On some generalizations of normal and pronormal subgroups

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Dedication

The aim of this talk

is a tribute

to the memory of

Francesco de Giovanni

Definitions

Definition

A subgroup H of a group G is said to be **subnormal in** G if there is a finite chain of subgroups of G:

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G.$$

Definition

A subgroup H of a group G is said to be **pronormal in** G if for every $g \in G$ there is an element $k \in < H, H^g >$ such that $H^g = H^k$.

Almost normal subgroups and nearly normal subgroups

Definitions

A subgroup H of a group G is said to be **nearly normal** if

$$|H^G : H| \text{ is finite.}$$

A subgroup H of a group G is said to be **almost normal** if

$$|G : N_G(H)| \text{ is finite.}$$

A subgroup H of a group G is said to be **normal-by-finite (or virtually normal)** if

$$|H : H_G| \text{ is finite.}$$

Almost subnormal and subnormal-by-finite subgroups

Definitions

A subgroup H of a group G is said to be **almost subnormal in G**

if there exists a subgroup K of finite index in G such that H is subnormal in K.

A subgroup H of a group G is said to be **subnormal-by-finite**

if there exists a subgroup K of G such that

K is subnormal in G and $|H : K|$ is finite.

Almost pronormal and pronormal-by-finite subgroups

Definitions

A subgroup H of a group G is said to be **almost pronormal in G**

if there exists a subgroup K of finite index in G such that H is pronormal in K.

A subgroup H of a group G is said to be **pronormal-by-finite**

if it contains a subgroup K such that

K is pronormal in G and $|H : K|$ is finite.

Some Francesco's papers

- F. de Giovanni, S. Franciosi, L.A. Kurdachenko, Groups with many almost normal subgroups, *Ann. Mat. Pura Appl.*, **169** (1995), 35-65.
- F. de Giovanni, S. Franciosi, Groups with restrictions on infinite subnormal subgroups, *Rend. Circ. Mat. Palermo*, **45** (1996), 256-266.
- F. de Giovanni, S. Franciosi, Groups with many normal-by-finite subgroups, *Proc. Amer. Math. Soc.*, **125** (1997), 323-327.
- F. de Giovanni, S. Franciosi, Y.P. Sysac, On subnormal subgroups of factorized groups, *J. Algebra*, **198** (1997), 469-480.
- F. de Giovanni, S. Franciosi, Y.P. Sysac, On ascendant subgroups of infinite groups, *Comm. Algebra*, **26** (1998), 3313-3333.

Some Francesco's papers

- F. de Giovanni, S. Franciosi Groups without nearly abnormal subgroups, *Glasgow Math. J.*, **41** (1999), 283-288.
- F. de Giovanni, G. Vincenzi, Pronormality in infinite groups, *Proc. Roy. Irish Acad.*, **100A** (2000), 189-203.
- F. de Giovanni, C. Musella, Groups with nearly modular subgroup lattice, *Colloquium Math.*, **88** (2001), 738-764.
- F. de Giovanni, C. Musella, Y.P. Sysak, Groups with Almost Modular Subgroup Lattice, *J. Algebra*, **243** (2001), 738-764.
- F. de Giovanni, G. Vincenzi, Some topics in the theory of pronormal subgroups of groups, *Quaderni Math.*, **8** (2002), 175-202.

Some Francesco's papers

- M. De Falco, F. de Giovanni, C. Musella, R. Schmidt, Groups in which every non-abelian subgroup is permutable, *Rend. Circ. Mat. Palermo* , **LII** (2003), 70-76.
- M. De Falco, F. de Giovanni, C. Musella, Groups in which every subgroup is modular-by-finite, *Bull. Austral. Math. Soc.*, **69** (2004), 441-450.
- M. De Falco, F. de Giovanni, C. Musella, Y.P. Sysak , The structure of groups whose subgroups are permutable-by-finite, *J. Austral. Math. Soc.*, **81** (2004), 1007-1017.
- M. De Falco, F. de Giovanni, C. Musella, Groups in which every subgroup is permutable-by-finite, *Comm. Algebra*, **32** (2006), 35-47.
- F. de Giovanni, A. Russo, G. Vincenzi, Groups with all subgroups pronormal-by-finite, *Mediterranean J. Math.*, **1** (2007), 65-71.

