On finite groups in which the twisted conjugacy classes satisfy some conditions

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The relation of φ-conjugation

Let G be a group, $\varphi \in \mathrm{Aut}(G)$ and $x, y \in G$:

$$y \text{ is } \varphi\text{-conjugate to } x \iff \exists z \in G : y = z^{-1} x z^{\varphi}$$

- φ-conjugation is an equivalence relation in G
- equivalence classes are called twisted conjugacy classes or φ-conjugacy classes
- For every $g \in G$, $[g]_{\varphi} = \{x^{-1} g x^{\varphi} | x \in G\}$ and $|[g]_{\varphi}| = |G : R_{\varphi}(g)|$, where $R_{\varphi}(g) = \{x \in G | x^{\varphi} = x^{g}\}$.
- In particular $[1]_{\varphi} = \{x^{-1} x^{\varphi} | x \in G\}$ and $|[1]_{\varphi}| = |G : C_G(\varphi)|$, because $R_{\varphi}(1) = \{x \in G | x^{\varphi} = x\} = C_G(\varphi)$
- $[1]_{\varphi} = \{1\}$ if and only if $\varphi = id_G$ and, if G is a finte group, then $[1]_{\varphi} = G$ if and only if $C_G(\varphi) = 1$ that is φ is fixed–point–free.

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Example

If $\varphi = id_G$ we have that

$$y \text{ is } \varphi\text{-conjugate to } x \iff \exists z \in G : y = z^{-1}xz$$

then, for every $g \in G$:

- $[g]_\varphi = \{x^{-1}gx | x \in G\} = Cl_G(g)$ the conjugacy class of g
- $R_\varphi(g) = C_G(g) = \{x \in G | g^x = g\}$ the centralizer of g in G.

Problem

If G is a finite group, then the set

$$n(G) := \{|Cl_G(g)|| g \in G\} \subseteq \mathbb{N}$$

has an influence on the structure of G.
Now, let $id_G \neq \varphi \in \mathrm{Aut}(G)$ and consider

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Theorem (C. N. 2025)

Let G be a group and $\varphi \in \mathrm{Aut}(G)$; then

- $1 \in n_\varphi(G)$ if and only if $\varphi \in \mathrm{Inn}(G)$
- For every $\varphi \in \mathrm{Inn}(G)$, we have $n_\varphi(G) = n(G)$
- For every $\varphi \in \mathrm{Inn}(G)$ and for every $t \in n(G)$ the number of conjugacy classes of order t is equal to the number of φ-conjugacy classes of order t.

So we consider $\varphi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$. Of course $n_\varphi(G) = \{|G|\}$ if and only if φ is fixed point free.

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There exist non-abelian groups having an automorphism φ such that φ-conjugation is a congruence.

Let $G = P \times \langle x \rangle$, where P is an extraspecial p-group and $\langle x \rangle \simeq \mathbb{Z}_p$; put $P' = Z(P) = \langle z \rangle$ and consider the automorphism $\varphi \in \mathrm{Aut}(G)$ such that $y^\varphi = y$ for every $y \in P$ and $x^\varphi = xz$. Then we have:

- for every $g = yx^t \in G$, with $y \in P$, we have that

$$g^{-1}g^\varphi = x^{-t}y^{-1}y(x^t)^\varphi = x^{-t}x^tz^t = z^t \in Z(P) \subseteq Z(G)$$

  then φ is a central automorphism and $[1]_\varphi = Z(P) \leq G$
- for every $g = yx^t \in G$, with $y \in P$, we have that $g[1]_\varphi \subseteq [g]_\varphi$. On the other hand $[g]_\varphi \subseteq g[1]_\varphi$ because

$$g_1^{-1}gg_1^\varphi = g_1^{-1}yx^tg_1^\varphi = x^ty[y, g_1]g_1^{-1}g_1^\varphi \in gZ(P) = g[1]_\varphi$$

- Then $[g]_\varphi = g[1]_\varphi$ that is φ-conjugation is a congruence in G.

Notice that $n_\varphi(G) = \{p\}$.

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Let $G = \langle x, y | x^p = y^2 = 1, x^y = x^{-1} \rangle$ the dhiedral group of degree a prime number $p \geq 5$; then for every $\varphi \in Aut(\mathrm{G}) \setminus Inn(\mathrm{G})$:

- $R(\varphi) = 2$
- φ-cojugacy classes are $[1]_\varphi = \langle x \rangle$ and $[y]_\varphi = y[1]_\varphi$
- $n_\varphi(G) = \{p\}$

Non-abelian groups of "φ-conjugacy type $\{p\}$"

Theorem (C.N. 2025)

Let G be a finite non-abelian group; if there exists $\varphi \in \mathrm{Aut}(G)$ such that $n_\varphi(G) = \{p\}$ where p is the smallest prime divisor of $|G|$, then:

- $C = C_G(\varphi)$ is a non-abelian group such that $n(C) = \{1, p\}$;
- $\varphi \in \mathrm{Aut}_C(G)$ is central;
- G is nilpotent of class 2.

