On the irreducibility of monomial representations for set-theoretical solutions to the Yang–Baxter equation

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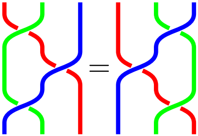
YANG–BAXTER EQUATION

Set-theoretic solutions of the YBE:



A set X and a bijective map
$r : X \times X \to X \times X$ such that

$$(r \times \mathrm{id}) \circ (\mathrm{id} \times r) \circ (r \times \mathrm{id}) = (\mathrm{id} \times r) \circ (r \times \mathrm{id}) \circ (\mathrm{id} \times r).$$

We will use the notation $r(x, y) = (\lambda_x(y), \rho_y(x))$. A solution is:

- **non-degenerate** if λ_x, ρ_x are bijective for all $x \in X$;
- **involutive** if $r^2 = \mathrm{id}$.

With **solution** we mean finite involutive non-degenerate set-theoretic solution to the Yang–Baxter equation.

$$\rho_y(x) = \lambda^{-1}_{\lambda_x(y)}(x).$$

EXAMPLES

Note that, given any set X, $r = \mathrm{id}_{X^2}$ is an involutive degenerate solution.

- Flip solution on a finite set X:
  $r(x, y) = (y, x)$, i.e. $\lambda_x = \mathrm{id}_X$ for every $x \in X$
- (Involutive) Lyubashenko solution on a finite set X:
  $r(x, y) = (\lambda(y), \lambda^{-1}(x))$ for λ a permutation of X.
- Our main example: (Involutive) Lyubashenko solution on $X_n = \{x_1, \ldots x_n\}$ with $\lambda : x_i \mapsto x_{i+1}$, where the indices are considered modulo n (λ is an n-cycle of X_n):

$$r(x_i, x_j) = (x_{j+1}, x_{i-1}).$$

PERMUTATION GROUP

A solution (X,r) is **indecomposable** if there is no non-trivial partition $X = A \sqcup B$ such that $r(A^2) = A^2$ and $r(B^2) = B^2$.

The **permutation group** of a solution (X,r) is

$$\mathcal{G}(X,r) = \langle \lambda_x \mid x \in X \rangle \leq Sym_X.$$

Proposition (Etingof-Schedler-Soloviev, 1999)

(X,r) is indecomposable $\iff$ $\mathcal{G}(X,r)$ acts transitively on X.

PERMUTATION BRACE

Let (X, r) be a solution, then $\mathcal{G}(X, r) = \langle \lambda_x \mid x \in X \rangle$ has a **brace structure** $(\mathcal{G}(X, r), +, \circ)$ with addition induced by

$$\lambda_x + \lambda_y = \lambda_x \circ \lambda_{\lambda_x^{-1}(y)} \text{ for } x, y \in X.$$

The **Dehornoy class** of a solution (X, r) is the exponent of the group $(\mathcal{G}(X, r), +)$.

Proposition (Feingesicht, 2024)

Let (X, r) be a solution of size n and Dehornoy class d, then

$$d \text{ divides } |\mathcal{G}(X, r)| \text{ divides } d^n.$$

Example:

- X_n is indecomposable and it has Dehornoy class n:

$$\mathcal{G}(X_n) = \langle \lambda \rangle \cong C_n \text{ and } + = \circ.$$

STRUCTURE BRACE

Etingof–Schedler–Soloviev defined the **structure group** of a solution (X, r) is

$$\mathbf{G(X,r)} = \langle X \mid x \circ y = u \circ v,\ \text{for } x, y, u, v \in X \text{ s.t. } r(x, y) = (u, v)\rangle.$$

Examples:

- For (X, r) the flip solution on a set X of size n, the structure group is

  $$G(X) = \langle X \mid x \circ y = y \circ x \text{ for } x, y \in X\rangle,$$

  i.e. the free abelian group on n generators.