Some Francesco's papers

- M. De Falco, F. de Giovanni, C. Musella, Y.P. Sysak, Groups with normality conditions for non-abelian subgroups, *J. Algebra*, **315** (2007), 665-682.
- F. de Giovanni, A. Russo, G. Vincenzi, Groups in which every subgroup is almost pronormal, *Note di Matematica*, **1** (2008), 583-594.
- M. De Falco, F. de Giovanni, C. Musella, Groups with normality conditions for non-periodic subgroups, *Boll.Uni. Mat. It*, **4** (2011), 109-121.
- M. De Falco, F. de Giovanni, C. Musella, Groups with normality condition for subgroups of infinite rank, *Pubbl. Mat*, **58**, (2014), 331-340.
- F. de Giovanni, F. Saccomanno, A note on infinite groups whose subgroups are closee to be normal-by-finite, *Turkish J. Math*, **39**, (2015), 49-53 107185.

Some Francesco's papers

- M. Brescia, F. de Giovanni, Groups satisfying the double chain condition on non-subnormal subgroups, *Ric. Mat.*, **65**, (2016), 255-261.
- M. Brescia, F. de Giovanni, Groups satisfying the double chain condition on non-pronormal subgroups, *Riv. Mat. Univ. Parma*, **8**, (2017), 353-366.
- M. De Falco, F. de Giovanni, H. Heineken, C. Musella, Normality in uncountable groups, *Adv. in Group Theory Appl.*, **3**, (2017), 13-29.
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Some Francesco's papers

- F. de Giovanni, M. Trombetti, Pronormality in group theory, *Adv. in Group Theory Appl.*, **9**, (2020), 123-149.
- M. De Falco, F. de Giovanni, C. Musella, Groups with normality condition for uncountable subgroups, *J. Austral. Math. Soc.*, **111**, (2021), 268-277.
- F. de Giovanni, M. Trombetti, Subnormality in linear groups, *J. Pure and Appl. Algebra*, **227**, (2023), 107185.
- M. De Falco, F. de Giovanni, C. Musella, Pronormality in uncountable groups, *Comm. Algebra*, **53**, (2025), 268-277.

Near normal, near subnormal and near pronormal subgroups

Definitions

A subgroup H of a group G is said to be
near normal in G
if it is normal in
every proper subgroup K of G containing it.

A subgroup H of a group G is said to be
near subnormal in G
if it is subnormal in
every proper subgroup K of G containing it.

Near normal, near subnormal and near pronormal subgroups

Definition

A subgroup H of a group G is said to be
near pronormal in G

if it is pronormal in
every proper subgroup K of G containing it.

Near normal, near subnormal and near pronormal subgroups

- R.M. Guralnick, H.P. Tong-Viet, G. Tracey, Weakly subnormal subgroups and variations of the Baer-Suzuki theorem, *J. London Math. Soc.* **111** (2025), e70057.

- B. Baumeister, T.C. Burness, R.M. Guralnick, H.P. Tong-Viet, On the maximal overgroups of Sylow subgroups of finite groups, *Advances in Math.* **444** (2024), 109632.

New paper

L.A. Kurdachenko, P. Longobardi, M. M.

On some generalizations of normal and pronormal subgroups

in preparation.

Outline of the talk

- Groups with many near normal (near subnormal) subgroups

    - Groups with many near pronormal subgroups

- Groups in which the property of being near normal (near pronormal) is transitive

- Groups in which near normal subgroups form a sublattice of the lattice of all subgroups

Outline of the talk

- Groups with many near normal (near subnormal) subgroups
  - Groups with many near pronormal subgroups
- Groups in which the property of being near normal (near pronormal) is transitive
  - Groups in which near normal subgroups form a sublattice of the lattice of all subgroups

Examples

Examples

If H is a normal subgroup of a group G,
then H is near normal in G.

If H is a subnormal subgroup of a group G,
then H is near subnormal in G.

If H is a maximal subgroup of a group G,
then H is near normal and near subnormal in G.

A Sylow 2-subgroup of $PSL(2,8)$
is near normal and non maximal in $PSL(2,8)$.

Some remarks

Let G be a group and H be a subgroup of G.

Remarks

If H is a near normal subgroup of G, not contained in a maximal subgroup of G,
then H is normal in G.

If H is a non central subgroup of order 2
in a generalized dihedral 2-group,
then H is near subnormal and non subnormal in G.

Some remarks

Let G be a group and H be a subgroup of G.

Remarks

If H is a near normal subgroup of G, and K, L are proper subgroups of G, containing H such that

$$\langle L, K\rangle = G.$$

then H is normal in G.

If H is a non normal, near normal subgroup of G, then H is contained in a unique maximal subgroup of G.

Some remarks

Let G be a finite group and H be a subgroup of G.

Remark

If H is a non subnormal, near subnormal subgroup of G, then H is contained in a unique maximal subgroup of G.
(Wielandt's Zipper Lemma)

I.M. Isaacs, Finite group theory, *Graduate Studies in Mathematica* **92**, American Math. Soc., Providence (2008), 97-112.

Some remarks

Remark

If G is a simple group with no maximal subgroups, then G has no proper non-trivial near normal subgroups.

