From Vaughan Jones' connections to new infinite simple groups

Ryan Seelig (joint work with Arnaud Brothier)

UNSW Sydney

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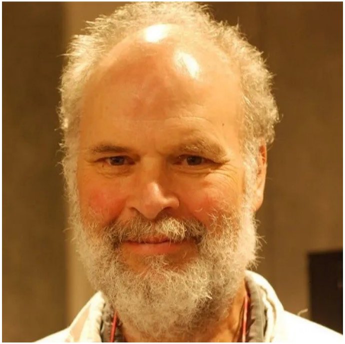
AGTA Conference, Napoli, Italy

Main characters



Vaughan Jones

Arnaud Brothier

Plan of the talk

1. Tell the story of Jones' connections.

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2. Introduce Brothier's forest-skein groups.

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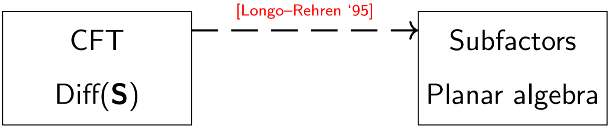
2. Introduce Brothier's forest-skein groups.
3. Investigate a new example.

Between subfactors and conformal field theory

CFT
Diff($\mathbf{S}$)

Subfactors
Planar algebra

Between subfactors and conformal field theory

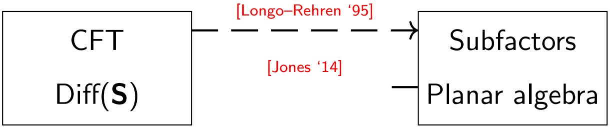


Between subfactors and conformal field theory

$$\boxed{\begin{array}{c}\text{CFT}\\ \mathrm{Diff}(\mathbf{S})\end{array}} \xrightarrow{\text{[Longo–Rehren '95]}} \boxed{\begin{array}{c}\text{Subfactors}\\ \text{Planar algebra}\end{array}}$$

"Does every subfactor have something to do with CFT?" – Jones.

Between subfactors and conformal field theory



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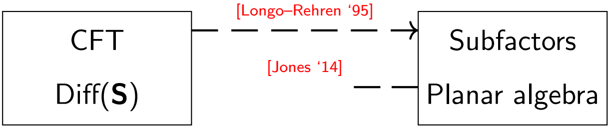

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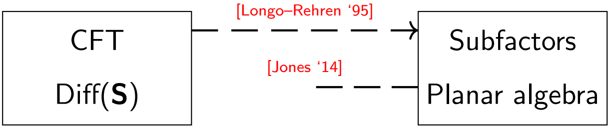

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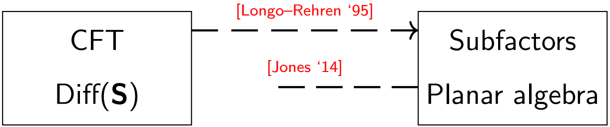

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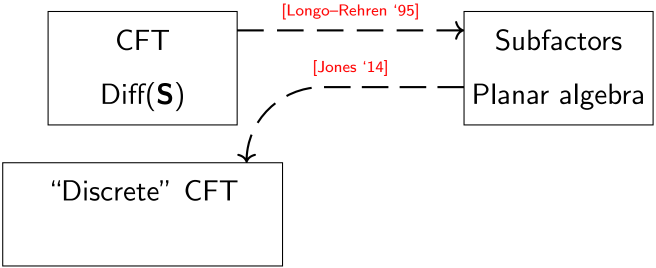

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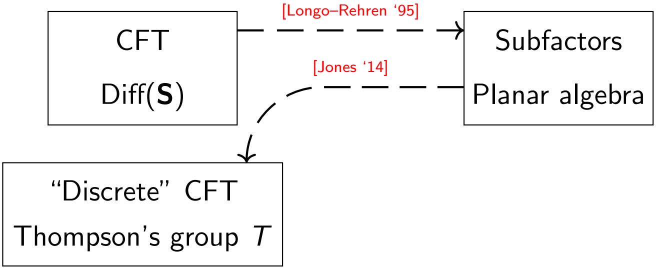

