A Study of Non-inner Automorphisms in Certain Finite p-Groups

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June 24, 2025

Introduction

Let G be an arbitrary group and let G' and $Z(G)$ respectively denote the commutator subgroup and the center of G. We denote by $\Phi(G)$ and $d(G)$, respectively, the Frattini subgroup and the rank of the group G. By $Z_i(G)$, where $i \geq 2$, we denote the ith term of the upper central series of G and $Z_1(G)$ is denoted by $Z(G)$, the center of G. By $\gamma_i(G)$, where $i \geq 2$, we denote the ith term of the lower central series of G and $\gamma_2(G)$ is denoted by G', the commutator subgroup of G. For a nilpotent group G, the smallest positive integer c such that $Z_c(G) = G$ (or $\gamma_{c+1}(G) = 1$) is called the nilpotency class of G. For a finite p-group G, let $\Omega_i(G) = \langle g \in G \mid g^{p^i} = 1 \rangle$, where i is a positive integer.

$\mathrm{Aut}_c(G)$ and Inner Automorphisms

An automorphism α of G is called a class-preserving automorphism if for each $x \in G$, there exists an element $g_x \in G$ such that $\alpha(x) = g_x^{-1}xg_x$; and is called an inner automorphism if for all $x \in G$, there exists a fix element $g \in G$ such that $\alpha(x) = g^{-1}xg$. The group $\mathrm{Inn}(G)$ of all inner automorphisms of G is a normal subgroup of the group $\mathrm{Aut}_c(G)$ of all class-preserving automorphisms of G.

Non-inner Automorphism Conjecture

Non-Inner Automorphism Conjecture of Finite p-Groups

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Some Important Definitions

Bibliography

Non-inner Automorphism Conjecture

A long-standing conjecture asserts that every finite non-abelian p-group admits a non-inner automorphism of order p (see also [**?**, Problem 4.13]). This conjecture arises from Gaschutz's result (1996) that finite p-groups G of order greater than p possess non-inner automorphisms of p-power order [15].

This conjecture was first attacked by Lieback [24] in 1965. He proved the following result:

Theorem

A finite p-group G of class 2 has a non-inner automorphism of order p if p is an odd prime, and of order 2 or 4 if $p = 2$. Moreover, such an automorphism leaving $\Phi(G)$ element-wise fixed.

In 2006, Abdollahi [1] proved the result for class 2. He proved that

Theorem

Every finite non-abelian p-group G of class 2 has a non-inner automorphism of order p leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ element-wise fixed.

In 2013, Abdollahi et al. [6] also proved the conjecture for finite p-groups of class 3.

Results on Co-Classes

Definition

A finite p-group G of order p^n and of nilpotency class $n - c$ is said to be of co-class c.

In 2014, Fouladi and Orfi [12] proved the conjecture for all finite p-groups of co-class 2 with $p > 2$.
In same year, Abdollahi and Ghoraishi [4] proved the result for 2-groups also. They proved that:

Theorem

Let G be a finite 2-group of co-class 2. Then G admits a non-inner automorphism of order 2 or 4 leaving $\Phi(G)$ element-wise fixed.

In 2016, Ruscitti et al. [27] proved the conjecture for all non-abelian finite p-groups of co-class 3, where p is a prime integer such that $p \neq 3$.

Recently, Komma [23] proved the conjecture for all non-abelian finite p-groups of co-class 4 and 5, where p is a prime integer such that $p \neq 5$.

Results on Generators

In 2020, Fouladi and Orfi [13] proved the conjecture for all non-abelian finite d-generator p-groups G with $p > 2$ and $|Z_3(G)/Z(G)| \leq p^{d+1}$.

In 2021, Ghoraishi [21] proved the conjecture for all finite non-abelian p-groups G with **co-class of G is less than or equal to the minimum number of generators of G.**

Regular Group

A finite p-group G is called regular if for any two elements $x, y \in G$, there exists an element c in the commutator subgroup $[\langle x, y\rangle, \langle x, y\rangle]$ of $\langle x, y\rangle$ such that $(xy)^p = x^p y^p c^p$.

In 1980, Schmid [28] proved the conjecture for all regular p-groups.

It is well-known that for an odd prime p, every finite p-group G with cyclic G' is regular.
It follows that if G is a finite p-group ($p > 2$) with cyclic G', then G satisfies the conjecture.

For finite 2-groups, Jamali and Viseh [22] proved the following result:

Theorem

Let G be a finite non-abelian 2-group such that G' is cyclic. Then G has a non-inner automorphism of order 2 fixing either $\Phi(G)$ or $Z(G)$ element-wise.

Powerful Group

A finite p-group G is called powerful if $G' \leq G^p$ (for $p > 2$) and $G' \leq G^4$ (for $p = 2$).

In 2010, Abdollahi [2] proved the conjecture for all finite non-abelian p-groups G with $G/Z(G)$ is powerful.