Jónsson groups of cardinality $\aleph_1$ are examples of simple groups with no maximal subgroups.

Some remarks

Remark

An uncountable group is called a Jónsson group if it is of cardinality α, and all its proper subgroups have cardinality strictly less than α.

Examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by Shelah and Obraztov.

- S. Shelah, On a problem of Kurosh, Jónsson groups, and applications, in *Word Problems II-The Oxford Book*, North-Holland, Amsterdam, 1980, 373-394.
- V. N. Obraztov, An embedding theorem for groups and its corollaries, *Math. USSR-Sb.* **66** (1990), 541-553.

Groups with all subgroups normal - Dedekind groups

Theorem (Dedekind, Baer)

Let G be a group with all subgroups normal.

If G is not periodic, then G is abelian.

If G is periodic,
then either G is abelian,
or $G = Q_8 \times A \times D$,
where A is an elementary abelian 2-group and D is an abelian group with all elements of odd order.

R. Dedekind, Über Gruppen, deren sämtliche Teiler Normalteiler sind, *Math. Ann.***48** (1897), 548-561.

R. Baer, Nilpotent groups and their generalizations, *Trans. Amer. Math. Soc.* **47** (1940), 393-434.

Two famous theorems of B.H. Neumann

Theorem (B.H. Neumann, 1955)

Let G be a group.

H is nearly normal, $\forall H \leq G \iff G'$ is finite.

Theorem (B.H. Neumann, 1955)

Let G be a group.

H is almost normal, $\forall H \leq G \iff G/Z(G)$ is finite.

B.H. Neumann, Groups with finite classes of conjugate subgroups, *Math. Z.* **63** (1955), 76-96.

Groups with all subgroups normal-by-finite

Remark

Tarski groups (infinite simple p-groups, in which every proper subgroup has order p, p a suitable prime) are examples of groups G

such that H is normal-by-finite, $\forall H \leq G$.

Theorem (Buckley, Lennox, Neumann, Smith, Wiegold 1995)

Let G be a locally finite group.

If H is normal-by-finite, $\forall H \leq G$, then G is abelian-by-finite.

J.T. Buckley, J.C. Lennox, B.H. Neumann, H. Smith, J. Wiegold, Groups with all subgroups normal-by-finite , *J. Austral. Math. Soc.* **59** (1995), 384-398.

Groups with all subgroups normal-by-finite

Theorem (Smith, Wiegold, 1996)

Let G be a locally graded group.

If there exists an integer k such that $|H/H_G| \leq k$, $\forall H \leq G$, then G is abelian-by-finite.

H. Smith, J. Wiegold, Locally graded groups with all subgroups normal-by-finite , *J. Austral. Math. Soc.* **60** (1996), 222-227.

Groups with all subgroups near normal

Remark

Let G be a group.

If all subgroups of G are near normal in G, then every proper subgroup of G is a Dedekind group.

Minimal non-abelian groups

Theorem (Miller, Moreno, 1903)

Let G be a finite group with all proper subgroups abelian.

Then either G is abelian or G is minimal non-abelian, of one of the following types:

- $G = V_q \rtimes C_{r^s}$, where q, r are different primes, s a positive integer, and V_q is an irreducible C_{r^s}-module over the field with q elements with kernel the maximal subgroup of C_{r^s},
- Q_8,
- $G = \langle a, b \mid a^{q^m} = b^{q^n} = 1, a^b = a^{q^{m-1}} \rangle$, where q is a prime number, $m \geq 2$, $n \geq 1$, of order q^{m+n}, and
- $G = \langle a, b \mid a^{q^m} = b^{q^n} = [a, b]^q = [a, b, a] = [a, b, b] = 1 \rangle$, where q is a prime number, $m \geq n \geq 1$, of order q^{m+n+1}.

Minimal non-Dedekind groups

Theorem (Miller, 1907)

Let G be a finite group whose proper subgroups are Dedekind .

Then either G is a Dedekind group, or G is minimal non-abelian, or G is of one of the following types:

- Q_{16}, the generalized quaternion group of order 16,
- $G = Q_8 \rtimes \langle x \rangle$, where $\langle x \rangle$ is of order 3^m, $m > 0$, that induces an automorphism permuting cyclically the three maximal subgroups of Q_8.

- G.A. Miller, H.C. Moreno Non-abelian Groups in Which Every Subgroup is Abelian, *Trans. Amer. Math. Soc.* **4** (1903), 398-404.
- G.A. Miller On Groups in Which Every Subgroup is either Abelian or Hamiltonian, *Trans. Amer. Math. Soc.* **8** (1907), 25-29.
- A. Ballester-Bolinches, R. Esteban-Romero, Minimal non-supersoluble groups, *Rev. Mat. Ibroamericana* **23** (2007), 127-142.