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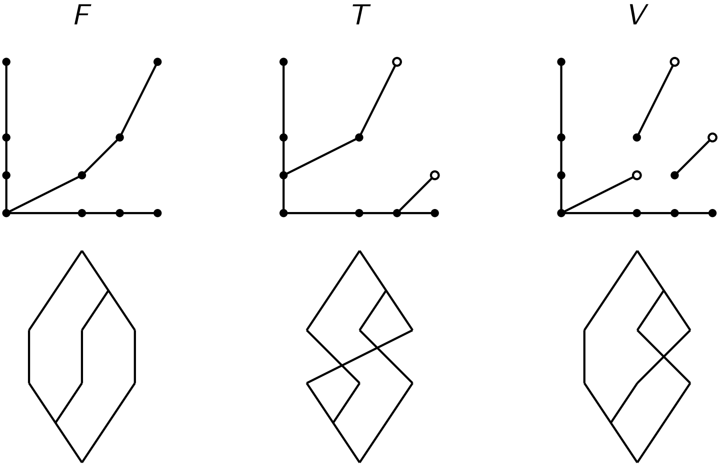
Tree-diagrams and Thompson's groups



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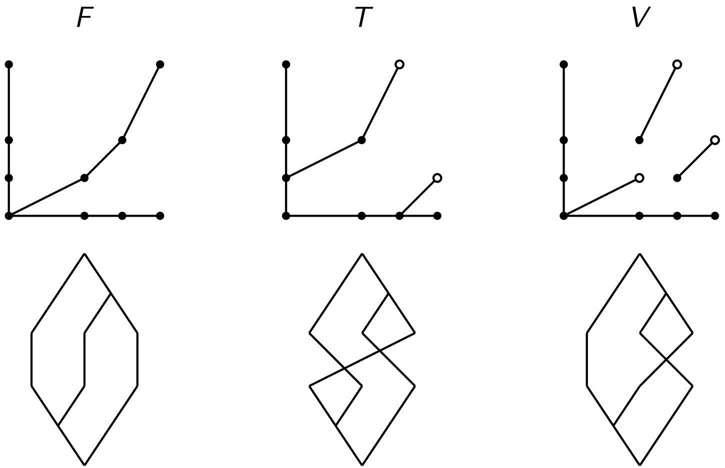

- T and V are finitely presented infinite simple. [Thompson '65]

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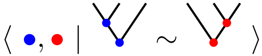
Idea: Pick a "colour set" and a set of "skein relations", e.g.,



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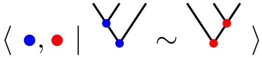

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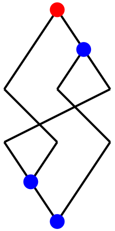

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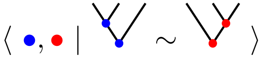

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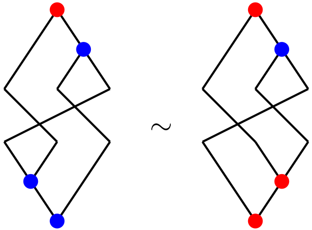

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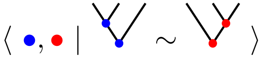

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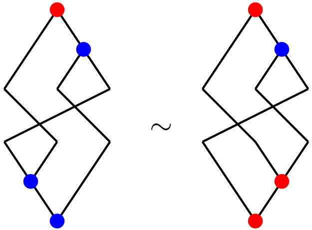

Caveat: Not every skein relation gives a group.

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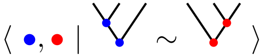

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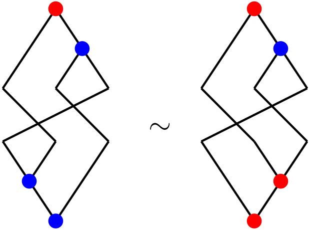

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Theorem (Brothier '22)

The data $\langle \, \textcolor{blue}{\bullet}, \textcolor{red}{\bullet} \mid \textcolor{blue}{t} \sim \textcolor{red}{s} \, \rangle$ *gives three "Thompson-like" groups.*

First examples

Consider two colours • and •.

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- 1st and 2nd examples gives simple, whereas 3rd does not.

