N is locally nilpotent.

Later they got two stronger results (for compact groups) which cover a wider class of groups (where weaker left/right Engel like conditions hold).

If G is nilpotent of class n then G is n-Engel.

But n-Engel groups need not be nilpotent.

Kim and Rhemtulla (1995)

Any orderable n-Engel group is nilpotent.

A group G is called orderable if there exists a full order relation $\leq$ on the set G such that $x \leq y$ implies $axb \leq ayb$ for all $a, b, x, y \in G$.

Shumyatsky (2018) considered orderable groups G such that $L_n(g)$ is polycyclic for each $g \in G$.

A group K is polycyclic if and only if it admits a finite subnormal series all of whose factors are cyclic.

The Hirsch length $h(K)$ of a polycyclic group K is the number of infinite factors in the subnormal series.

Theorem 1 - Shumyatsky (2018)

Let h, n be positive integers and G be an orderable group in which $L_n(g)$ is polycyclic with $h(L_n(g)) \leq h$ for every $g \in G$.
Then, there exist (h, n)-bounded numbers h^* and c^* such that G has a finitely generated normal nilpotent subgroup N with $h(N) \leq h^*$ and G/N nilpotent of class at most c^*.

What if we put $R_n(g)$ in place of $L_n(g)$?

Given h, n positive integers.
What can we say about an orderable group G in which $R_n(g)$ is polycyclic with $h(R_n(g)) \leq h$ for every $g \in G$?

Lemma 3.3, P. Shumyatsky, C. Sica and M. T. (2025)

Let h, n be positive integers and G be an orderable group in which $R_n(g)$ is polycyclic with $h(R_n(g)) \leq h$ for every $g \in G$.
Then, G' is nilpotent with (h, n)-bounded class.

Is there any connection between the subgroups $L_n(g)$ and $R_n(g)$?

$$L_n(g) \leq R_n(g)?$$

$$L_n(g) = \langle [x, {}_n g] \, : \, x \in G \rangle, \quad \gamma_n(G) = \langle [g_1, \ldots, g_n] \, : \, g_1, \ldots g_n \in G \rangle.$$

$$L_{n-1}(g) \leq \gamma_n(G), \; \forall g \in G.$$

Is there any connection between $\gamma_n(G)$ and $R_n(g)$?

Lemma 2.3, P. Shumyatsky, C. Sica and M. T. (2025)

Let $G = H\langle g \rangle$ where H is a nilpotent of class c normal subgroup. For any positive integers c, n there exists an integer $f = f(c, n)$ such that $\gamma_f(G) \leq R_n(g)$.

- Let $h, n \in \mathbb{N}$ and G be an orderable group in which $R_n(g)$ is polycyclic with $h(R_n(g)) \leq h$ for every $g \in G$.
- Given $g \in G$, put $H_g = G'\langle g \rangle$.
- Lemma 3.3 and 2.3 imply $\gamma_f(H_g) \leq R_n(g)$ for a certain f.
- On the other hand, $L_{f-1}(g) \leq \gamma_f(H_g) \leq R_n(g)$.
- It follows $L_{f-1}(g)$ polycyclic and $h(L_{f-1}(g)) \leq h$.

Hence, from Theorem 1 (Shumyatsky - 2018) on $L_n(g)$, we got the $R_n(g)$ analogue.

P. Shumyatsky, C. Sica and M. T. (2025)

Let h, n be positive integers and G be an orderable group in which $R_n(g)$ is polycyclic with $h(R_n(g)) \leq h$ for every $g \in G$.
Then, there exist (h, n)-bounded numbers h^* and c^* such that G has a finitely generated normal nilpotent subgroup N with $h(N) \leq h^*$ and G/N nilpotent of class at most c^*.

P. Shumyatsky, C. Sica and M. T., *Right Engel conditions for Orderable Groups*, Bull. Austral. Math. Soc., to appear.

THANK YOU!

If G is locally nilpotent then G is Engel.

But Engel groups need not be locally nilpotent (Golod's example).

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What about profinite groups in which $L_n(g)$ is "small"?
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Khukhro and Shumyatsky (2016)

If G is a profinite group such that for every $g \in G$ there is $n = n(g)$ such that $L_n(g)$ is finite, then G has a finite normal subgroup N such that G/N is locally nilpotent.

What if G is compact?

Medvedev (2003)

Any compact (Hausdorff) Engel group is locally nilpotent.

Khukhro and Shumyatsky (2016)

If G is a profinite group such that for every $g \in G$ there is $n = n(g)$ such that $L_n(g)$ is finite, then G has a finite normal subgroup N such that G/N is locally nilpotent.

This condition means that every element g of the group is *almost left n-Engel* for some $n = n(g)$ depending on g. Note, however, that Engel groups do not necessarily satisfy this condition.

A weaker condition is the following.

A group G is almost left Engel if for every $g \in G$ there is a finite set $\mathcal{E}(g)$ such that for every $x \in G$ there is a positive integer $n(x,g)$ such that $[x,_n g] \in \mathcal{E}(g)$ for all $n \geq n(x,g)$.

It includes Engel groups (when $\mathcal{E}(g) = 1$ for every $g \in G$).

Then the following result covers a wider class of groups and is stronger than the one in 2016.

Khukhro and Shumyatsky (2018)

If G is a compact (Hausdorff) almost left Engel, then G has a finite normal subgroup N such that G/N is locally nilpotent.

A similar result was proved for compact groups which are almost right Engel.

Khukhro and Shumyatsky (2019)

If G is a compact (Hausdorff) almost right Engel group, then G has a finite normal subgroup N such that G/N is locally nilpotent.

A group G is almost right Engel if for every $g \in G$ there is a finite set $\mathcal{R}(g)$ such that for every $x \in G$, there is apositive integer $r(x, g)$ such that $[g,_n x] \in \mathcal{R}(g)$ for all $n \geq r(x, g)$.

Thus, G is an Engel group if we can choose $\mathcal{R}(g) = 1$ for all $g \in G$.

If G is locally nilpotent then G is Engel.

But Engel groups need not be locally nilpotent (Golod's example).

Gruenberg (1966)

Any linear Engel group is hypercentral (and hence locally nilpotent).

Shumyatsky (2018 - 2019)

If G is a linear almost left (right) Engel group, then G has a finite normal subgroup N such that G/N is hypercentral (and hence locally nilpotent).