Groups with many near normal subgroups

Proposition 1

Let G be a finite group. All subgroups of G are near normal in G if and only if one of the following holds:

- G is a Dedekind group,
- G is minimal non-abelian,
- $G = Q_{16}$, the generalized quaternion group of order 16,
- $G = Q_8 \rtimes \langle x \rangle$, where x is an element of order 3^α, that induces an automorphism of order 3 permuting cyclically the three maximal subgroups of Q_8.

Groups with many near normal subgroups

Remark

Tarski groups (infinite simple p-groups, in which every proper subgroup has order p, p a suitable prime number) are examples of groups G

such that H is near normal, $\forall H < G$.

Proposition 2

Let G be a locally graded group.
If all subgroups of G are near normal,
then either G is finite or G is a Dedekind group .

Proof. The class of Dedekind groups is an *accessible class*, i.e. every locally graded minimal-non-Dedekind group is finite.

F. de Giovanni, M. Trombetti Cohopfian Groups And Accessible Group Classes, *Pac. J. Math.***312** (2021), 457-475.

F. de Giovanni, M. Trombetti A constructive approach to accessible group classes, *Ann. Mat. Pura Appl.*.**201** (2022), 985-1003.

Groups with many near normal subgroups

Proposition 3

Let G be a an infinite non-abelian group, all of whose subgroups are near normal in G. Then:

- $G/Z(G)$ is a 2-generated infinite simple group, with no elements of order 2,
- G is minimal non-abelian.

Outline of the talk

- Groups with many near normal (near subnormal) subgroups
  - Groups with many near pronormal subgroups
- Groups in which the property of being near normal (near pronormal) is transitive
  - Groups in which near normal subgroups form a sublattice of the lattice of all subgroups

Groups in which all Sylow p-subgroups are near normal

Theorem A (Kurdachenko, Longobardi,-, 2005)

Let G be a finite soluble group.
All Sylow p-subgroups of G are near normal in G if and only if
either G is nilpotent
or $G = P \rtimes Q$, where P is a Sylow p-subgroup, $Q = \langle w \rangle$ is a cyclic Sylow q-subgroup of G, p, q different primes, $w^q \in Z(G)$, Q centralizes $\Phi(P)$ and acts irreducibly on $P/\Phi(P)$.

Groups in which all Sylow p-subgroups are near subnormal

Theorem B (Kurdachenko, Longobardi, -, 2025)

Let G be a finite non-soluble group.
All Sylow p-subgroups of G are near normal in G if and only if there exists an integer $m \geq 0$ such that
$G/Z_m(G) \simeq PSL(2,17)$, with $Z_m(G) \leq \Phi(G)$,
(where $Z_m(G)$ is the m-th center of G).

Proof.

Theorem B

Let G be a finite non-soluble group. All Sylow p-subgroups of G are near subnormal in G if and only if there exists an integer $m \geq 0$ such that $G/Z_m(G) \simeq PSL(2,17)$, with $Z_m(G) \leq \Phi(G)$, (where $Z_m(G)$ is the m-th center of G).

Proof.

- Assume $G/Z_m(G) \simeq PSL(2,17)$.
- The order of $PSL(2,17)$ is $17 \cdot 3^2 \cdot 2^2$. A Sylow 2-subgroup is maximal, while a Sylow 3-subgroup is contained in a unique maximal subgroup which is its normalizer. The same is true for a Sylow 17-subgroup.
- Hence every Sylow p-subgroup of $PSL(2,17)$ is near normal in $PSL(2,17)$.

Proof.

- Let P be a Sylow p-subgroup of G.
- $PZ_m(G)/Z_m(G)$ is a Sylow p-subgroup of $G/Z_m(G) \simeq PSL(2,17)$, thus it is near normal in $G/Z_m(G)$,
- $PZ_m(G)$ is nilpotent, then P is subnormal in $PZ_m(G)$,
- Every maximal subgroup M of G containing P also contains $PZ_m(G)$, which is normal in M, thus P is subnormal in M, as required.

Conversely, assume that G is a non-soluble group with all Sylow p-subgroups near normal in G.

- If G is simple, then, using the results of Baumeister, Burness, Guralnick, Tong-Viet and of Guralnick, Tong-Viet and Tracey, it is possible to prove that $G \simeq PSL(2,17)$.

Proof.