Camina pair

For a subgroup H of G, let x^H denote the subset $\{h^{-1}xh \mid h \in H\}$ of G. Let G be a finite p-group and let N be a non-trivial proper normal subgroup of G. The pair (G, N) is called a Camina pair if $xN \subseteq x^G$ for all $x \in G - N$. In 2012, Ghoraishi [17] proved the following result:

Theorem

Let G be a finite p-group, where p is an odd prime, such that $(G, Z(G))$ is a Camina pair. Then G has a non-inner automorphism of order p leaving $\Phi(G)$ element-wise fixed.

Exponent

In the same year, Shabani-Attar [30] proved the conjecture for all finite non-abelian p-groups of order p^m and exponent p^{m-2}.

Moreover, He proved that if G is a finite p-group of maximal class, then G has at least $p(p-1)$ non-inner automorphisms of order p that fixing $\Phi(G)$ element-wise.

Maximal Subgroups

For a group a finite p-group G with $|G| > p$ if there is a maximal subgroup M such that $Z(M) \subseteq Z(G)$, then there exists a non-inner automorphism of G of order p (see, Rotman [Lemma 9.108]).

In 2002, Deaconescu and Silberberg [10] proved that if the conjecture is false for a finite p-group G, then $Z(G) < Z(M)$ for all maximal subgroups M of G. This raises the following natural question:

Question. *Given a finite p-group G with $Z(G) < Z(M)$ for all maximal subgroups M of G, does the conjecture hold?*

Recently in 2024, we have proved that every finite p-group G, ($p > 2$) of nilpotency class n such that $\exp(\gamma_{n-1}(G)) = p$, $|\gamma_n(G)| = p$ and $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$, has a non-inner automorphism of order p which fixes $\Phi(G)$ element-wise.

Proof Since $n = \mathrm{cl}(G) \geq 4$ and $\exp(\gamma_{n-1}(G)) = p$, there exists an element $x \in \gamma_{n-1}(G) \setminus Z(G)$ of order p. Thus $[x, G] \subseteq \gamma_n(G)$, and therefore the order of conjugacy class of x in G is p. It follows that $M = C_G(x)$ is a maximal subgroup of G. Let $g \in G \setminus M$. Then

$$(gx)^p = g^p x^p [x, g]^{p(p-1)/2} = g^p.$$

Consider the map β of G defined as $\beta(g) = gx$ and $\beta(m) = m$ for all $m \in M$.

The map β can be extended to an automorphism of G fixing $\Phi(G)$ element-wise and of order p.

We claim that β is a non-inner automorphism of G.

For a contradiction, assume that $\beta = \theta_y$, the inner automorphism of G induced by some $y \in G$, which implies that $y \in C_G(M)$. If $y \notin M$, then $G = M\langle y \rangle$. It follows that $y \in Z(G)$, which is a contradiction.

Therefore $y \in Z(M)$. Since $\beta = \theta_y$, we have $g^{-1}\theta_y(g) = [g, y] = x$. Now, by the given hypothesis $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$, we have $y \in \gamma_{n-1}(G)$. Therefore

$$x = [g, y] \in \gamma_n(G) \leq Z(G),$$

which contradicts our choice of x in G. Hence G has a non-inner automorphism of order p that fixes $\Phi(G)$ element-wise.

Let G be a finite p-group such that $|Z(G)| = p$. Let M be any maximal subgroup of G. Since $Z(M)$ is a characteristic subgroup of M and M is a normal subgroup of G, we have $Z(M)$ is a normal subgroup of G. Thus $Z(G) \leq Z(M)$ for all maximal subgroups M of G. We obtain the following Corollary from above Theorem:

Corollary

Let G be a finite p-group ($p > 2$) of class n such that $|Z(G)| = \exp(\gamma_{n-1}(G)) = p$ and $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Then G has a non-inner automorphism of order p that fixes $\Phi(G)$ element-wise.

Corollary

Let G be a finite 3-group of order 3^n and of co-class 3 such that $Z(M) = \gamma_{n-4}(G)$ is of exponent 3 for all maximal subgroups M of G. Then G has a non-inner automorphism of order 3.

Proof.

Given that cl$(G) = n$. It follows that $\gamma_n(G) \leq Z(G)$. Consequently, $|\gamma_n(G)| = p$. Considering the provided hypothesis $Z(M) = \gamma_{n-1}(G)$ is of exponent p for all maximal subgroups M of G and the proof of Theorem 2.1, we deduce that $Z(C_G(x)) \leq \gamma_{n-1}(G)$ for all $x \in \gamma_{n-1}(G) \setminus Z(G)$. Hence, it follows from Theorem 2.1 that G possesses a non-inner automorphism of order p that fixes $\Phi(G)$ element-wise. $\square$

References I

A. Abdollahi, *Finite p-groups of class 2 have noninner automorphisms of order p*, J. Algebra, **312**(2) (2007) 876-879.

A. Abdollahi, *Powerful p-groups have non-inner automorphisms of order p and some cohomology*, J. Algebra, **323**(3) (2010) 779-789.

A. Abdollahi and S. M. Ghoraishi, *Noninner automorphisms of finite p-groups leaving the center elementwise fixed*, Int. J. Group Theory, **2**(4) (2013) 17-20.

