Normality conditions in the Sylow p-subgroup of $\mathrm{Sym}(p^n)$ and its associated Lie algebra

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Let $V = \mathbb{F}_p^n$ and let W_n be the Sylow p-subgroup of $\mathrm{Sym}(V)$.

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The group $\mathrm{Fun}(\mathbb{F}_p^{n-1}, \mathbb{F}_p)$ is the last base group of the wreath product and it may be identified with the additive group of the polynomials in $n-1$ variables in which every variable appears with degree at most $p-1$.

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In particular, the k-th base subgroup B_k of W_n is defined as

$$B_k := \mathrm{Fun}(\mathbb{F}_p^{k-1}, \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \dots, x_k]/(x_1^p - x_1, \dots, x_{k-1}^p - x_{k-1})$$

We will denote an homogeneous element f in B_k as $f\Delta_k$.

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This operator can be used to express the conjugation action of an element $h\Delta_i \in B_i$ on an element $f\Delta_k \in B_k$ by way of the commutator

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The element $f\Delta_k \in B_k$ acts on $(x_1, \ldots, x_n) \in V$ via the translation

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_n) - e_k f(x_1, \ldots, x_{k-1}).$$

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Notice that T is the image of the right regular representation $\sigma\colon V \to \mathrm{Sym}(V)$.

Our aim is to determine the growth of the normalizer chain originating from T and defined as follows.

$$\mathbf{N}_i = \begin{cases} T & \text{if } i = -1 \\ U_n & \text{if } i = 0 \\ N_{\mathrm{Sym}(V)}(N_{i-1}) & \text{if } i \geq 1, \end{cases} \tag{3}$$

where U_n is the p-Sylow subgroup of $\mathrm{AGL}(V)$.

Theorem ([Aragona et al., 2021])

For every $k \geq 1$, *we have* $N_{\mathrm{Sym}(V)}(\mathbf{N}_{k-1}) = N_{W_n}(\mathbf{N}_{k-1})$.

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Thus, the chain of normalizers for $i \geq 1$ of T in $\mathrm{Sym}(V)$ is equal to the chain of normalizers of T in W_n.

Power Monomials

Let $\Lambda = (\lambda_1, \dots, \lambda_i, \dots)$ be a sequence of non-negative integers with finite support and weight

$$\mathrm{wt}(\Lambda) := \sum_{i=1}^{\infty} i\lambda_i < \infty,$$

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we shall say that Λ is a partition of N if $\mathrm{wt}(\Lambda) = N$. The power monomial x^{Λ} , where Λ is a partition, is defined as

$$x^{\Lambda} = \prod_{i=1}^{\infty} x_i^{\lambda_i}.$$

Central series of W_n

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Lemma

Let $i \geq 1$, then $\gamma_i(W_n) \cap B_k = \langle x^\Lambda \Delta_k \mid \mathrm{pdeg}(x^\Lambda \Delta_k) \leq p^{n-1} - i \rangle$.

And so

$$\gamma_i(W_n) = \gamma_{i+1}(W_n) \rtimes \langle c x^\Lambda \Delta_k \mid \mathrm{pdeg}(x^\Lambda) = p^{n-1} - i,\ 1 \leq k \leq n \text{ and } c \in \mathbb{F}_p \rangle.$$

Corollary

Two consecutive terms of the lower central series are elementary abelian.

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We have that

$$\mathfrak{L}_n = \wr^n \mathfrak{L}_1,$$

where $\mathfrak{L}_1$ is the one dimensional algebra over $\mathbb{F}_p$.

We identify $\mathfrak{L}_n$ as the subalgebra of the Witt algebra over $\mathbb{F}_p$ in n variables spanned by the basis

$$\mathfrak{B} = \bigcup_{k=1}^{n} \mathfrak{B}_k \text{ where } \mathfrak{B}_k = \left\{ x^{\Lambda} \partial_k \mid \Lambda \in \mathcal{P}_p(k-1) \right\}.$$

where $\mathcal{P}_p(k-1)$ denotes the set of all the partitions with values in $\{1, \ldots, p-1\}$ and whose maximal part is less then or equal to $k-1$.

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The product of $\mathfrak{L}_n$ is defined on the basis $\mathfrak{B}$ as follows

$$\begin{aligned} \left[x^{\Lambda}\partial_k, x^{\Theta}\partial_j\right] := & \quad \partial_j(x^{\Lambda})x^{\Theta}\partial_k - x^{\Lambda}\partial_k(x^{\Theta})\partial_j \\ = & \quad \begin{cases} \partial_j(x^{\Lambda})x^{\Theta}\partial_k & \text{if } j < k, \\ -x^{\Lambda}\partial_k(x^{\Theta})\partial_j & \text{if } j > k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem

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This result was already proven by Kaloujnine in group context. We provided an alternative proof of it and we proved that the same holds also for $\mathfrak{L}_n$.

Normal subgroups

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$$|N : \gamma_{p^{k-1}+1}(W_n)| \leq \left(p^{p^{k-1}}\right)^{n-k+1}.$$

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In the special case in which $k=n$, the normal subgroup N coincides with a term of the lower central series.

The map φ

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Let $x^\Lambda \Delta_k \in W_n$ be a monomial element in the k-th base subgroup of W_n. We define

$$\varphi_i(x^\Lambda \Delta_k) = \begin{cases} x^\Lambda \partial_k & \text{if } x^\Lambda \Delta_k \in \gamma_i(W_n) \setminus \gamma_{i+1}(W_n) \\ 0 & \text{otherwise} \end{cases}$$

For a polynomial element $f\Delta_k$, we define $\varphi_i(f\Delta_k) \coloneqq \varphi_i(\mathrm{lt}(f\Delta_k))$, where $\mathrm{lt}(f\Delta_k)$ is the leading term of the polynomial f with respect to a p-weighted degree.

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And, in general, for $g = g_1 \dots g_n \in W_n$, we set

$$\varphi_i(g) = \begin{cases} \sum_{j=1}^{n} \varphi_i(\mathrm{lt}(g_j)) & \text{if } g \in \gamma_i(W_n) \setminus \gamma_{i+1}(W_n), \\ 0 & \text{otherwise.} \end{cases}$$

Normalizer chain

We recall that we wanted to compute the following chain of normalizers originating from T.

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$$|\mathfrak{N}_{\mathfrak{L}_n}(H^{\varphi})| = |N_{W_n}(H)|.$$

As a consequence, the growth of the chain of normalizers in W_n matches the growth of the chain of idealizers in the algebra $\mathfrak{L}_n$.

Consider

$$q_{p,i} = \sum_{j=1}^{i} t_{p,j}$$

where $t_{p,j}$ denotes the number of partitions of j into at least two parts, where each part can be repeated at most $p-1$ times.

Theorem ([Aragona et al., 2024a])

Let $1 \leq i \leq n-1$, then $|\mathbf{N}_i/\mathbf{N}_{i-1}| = p^{q_{p,i+1}}$.

We find a correspondence between the normalizer chain and particular partitions of integers.

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Thank you!