- Now assume G non simple.
- If $G' < G$, then there exists a maximal subgroup V of G normal in G of index a prime number q. Every Sylow p-subgroup of G with p a prime different from q is contained in V; then it is normal in V, hence in G, thus it is contained in the Fitting subgroup F of G. Then G/F is a q-group and G is soluble, a contradiction.
- Therefore $G = G'$. In particular $Z_i(G) \leq \Phi(G)$, for every positine integer i.
- Let N be a minimal normal subgroup of G. We show that N is soluble.
- In fact, let L be a maximal subgroup of G containing a Sylow 2-subgroup D of G. Then D is normal in L and L is soluble by Feit-Thompson's theorem. If $N \not\leq L$, then $G = LN$ and G/N is soluble, a contradiction.

Proof.

- Write R the soluble radical of G. Then G/R again has all Sylow subgroups near normal, therefore it is a simple group by the previous remark, and then $G/R \simeq PSL(2, 17)$.
- We show that $R \leq Z_m(G)$, for a suitable integer m.
- By induction, it suffices to show that a minimal normal subgroup N of G is contained in Z(G).
- For, let N be a minimal normal subgroup of G. Write M_i a maximal subgroup of G containing P_i, where P_i is a Sylow p_i-subgroup of G, $p_i \in \{2, 3, 17\}$.
- Then P_i is normal in M_i, then, by Burnside's theorem M_i is soluble. Hence $N \leq M_i$. Then N is an elementary abelian p-group, where p is a prime.
- Let $p_j \in \{2, 3, 17\}$, $p_j \neq p$. Then we have $N \leq C_{M_j}(P_j)$, thus $P_j \leq C_G(N)$ which is normal in G.
- If $C_G(N) < G$, then P_j is normal in $C_G(N)$ and then it is a normal in G, a contradiction.
- Thus $C_G(N) = G$ and $N \leq Z(G)$, as required.

Outline of the talk

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Pronormal subgroups

Problem (19.74 of Kourovka notebook, 2025)

Does there exist an infinite group G
that does not contain nontrivial proper pronormal subgroups?

E.I. Khukhro, V.D. Mazurov, *Unsolved Problems in Group Theory. The Kourovka Notebook*, **34th edition**, (2025).

Groups with all subgroups pronormal

Remark

Let G be a group, H a subgroup of G

If H is pronormal and subnormal in G, then H is normal in G.

If H is pronormal in G, $\forall H \leq G$, then G is a $\bar{T}$-group.

Definitions

Let G be a group.

G is said to be a T-**group** if every subnormal subgroup of G is normal.

G is said to be a $\bar{T}$-**group** if every subgroup of G is a T-group.

Groups with all subgroups pronormal

Remarks

- A finite soluble T-group is a $\bar{T}$-group.
- A soluble T-group is metabelian.
- A finitely generated soluble T-group is either finite or abelian.

W. Gaschütz, Gruppen, in denen das Normalteilersein transitiv ist, *J. Reine Angew. Math.*, **198**, (1957), 87-92.

D.J.S. Robinson, Groups in which normality is a transitive relation, *Proc. Cambridge Philos. Soc.*, **60**, (1964), 21-38.

Groups with all subgroups pronormal

Theorem (Peng, 1969)

Let G be a finite group.

H is pronormal in G, $\forall H \leq G$ if and only if
G is a soluble $\bar{T}$-group in G.

T.A. Peng, Finite groups with pro-normal subgroups, *Proc. Amer. Math. Soc.* **20** (1969), 232-234.

Groups with all subgroups pronormal

Theorem (de Giovanni, Vincenzi, 2000)

Let G be an FC-group.

H is pronormal in G, $\forall H \leq G$ if and only if G is a soluble $\bar{T}$-group.

F. de Giovanni, G. Vincenzi, Pronormality in infinite groups, *Math. Proc. R. Irish Acad..* **100A** (2000), 189-203.

Groups with all subgroups pronormal

Theorem (de Giovanni, Trombetti, 2020)

Let G be a linear group.

H is pronormal in G, $\forall H \leq G$ if and only if
G is a soluble $\bar{T}$-group in G.

Theorem (de Giovanni, Trombetti, 2020)

Let G be a locally graded group .

If H is pronormal in G, $\forall H \leq G$, then G is metabelian.

F. de Giovanni, M. Trombetti, Pronormality in group theory, *Adv. in Group Theory Appl.*, **9**, (2020), 123-149.

Groups with all subgroups pronormal

Theorem (Kuzennyi, Subbotin, 1987)

Let G be a periodic locally graded group.

H is pronormal in G, $\forall H \leq G$ if and only if

$G = A \rtimes B$, where

- B is a Dedekind group, A an abelian group with every subgroup normal in G,
- $\pi(A) \cap \pi(B) = \emptyset$ and $2 \notin \pi(A)$,
- $G' = A' \times B'$,
- every Sylow $\pi(B)$-subgroup of G is a a complement of A.

Groups with all subgroups pronormal

Theorem

Let G be a locally graded non-periodic group.