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Twisted conjugacy classes of the unit element

Let G be a group and $\varphi \in \mathrm{Aut}(G)$; the φ-conjugacy class of 1 that is the set $[1]_\varphi = \{[x,\varphi] = x^{-1}x^\varphi | x \in G\} =: [G,\varphi]$ is a subgroup if:

- $\varphi = id_G$
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Usually $[G,\varphi]$ is not a subgroup even if φ is inner, for instance: If $G = \mathbb{S}_3$ and $\varphi = \overline{(123)}$, then $[G,\varphi] = \{1, (132)\} \not\leq G$

For every $x, y \in G$ we have that

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Hence if $[G,\varphi]$ is a subgroup, then it is a normal subgroup

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For every integer $n > 2$ and for every odd prime p there exists a finite p-group G of class $\geq n$ such that $[G,\varphi] \leq G$ for every $\varphi \in \mathrm{Inn}(G)$.

Notice

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If $[G, \varphi]$ is a subgroup for every $\varphi \in Aut(G)$, then the group G is nilpotent. If in addition G is finitely generated, then G is abelian.

The last part of this conjecture is certainly false:

there exist finite non-abelian p-groups in which every automorphism is central (M.J. Curran $p = 2$, J.J. Malone p odd).

Notice: these groups are nilpotent of class 2.

Question

Is it possible to bound the nilpotency class of a finite group G such that $[G, \varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$?

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Notice: these groups are nilpotent of class 2.

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Is it possible to bound the nilpotency class of a finite group G such that $[G, \varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$?

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For every integer $n \in \mathbb{N}$ and for every odd prime p, there exists a finite p-group G of class n, such that $[G, \varphi] \leq G$ for every $\varphi \in \text{Aut}(G)$.

sketch of proof:

Let n be an integer $n \geq 2$, p be an odd prime and $G = A \rtimes \langle x \rangle$, where $A = \langle a \rangle \times \langle b \rangle \simeq \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$, $\langle x \rangle \simeq \mathbb{Z}_{p^{n-1}}$ and $c^x = c^{1+p}$ for every $c \in A$.

It is easy to prove that:

- $G' = \langle a^p \rangle \times \langle b^p \rangle$
- G has class n
- G is a regular p-group, that is $(hg)^p = h^p g^p z^p$, with $z \in G'$, for every $h, g \in G$

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Let $\varphi \in \text{Aut}(G)$;

- For every $c \in A$ we have that $[c, \varphi] = c^{-1}c^{\varphi} \in A$ and $B = \{[c, \varphi] | c \in A\} \leq A$ because for every $c, d \in A$ $[c, \varphi][d, \varphi] = [c, \varphi]^{d}[d, \varphi] = [cd, \varphi] \in B$ and $[c, \varphi]^{-1} = [c^{-1}, \varphi] \in B$
- $[x, \varphi] \in A$ and $V = \{[x^{\alpha}, \varphi] | \alpha \in \mathbb{Z}\} = \langle [x, \varphi] \rangle$
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Corollary

There exists an infinite non-nilpotent group G such that $[G, \varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$.

proof:

For every $n \in \mathbb{N}$,

- fix an odd prime p_n such that $p_n \neq p_m$ for every $m < n$;
- consider a finite p_n-group P_n of class n such that $[P_n, \varphi] \leq P_n$ for every $\varphi \in \mathrm{Aut}(P_n)$.

Then $G := Dir_{n \in \mathbb{N}} P_n$ is non-nilpotent and $[G, \varphi] \leq G$ for every $\varphi \in \mathrm{Aut}(G)$.

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What happens if $p = 2$?

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There exist non-abelian finite 2-groups G such that $[G, \varphi] \leq G$ for every $\varphi \in \operatorname{Aut}(G)$.

These 2-groups have class 2

Open question

Is it possible to construct such a finite 2-group of class > 2?

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Is it possible to construct such a finite 2-group of class > 2?

If $G = A \rtimes \langle x \rangle$, where $A = \langle a \rangle \times \langle b \rangle \simeq \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, $\langle x \rangle \simeq \mathbb{Z}_{2^{n-1}}$ and $c^x = c^3$ for every $c \in A$; then there exists $\varphi \in \mathrm{Aut}(G)$ such that $[G, \varphi] \not\trianglelefteq G$.

It is possible to prove that

if $c \in A$ has order 2^n, then xc has order 2^{n-1}

Hence

- for all $c \in A$ of order 2^n there exists $\varphi \in \mathrm{Aut}(G)$ such that $y^\varphi = y$ for all $y \in A$, and $x^\varphi = xc$
- $[G, \varphi] = [\langle x \rangle, \varphi]$
- $|[G, \varphi]| = |[\langle x \rangle, \varphi]| = |\langle x \rangle : C_{\langle x \rangle}(\varphi)| \leq 2^{n-1}$
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Refereces

1. C. Nicotera, *On finite groups in which the twisted conjugacy classes of the unit element are subgroups*, Arch. Math. Vol **123** (2024) pp. 225-232
2. C. Nicotera, *On twisted conjugacy classes in finite groups*, to appear.