A. Abdollahi and S. M. Ghoraishi, *On non-inner 2-automorphisms of finite 2-groups*, Bull. Aust. Math. Soc., **90**(2) (2014) 227-231.

References II

A. Abdollahi and S. M. Ghoraishi, *On noninner automorphisms of 2-generator finite p-groups*, Comm. Algebra, **45**(8) (2017) 3636-3642.

A. Abdollahi, M. Ghoraishi and B. Wilkens, *Finite p-groups of class 3 have noninner automorphisms of order p*, Beitr. Algebra Geom., **54**(1) (2013) 363-381.

A. Abdollahi, S. M. Ghoraishi, Y. Guerboussa, M. Reguiat and B. Wilkens, *Noninner automorphisms of order p for finite p-groups of coclass 2*, J. Group Theory, **17**(2) (2014) 267-272.

J. L. Alperin, *Groups with finitely many automorphisms*, Pacific J. Math., **12**(1) (1962), 1-5.

References III

- M. T. Benmoussa and Y. Guerboussa, *Some properties of semi-abelian p-groups*, Bull. Aust. Math. Soc., **91**(1) (2015) 86-91.
- M. Deaconescu and G. Silberberg, *Noninner automorphisms of order p of finite p-groups*, J. Algebra, **250**(1) (2002) 283-287.
- E.I. Khukhro and V.D. Mazurov, *Unsolved Problems in Group Theory: The Kourovka Notebook*, No. 20, Russian Academy of Sciences, Siberian Branch, Sobolev Institute of Mathematics, Novosibirsk (2022).
- S. Fouladi and R. Orfi, *Noninner automorphisms of order p in finite p-groups of coclass 2, when* $p > 2$, Bull. Aust. Math. Soc., **90**(2) (2014) 232-236.

References IV

S. Fouladi and R. Orfi, *A note on the existence of noninner automorphisms of order p in some finite p-groups*, J. Algebra Appl., **20**(4) (2021) 2150067.

R. Garg and M. Singh, *Finite p-groups with non-cyclic center have a non-inner automorphism of order p*, Arch. Math., **117**(2) (2021) 129-132.

W. Gaschütz, *Nichtabelsche p-Gruppen besitzen äussere p-Automorphismen*, J. Algebra, **4**(1) (1966) 1-2.

N. Gavioli, L. Legarreta and M. Ruscitti *A note on non-inner automorphisms in finite normally constrained p-groups*, Publ. Math. Debr., **94**(1-2) (2019) 187-195.

References V

- S. M. Ghoraishi, *A note on automorphisms of finite p-groups*, Bull. Aust. Math. Soc., **87**(1) (2013) 24-26.
- S. M. Ghoraishi, *On noninner automorphisms of finite nonabelian p-groups*, Bull. Aust. Math. Soc., **89**(2) (2014) 202-209.
- S. M. Ghoraishi, *On noninner automorphisms of finite p-groups that fix the Frattini subgroup elementwise*, J. Algebra Appl., **17**(1) (2018) 1850137.
- S. M. Ghoraishi, *On noninner automorphisms of finite p-groups that fix the center elementwise*, Int. J. Group Theory, **8**(1) (2019) 1-9.

References VI

S. M. Ghoraishi, *Noninner automorphisms of order p for finite p-groups of restricted coclass*, Arch. Math., **117**(4) (2021) 361-368.

A. R. Jamali and M. Viseh, *On the existence of noninnner automorphisms of order two in finite 2-groups*, Bull. Aust. Math. Soc., **87**(2) (2013) 278-287.

P. Komma, *Non-inner automorphisms of order p in finite p-groups of coclass 4 and 5*, Int. J. Group Theory (2023) 280-287. http://dx.doi.org/10.22108/ijgt.2023.135843.1816.

H. Liebeck, *Outer automorphisms in nilpotent p-groups of class 2*, J. London Math. Soc., **40**(1) (1965) 268-275.

References VII

- R. Orfi and S. Fouladi, *Noninner automorphisms of order p in some finite p-groups*, Bull. Iran. Math. Soc., **44**(5) (2018) 1295-1301.
- M. Ruscitti and L. Legarreta, *The existence of non-inner automorphisms of order p in finite thin p-groups*, (2016). https://doi.org/10.48550/arXiv.1604.07267.
- M. Ruscitti, L. Legarreta and M. K. Yadav, *Non-inner automorphisms of order p in finite p-groups of coclass 3*, Monatsh. Math., **183**(4) (2017) 679-697.
- P. Schmid, *A cohomological property of regular p-groups*, Math. Z., **175**(1) (1980) 1-3.

References VIII

M. Shabani-Attar, *On a conjecture about automorphisms of finite p-groups*, Arch. Math., **93**(5) (2009) 399-403.

M. Shabani-Attar, *Existence of noninner automorphisms of order p in some finite p-groups*, Bull. Aust. Math. Soc., **87**(2) (2013) 272-277.

THANK YOU