If H is pronormal in G, $\forall H \leq G$, then
G is abelian.

N.F. Kuzennyi, I.Yu Subbotin, Groups in which all the subgroups are pronormal, *Ukrainian Math. J.*, **39** (1987), 251-254.

Groups with all subgroups almost pronormal

Theorem (de Giovanni, Russo, Vincenzi, 2008)

Let G be a group. Suppose that G has an ascending series whose factors are finite or locally nilpotent.

If H is almost pronormal, $\forall H \leq G$
Then G is metabelian-by-finite.

Theorem (de Giovanni, Russo, Vincenzi, 2008)

Let G be a finitely generated soluble-by-finite group.
If H is almost pronormal, $\forall H \leq G$, then $G/Z(G)$ is finite.

Groups with all subgroups pronormal by-finite

Theorem (de Giovanni, Russo, Vincenzi, 2007)

Let G be a finitely generated soluble group, with no infinite dihedral sections.

If H is pronormal-by-finite, $\forall H \leq G$, then $G/Z(G)$ is finite.

- F. de Giovanni, A. Russo, G. Vincenzi, Groups with all subgroups pronormal-by-finite, *Mediterranean J. Math.*, **4** (2007), 65-71.
- F. de Giovanni, A. Russo, G. Vincenzi, Groups in which every subgroup is almost pronormal, *Note di Mat.*, **1** (2008), 95-103

Groups with all subgroups near pronormal.

Remarks

Let G be a group.

If all subgroups of G are near pronormal in G, then every proper subgroup of G is a $\bar{T}$-group.

Let G be a finite group.

All subgroups of G are near pronormal in G if and only if either G is a $\bar{T}$-group or G is a minimal non-T-group.

Minimal non-T-groups

Theorem (Robinson, 1969)

Let G be a finite group with all proper subgroups T-groups. Then either G is a $\bar{T}$-group, or G is one of seven types:

- $G = Q_{16}$, the generalized quaternion group of order 16,
- $G = \langle a, b \mid a^{q^m} = b^{q^n} = 1, a^b = a^{q^{m-1}} \rangle$, where q is a prime number, $m \geq 2$, $n \geq 1$, of order q^{m+n},
- $G = \langle a, b \mid a^{q^m} = b^{q^n} = [a, b]^q = [a, b, a], [a, b, b] = 1 \rangle$, where q is a prime number, $m \geq n \geq 1$, of order q^{m+n+1},
- $G = Q_8 \rtimes \langle x \rangle$, where x is an element of order 3^m, that induces an automorphism permuting cyclically the three maximal subgroups of Q_8,

Minimal non-T-groups

- $G = P \rtimes Q$, where $P = \langle a \rangle \times \langle b \rangle$ is an elementary abelian p-group of order p^2, $Q = \langle x \rangle$ is cyclic of order $q^s > 1$, p, q different primes, $p \equiv 1 \ (mod \ q^f)$, $f > 0$, $f \leq s$, and $a^x = a^\xi$, $b^x = b^\eta$, where ξ is a q^f-primitive root of the unity, and $\eta = \xi^{1+kq^{f-1}}$, $0 < k < q$,
- $G = P \rtimes Q$, where P is an elementary abelian p-group, $Q = \langle x \rangle$ is cyclic of order $q^s > 1$, p, q different primes, $p \not\equiv 1 \ (mod \ q)$, and P is an irreducible Q-module over the field with p elements, with centralizer $\langle x^q \rangle$ in Q,
- $G = P \rtimes Q$, where $P = \langle a_0, a_1, \cdots, a_{q-1} \rangle$ is an elementary abelian p-group of order p^q, $Q = \langle x \rangle$ is cyclic of order $q^s > 1$, p, q different primes, $p \equiv 1 \ (mod \ q^f)$, $f < s$, and $a_j^x = a_{j+1}$ for $0 \leq j < q-1$, $a_{q-1}^x = a_0^\xi$, where ξ is a q^f-primitive root of the unity.

D.J.S. Robinson, Groups which are minimal with respect to normality being intransitive , *Pacific J. Math.* **31** (1969), 777-785.

Groups with all near pronormal subgroups

Proposition 6

Let G be a finite group. All subgroups of G are near pronormal if and only if one of the following holds:

- G is a $\bar{T}$-group,
- G is minimal non-T-group, as in the previous theorem.

Groups with all near pronormal subgroups

Proposition 7

Let G be a locally graded group. If all subgroups of G are near pronormal,
then either G is finite or every subgroup of G is pronormal .

Proof. The class of groups with all pronormal subgroups is an accessible class.

