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Finite p-groups as Generalized Frobenius Complements

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- Other example when P is a Frobenius complement for G.

Frobenius theorem and generalizations

- (Frobenius 1901, in our particular case) Assume $P \cap P^g = 1$ for all $g \in G \setminus P$. Then $D = \Delta \cup \{1\}$, and $G = PD$, $P \cap D = 1$, $1 \neq x \in P$ acts f.p.f. on D, and P is cyclic or quaternionic (for $p = 2$).

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- P has an irreducible complex module on which the elements of $P \setminus P_1$ act f.p.f.

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- THANK YOU!