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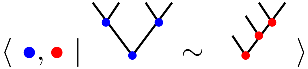
- Sometimes simple, sometimes not. [Brothier–S '24]

A strange new example

Remainder of talk is focused on T-type group G of



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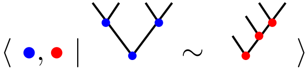

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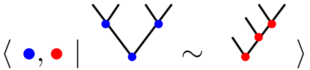

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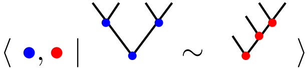

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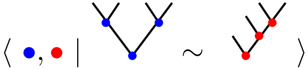

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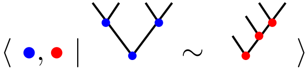

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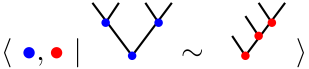

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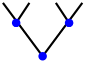
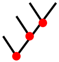
$\langle \bullet, \bullet \mid$  $\sim$  $\rangle$

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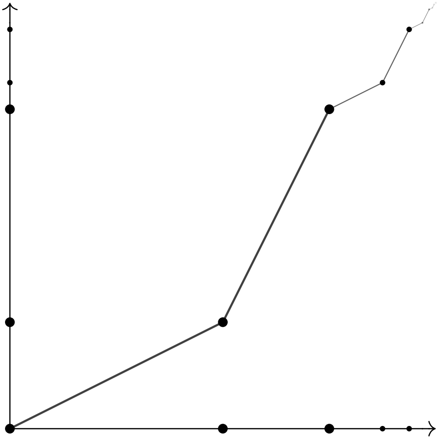
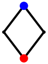
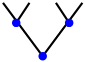
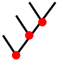
Graph of the action



Action of  on **S** for  $\sim$  .

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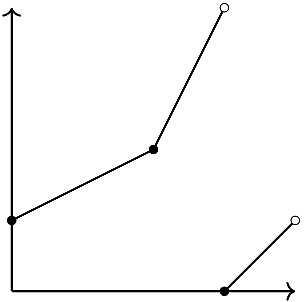

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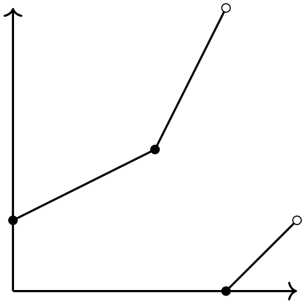

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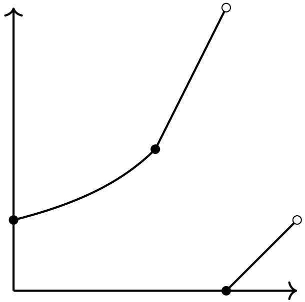

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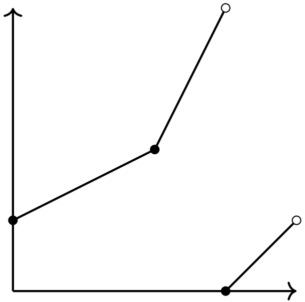

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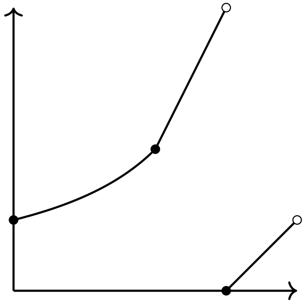

Piecewise projective

- All finitely presented simple groups in the literature acting on **S** by homeomorphisms are atleast piecewise projective.

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- Implies $\mathbf{Z} * \mathbf{Z} \hookrightarrow D(K)$.
- Bounded cohomology: any $D(K) \curvearrowright \mathbf{S}$ unrolls to $D(K) \curvearrowright \mathbf{R}$.
  [Ghys '87, Fournier-Facio–Lodha '23]

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  [Ghys '87, Fournier-Facio–Lodha '23]
- The subgroup structure of $\mathrm{PP}_+(\mathbf{R})$ implies a contradiction.

  [Brin–Squier '85, Monod '13]

Thanks for listening!

- A. Brothier. Forest-skein groups I: between Vaughan Jones' subfactors and Richard Thompson's groups. *Preprint, arXiv:2207.03100*, 2022.
- A. Brothier. Forest-skein groups II: Construction from homogeneously presented monoids. *Internat. J. Math.*, 34(8):Paper No. 2350042, 41, 2023.
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