Lower central series in some groups acting on rooted trees

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Table of contents

Introduction

Finite width: definition

A group G in which $|\gamma_i(G) : \gamma_{i+1}(G)|$ is bounded for all $i \geq 1$ is said to be of finite width.

Finite width: some historical facts

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Grigorchuk-Gupta-Sidki groups

Automorphisms of regular rooted trees

- The root is a distinguished vertex.
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- Also $\mathrm{St}_G(n) \trianglelefteq G$.

Describing elements of $\operatorname{Aut} \mathcal{T}_d$

Every $f \in \operatorname{Aut} \mathcal{T}_d$ can be written as

$$f = (f_1, \ldots, f_d)\sigma,$$

where $f_i \in \operatorname{Aut} \mathcal{T}_d$ and $\sigma \in \mathrm{Sym}(d)$.

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The group $G_{\mathbf{e}} = \langle a, b \rangle$ is the GGS-group corresponding to the defining vector $\mathbf{e}$.

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A GGS-group is periodic if and only if $\sum_{i=1}^{p-1} e_i \equiv 0 \mod p$.

A specific example: the Gupta-Sidki p-group

Let $\mathbf{e} = (1, -1, 0, \ldots, 0)$. The Gupta-Sidki group $G_{(1,-1,0,\ldots,0)}$ is generated by a, b, where

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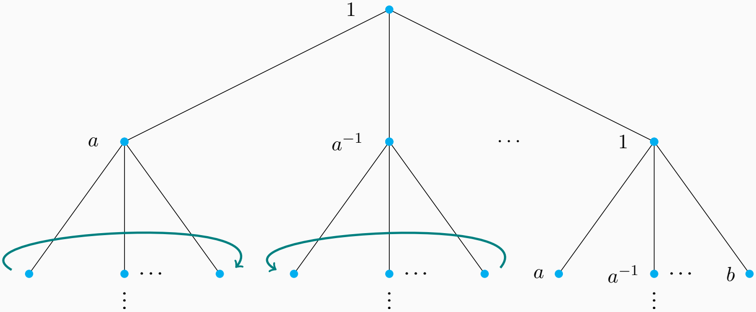

Note that the Gupta-Sidki p-groups are periodic for every $p > 2$.

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In our case: FG-type groups I

Let G be a GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$, where p is an odd prime.

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Let G be a GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$, where p is an odd prime. Define the following two elements of $\mathbb{F}_p$:

$$\varepsilon(\mathrm{e}) = \sum_{i=1}^{p-1} e_i \qquad \text{and} \qquad \delta(\mathrm{e}) = \sum_{i=1}^{p-1} i e_i.$$

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The group G is of FG-type if $\varepsilon(\mathrm{e}) \neq 0$ and $\delta(\mathrm{e}) \neq 0$.

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- The Fabrykowski-Gupta p-groups are of FG-type, since $\mathbf{e} = (1, 0, \overset{p-2}{\ldots}, 0)$
- The GGS-group with constant defining vector is not of FG-type.
- All groups of FG-type are non-periodic.

Lower central series, uniseriality and our results

Back to the conjecture

Conjecture (Zelmanov, 1996)

Is it true that a just infinite pro-p group of finite lower central width is either soluble, p-adic analytic (so linear over $\mathbb{Q}_p$), or commensurable to a positive part of a loop group or to the Nottingham group?

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- $|\Gamma : \Gamma'| = 8$
- $|\gamma_i(\Gamma) : \gamma_{i+1}(\Gamma)| = 2$ or 4 for all $i \geq 2$ (Rozhkov)

For the GGS-groups

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Conjecture (Bartholdi, Eick, Hartung)

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YES!

Uniseriality

Definition

Let G and N be two p-groups and suppose that G acts on N. We define $N_0 := N$ and $N_i := [N, G, \overset{i}{\ldots}, G]$. If $N_t = 1$ and $N_{t-1} \neq 1$, then we say that G acts uniserially on N if $|N_{i-1} : N_i| = p$ for all $i \in \{1, \ldots, t\}$.