F. de Giovanni, M. Trombetti, Pronormality in group theory, *Adv. in Group Theory Appl.*, **9** (2020), 123-149.

Groups with many subgroups with a property generalizing normality

Groups with **many** subgroups with a property generalizing normality have been (and are) studied by **many authors**:

M. Arshaduzzaman, A. Ballester-Bolinches, M. Brescia, C. Casolo, G. Cutolo, F. de Giovanni, M. De Falco, F. De Mari, E. Detomi, M.R. Dixon, R. Esteban-Romero, M. Evans, M. Ferrara, V.E. Goretskii, H. Heineken, L.A. Kurdachenko, M. Mainardis, C. Musella, J. Otal, T. Pedraza , M.C. Pedraza-Aguilera , V. Pérez-Calabuig, A.A. Pypka, A. Russo, H. Smith, M. Tota, A. Tortora, I.Y. Subbotin, M. Trombetti, G. Vincenzi, M. Viscosi, B.A.F. Wehrfritz, ...

Outline of the talk

- Groups with many near normal (near subnormal) subgroups
- Groups with many near pronormal subgroups
- Groups in which the property of being near normal (near pronormal) is transitive
- Groups in which near normal subgroups form a sublattice of the lattice of all subgroups

More Francesco's papers

- F. de Giovanni, S. Franciosi, Groups in which every infinite subnormal subgroup is normal, *J. Algebra*, **96** (1985), 566-580.
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- M. Brescia, F. de Giovanni, Groups Satisfying the Double Chain Condition on Subnormal non Normal Subgroups, *Adv. Group Theory Appl.*, **13** (2022), 83-102.
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- M. Brescia, F. de Giovanni, Groups Satisfying the Double Chain Condition on Subnormal non Normal Subgroups, *Adv. Group Theory Appl.*, **13** (2022), 83-102.

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Groups in which pronormality is transitive

Definitions

A group G is said to be **a transitive pronormal group**
(G is a TP-group)
if from H pronormal subgroup of K,
and K pronormal subgroup of G
it follows H pronormal in G.

A group G is said to be a $\overline{TP}$-**group**
if every subgroup of G is transitive pronormal.

Groups in which pronormality is transitive

Theorem (Kurdachenko, Subbotin, 2002)

Locally soluble $\overline{TP}$-groups,

are exactly

groups in which every subgroup is pronormal.

Groups in which near normality is transitive

Definitions

A group G is said to be **a transitive near normal group (G is a TNN-group)** if from H near normal subgroup of K, and K near normal subgroup of G, it follows H near normal in G.

A group G is said to be a $\overline{TNN}$**-group** if every subgroup of G is transitive near normal.

Groups in which near pronormality is transitive

Definitions

A group G is said to be **a transitive near pronormal group** (**G is a TNP-group**) if

from H near pronormal subgroup of K,
and K near pronormal subgroup of G,
it follows H near pronormal in G.

A group G is said to be a $\overline{TNP}$-**group**
if every subgroup of G is transitive near pronormal.

Groups in which near normality is transitive

Remark

If a finite group G is a TNN-group,
then every subgroup of G is near normal in G.

Lemma

If G is an infinite finitely generated locally graded TNN-group,
then G/G' is infinite.

Groups in which near normality is transitive

Proposition 8

Let G be an infinite finitely generated locally graded TNN-group.
Then G is a T-group, and every near normal subgroup of G is normal in G.

Corollary

Let G be an infinite locally graded $\overline{TNN}$-group.
Then G is a $\overline{T}$-group.

Groups in which the property to be near normal is transitive

Theorem C

Let G be a locally soluble $\overline{TNN}$-group.
Then either G is finite or every subgroup of G is normal in G.

Proposition 9

Let G be a periodic locally graded $\overline{TNN}$-group.
Then either G is finite or G is a Dedekind-group.

Groups in which the property to be near normal is transitive

Problem (de Giovanni, 14.36 of Kourovka notebook, 2025)

Is it true that

every non-periodic locally graded $\overline{T}$-group must be abelian?

E.I. Khukhro, V.D. Mazurov, *Unsolved Problems in Group Theory. The Kourovka Notebook*, **34th edition**, (2025).

Groups in which near pronormality is transitive

Remark

Let G be a finite TNP-group.
Then every subgroup of G is near pronormal in G.

Proposition 9

Let G be a finitely generated infinite locally graded TNP-group.
Then G is a T-group.

Groups in which near pronormality is transitive

Theorem D (Kurdachenko, Longobardi, -, 2025)

Let G be a locally soluble $\overline{TNP}$-group. Then either G is finite or every subgroup of G is pronormal in G.

Proof.

- Suppose G infinite. It suffices to show that G is a TP-group. Assume X pronormal in K and K pronormal in G.
- Then X is near pronormal in G. Suppose X non pronormal in G.
- Then there exists $g \in G$ such that X and X^g are not conjugate in $\langle X, X^g \rangle$.
- May assume $G = \langle X, g \rangle$.