- The structure group of X_n is

  $$G(X_n) = \langle x_1, \ldots, x_n \mid x_i \circ x_j = x_{j+1} \circ x_{i-1}.\rangle$$

STRUCTURE BRACE

Let (X, r) be a solution, then $G(X, r)$ has a **brace structure** $(G(X, r), +, \circ)$ with addition induced by

$$x + y = x \circ \lambda_x^{-1}(y) \text{ for } x, y \in X.$$

Moreover, if (X, r) has Dehornoy class d, the set

$$kdG(X, r) = \{(kd)g \mid g \in G(X, r)\}$$

is an ideal for every positive integer k. We define

$$\overline{G}_k(X, r) = G(X, r)/kdG(X, r).$$

Proposition

Let (X, r) be a solution, then

$$\mathcal{G}(X, r) = \overline{G}_k(X, r)/\operatorname{Soc}(\overline{G}_k(X, r)).$$

MONOMIAL REPRESENTATION

Theorem (Dehornoy, 2015)

Let (X, r) *be a solution of Dehornoy class* $\mathbf{d}$*, then the map* $x \mapsto P_{\lambda_x} D_x$ *extends to a faithful representation*

$$\Theta : G(X) \to M_X(\mathbb{C}(q))$$

$$g = \sum_{x \in X} g_x x \mapsto \left(\prod_{x \in X} D_x^{g_x} \right) P_{\lambda_g}$$

Moreover for every positive integer $\mathbf{k}$*, the evaluation* $\mathbf{q} = \zeta_{kd}$ *yields a faithful representation*

$$\overline{\Theta}_k : \overline{G_k(X)} \to M_X(\mathbb{C}),$$

INDECOMPOSABILITY VS IRREDUCIBILITY

Recall that (X, r) is **indecomposable** if there is no non-trivial partition $X = A \sqcup B$ such that $r(A^2) = A^2$ and $r(B^2) = B^2$.

What is the connection between indecomposability of the solution and irreducibility of the representations?

- If (X, r) is decomposable, then Θ and $\overline{\Theta_k}$ are reducible:

  if $X = A \sqcup B$ is a decomposition, then the subspaces of $\mathbb{C}(q)^X$ spanned by A and B are invariant.

INDECOMPOSABILITY VS IRREDUCIBILITY - CASE $K > 1$

Let (X, r) be a solution of size n and Dehornoy class d. Recall

$$\Theta : G(X) \to M_X(\mathbb{C}(q)) \qquad \overline{\Theta}_k : \overline{G_k(X)} \to M_X(\mathbb{C})$$

Theorem (C. Dietzel, E. Feingesicht, S.P., 2025)

Suppose that $k > 1$, then the following are equivalent:

1. *X is indecomposable,*
2. *Θ is irreducible,*
3. *$\overline{\Theta}_k$ is irreducible.*

INDECOMPOSABILITY VS IRREDUCIBILITY - CASE $K = 1$

What about the case $k = 1$?

Let (X, r) be a solution of size n and Dehornoy class d. Recall

$$\Theta : G(X) \to M_X(\mathbb{C}(q)) \qquad \overline{\Theta}_k : \overline{G_k(X)} \to M_X(\mathbb{C})$$

Theorem (C. Dietzel, E. Feingesicht, S.P., 2025)

If (X, r) is indecomposable, $\overline{\Theta}_1$ is irreducible if and only if

$$d > 2 \qquad or \qquad d = 2 \text{ and } |\mathcal{G}(X, r)| < 2^{\frac{n}{2}}.$$

Examples:

- X_2 is indecomposable but $\overline{\Theta}_1$ is reducible:

$$\overline{\Theta}_1(x_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \overline{\Theta}_1(x_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\overline{\Theta}(x_1).$$

REFERENCES

- C. Dietzel, E. Feingesicht, S. Properzi, Indecomposability and irreducibility of monomial representations for set-theoretical solutions to the Yang-Baxter equation, arXiv:2409.10648 (2025).

- F.Cedó, Left Braces: Solutions of the Yang-Baxter Equation, Advances in Group Theory and Applications 5 (2018), 33 – 90.

- P. Dehornoy, Set-theoretic solutions of the Yang–Baxter equation, RC-calculus, and Garside germs, Advances in Mathematics 282 (2015), 93 – 127.

- V. Drinfeld, On Some Unsolved Problems in Quantum Group Theory, Lecture Notes in Mathematics 1510 (1992), 1 – 8.

- P. Etingof, T. Schedler, A Soloviev. Set-Theoretical Solutions to the Quantum Yang–Baxter Equation, Duke Mathematical Journal 100 (1999), 169—209

- E. Feingesicht. Dehornoy's Class and Sylows for Set-Theoretical Solutions of the Yang—Baxter Equation, International Journal of Algebra and Computation 34 (2024), 147—173