In our case

Let G be a non-periodic GGS-group, then we define the quotient $G_n := G/\operatorname{St}_G(n)$ and so $\operatorname{St}_{G_n}(n-1) := \operatorname{St}_G(n-1)/\operatorname{St}_G(n)$.

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Theorem (Fernández-Alcober, Garciarena-Peréz, N)

G_n acts uniserially on $\operatorname{St}_{G_n}(n-1)$.

Advantages: work with G_n

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- If G has the CSP (i.e. every normal subgroup contain some level stabilizer)
- If $|G : \gamma_i(G)| < \infty$

Then $|G : \gamma_i(G)| = |G_n : \gamma_i(G_n)|$.

Our goal

Understand the lower central series of G_n for every $n \in \mathbb{N}$.

C_p

Not much to say!

- G_2 is a p-group of maximal class of order p^{t+1} where t is the rank of the circulant matrix $C = C(e_1, \ldots, e_{p-1}, 0)$. (Fernández-Alcober, Zugadi-Reizabal)

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- The indices of the lower central series of G_2 are completely determined by the fact that G_2 is a p-group of maximal class.

Theorem (Fernández-Alcober, Garciarena-Peréz, N)

Let G be a non-periodic GGS-group. Then the following hold:

1. *We have $|G_2 : G_2'| = p^2$ and $|\gamma_i(G_2) : \gamma_{i+1}(G_2)| = p$ for every $i = 2, \ldots, p$. Also $\gamma_{p+1}(G_2) = 1$ and G_2 has nilpotency class p.*

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2. *For every $i \geq 2$ and every $g \in G \smallsetminus \mathrm{St}_G(1)$, we have*

$$\gamma_i(G_2) = \langle [b, g, \overset{i-1}{\ldots}, g] \rangle \gamma_{i+1}(G_2).$$

In particular, $\gamma_i(G_2) = \langle [b, a^\varepsilon b, \overset{i-1}{\ldots}, a^\varepsilon b] \rangle \gamma_{i+1}(G_2)$.

- G_3 acts uniserially on the level stabilizer $\mathrm{St}_{G_3}(2)$.
- We investigate the position of some commutators in the generators of G_3.
- If $\delta = 0$, the structure of the lower central series of G_3 is not clear.

Theorem (Fernández-Alcober, Garciarena-Peréz, N)

Let G be GGS-group of FG-type. Then the following hold:

1. *The indices between consecutive terms are:*

$$|\gamma_i(G_3):\gamma_{i+1}(G_3)| = \begin{cases} p^2 & \text{if } i \in \{1, p\}, \\ p & \text{if } i \in \{2, \dots, p-1, p+1 \dots, p^2-1\}, \\ 1 & \text{if } i \geq p^2. \end{cases}$$

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2. *Furthermore, for $i \in \{2,\dots,p-1,p+1\dots,p^2-1\}$, we have*

$$\gamma_i(G_3) = \langle [b, a^\varepsilon b, \overset{i-1}{\ldots}, a^\varepsilon b] \rangle \gamma_{i+1}(G_3),$$

   and,

$$\gamma_p(G_3) = \langle [b, a^\varepsilon b, \overset{p-1}{\ldots}, a^\varepsilon b], [b, a^\varepsilon b, \overset{p-2}{\ldots}, a^\varepsilon b, b] \rangle \gamma_{p+1}(G_3).$$

$G_4, G_5, \ldots$

- We compare the lower central series of G_n with that of the wreath product $W(G_{n-1}) = G_{n-1} \wr C_p$.

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- Some terms of the lower central series of $\psi(G_n)$ coincide with terms of the lower central series of $W(G_{n-1})$.
- Some other terms appear as "sandwiches" of the series of $W(G_{n-1})$.

Our main result

Theorem (Fernández-Alcober, Garciarena-Peréz, N)

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Let p be an odd prime and let G be a GGS-group of FG-type defined over the p-adic tree.

1. *We determine all the generators of consecutive terms of the lower central series.*
2. *We conclude that G is a group of lower central width 2.*

Conclusions

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Further work

Use the information above to determine the structure of the Lie algebra associated with the group.

Grazie mille per l'attenzione :)