Groups in which near pronormality is transitive

- G is a $\bar{T}$-group, therefore G is metabelian and periodic.
- Without loss of generality we can suppose $X_G = \{1\}$.
- Every subgroup of G' is normal in G, hence $X \cap G' = \{1\}$.
- Then X is abelian, and $X \cap \langle g \rangle = \{1\}$.
- Write $L = [G', G]$. Then G/L is a Dedekind group, every subgroup of L is normal in G and $\pi(L) \cap \pi(G/L) = \emptyset$.
- Moreover $C_X(L) = \{1\}$, in fact $C_X(L)$ is normal in XL which is normal in G and G is a T-group.
- Then X is isomorphic to a group of power automorphisms on L.

Groups in which near pronormality is transitive

- Then X induces a finite group of automorphisms in each p-component of L, thus it is residually finite.
- As $G = \langle X, g \rangle$ is infinite, then X is infinite.
- Since $L \cap \langle g \rangle = \{1\}$ and G is periodic, then the intersection $X_0 = L\langle g \rangle \cap X$ is finite.
- Then there exists a normal subgroup Y of X such that $|X/Y| > |X_0|$.
- Consider the subgroup $H = YL\langle g \rangle$.
- If $H = G$, then we have $X = YX_0$, and $|X/Y| < |X_0|$, which is not true.
- Therefore $H < G$, and Y is pronormal in H since it is normal in the near pronormal subgroup X.
- Then there exists $z \in \langle Y, Y^g \rangle$ such that $Y^z = Y^g$.

- Then $v = gz^{-1} \in N_G(Y)$.
- If $K = \langle X, v\rangle = G$, then Y is normal in G, a contradiction.
- Thus $K < G$, hence X is pronormal in K.
- Then there exists $w \in \langle X, X^v\rangle$ such that $X^v = X^w$.
- Thus $X^{wz} = X^g$.
- But $z \in \langle Y, Y^g\rangle \leq \langle X, X^g\rangle$, $w \in \langle X, X^v\rangle \leq \langle X, X^g\rangle$.
- Then $wz \in \langle X, X^g\rangle$, a contradiction.

Outline of the talk

- Groups with many near normal (near subnormal) subgroups
  - Groups with many near pronormal subgroups
- Groups in which the property of being near normal (near pronormal) is transitive
  - Groups in which near normal subgroups form a sublattice of the lattice of all subgroups

Francesco's papers on lattices

- F. de Giovanni, Alcuni omomorfismi tra reticoli di sottogruppi, *Rend. Accad. Sci. Fis. Mat. Napoli* **45** (1978), 437-441.
- F. de Giovanni, S. Franciosi, Alcuni epimorfismi tra reticoli di sottogruppi e reticoli di sottogruppi normali, *Rend. Mat.* (4)**13** (1980), 531-540.
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The poset of near normal subgroups

Let G be a group.

Notation

Write $nn(G)$ the poset of all

near normal subgroups of G,

ordered by inclusion.

Remark

Obviously:

if $H, K \in nn(G)$, then $\langle H, K \rangle \in nn(G)$.

The poset of near normal subgroups

Let G be a group.

Problem

When $nn(G)$ is a **sublattice**
of the lattice $\mathcal{L}(G)$ of all subgroups of G ?

The poset of near normal subgroups of G

Proposition 10

Let G be a **finite nilpotent group**.

Then $nn(G)$ is a **sublattice** of $\mathcal{L}(G)$ if and only if either

- $nn(G) = \mathcal{N}(G)$, the lattice of all normal subgroups or
- G is a p-group, p a prime,
  $|G/\Phi(G)| = p^2$, $H \cap \Phi(G) \trianglelefteq G$, $\forall H \in nn(G)$.

The poset of near normal subgroups of G

Theorem D (Kurdachenko, Longobardi, -, 2025)

Let G be **a finite group**.

If $nn(G)$ is a **sublattice** of $\mathcal{L}(G)$, then either

- G is nilpotent, or
- $G = P \rtimes Q$, where P is a p-Sylow subgroup of G, Q is a q-Sylow subgroup of G, p, q different primes, $Q = \langle x \rangle$ is cyclic, and $x^q \in Z(G)$, $P/\Phi(P)$ is minimal normal in $G/\Phi(P)$.

The poset of near normal subgroups of G

Theorem F (Kurdachenko, Longobardi, -, 2025)

Let G be a **finitely generated infinite soluble group**.

Then $nn(G)$ is a **sublattice** of $\mathcal{L}(G)$

if and only if

$nn(G) = \mathcal{N}(G)$.

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Thank you for the attention !