Finite and locally finite groups containing an element with small centralizer

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Small centralizers

One of main avenues in group theory:

small centralizer $C_G(g) \Rightarrow$ structure of G.

Brauer–Fowler (1955): if G is finite and $g \in G$ is an involution, then G has a soluble subgroup of index bounded in terms of $|C_G(g)|$.

Using the classification of finite simple groups, Hartley (1992) generalized Brauer–Fowler theorem for any order of g:

G has a soluble subgroup of index bounded in terms of $|C_G(g)|$.

These results can be stated in terms of automorphisms $\varphi \in \operatorname{Aut} G$:

structure of G in terms of φ and the fixed-point subgroup $C_G(\varphi)$.

Coprime automorphisms: solubility

There are especially nice results for finite groups

with coprime automorphisms $\varphi \in \operatorname{Aut} G$, when $(|\varphi|, |G|) = 1$.

Thompson (1959): if $|\varphi|$ is a prime and $C_G(\varphi) = 1$, the G is nilpotent.

Rowley (1995) using CFSG: $C_G(A) = 1$ and $(|G|, |A|) = 1 \Rightarrow G$ is soluble;
$C_G(\varphi) = 1 \Rightarrow G$ is soluble for any $|\varphi|$.

Fitting height

Recall that the Fitting subgroup $F(G)$ of a finite group G is the largest normal nilpotent subgroup of G.

The Fitting series of G is defined as $F_1(G) = F(G)$,

and by induction $F_{i+1}(G)$ is the inverse image of $F(G/F_i(G))$.

If G is soluble, then the least number h such that $F_h(G) = G$

is called the Fitting height $h(G)$ of G.

Coprime automorphisms: Fitting height

Once G is soluble, next task is bounding the Fitting height.

Let G be a soluble finite group and $A \leqslant \mathrm{Aut}\, G$ soluble with $(|G|, |A|) = 1$.

Thompson (1964): the Fitting height $h(G)$ is bounded in terms of $h(C_G(A))$ and the composition length $\alpha(A)$, namely, $h(G) \leqslant 5^{\alpha(A)} \cdot h(C_G(A))$.

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Bounds in Thompson's theorem improved in numerous papers:

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Turull (1984): $h(G) \leqslant 2\alpha(A) + h(C_G(A))$;

Hartley–Isaacs (1990): subgroup H of index $\leqslant f(|A|, |C_G(A)|)$ with $h(H) \leqslant 2\alpha(A) + 1$.

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Hartley–Isaacs (1990): subgroup H of index $\leqslant f(|A|, |C_G(A)|)$ with $h(H) \leqslant 2\alpha(A) + 1$.

A rank analogue of the Hartley–Isaacs theorem was proved by Mazurov–Khukhro (2006): there is $H \leqslant G$ with $|G : H|$ bounded in terms of $|A|$ and the (Prüfer) rank of $C_G(A)$ (instead of $|C_G(A)|$) and with $h(H) \leqslant 5(4^{\alpha(A)} - 1)/3$.

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Difficult problem of improving Dade's bound to a linear one, even for cyclic $A = \langle \varphi \rangle$. (Some special cases tackled by Ercan and Güloğlu.)

So far the best general result is the quadratic bound $7\alpha(\langle \varphi \rangle)^2$ obtained by Jabara (2017).

Statement of the results

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Because the automorphism in the Theorem is not assumed to be coprime, the result also applies to any element with given order of centralizer.

Corollary 1

If a finite group G contains an element g with centralizer of order $m = |C_G(g)|$, then G has a soluble subgroup whose index and Fitting height are m-bounded.

Application to locally finite groups

Recall: Corollary 1

If a finite group G contains an element g with centralizer of order $m = |C_G(g)|$, then G has a soluble subgroup whose index and Fitting height are m-bounded.

Hence we have an affirmative answer to Hartley's problem of 1995 (Problem 13.8(a) in the "Kourovka Notebook" recorded by Belyaev).

Corollary 2

If a locally finite group G has an element g with centralizer of order $m = |C_G(g)|$, then G has a locally soluble subgroup of finite m-bounded index which has a normal series of finite m-bounded length with locally nilpotent factors.

Obtained by inverse limit argument applying Corollary 1 to every finite subgroup of G containing g.

Tools in the proof

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The reduction to the soluble case is by Hartley's generalized Brauer–Fowler theorem based on the classification of finite simple groups.

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The proof is by induction on $|\varphi|$ and $|C_G(\varphi)|$, using the theorem of Busetto–Jabara (2016): if $G = UV = VW = UW$ is a finite soluble group for three different Hall subgroups U, V, W, then

$$h(G) \leqslant h(U) + h(V) + h(W) - 2.$$

(Jabara's quadratic bound for $h(G)$ when $C_G(\varphi) = 1$ used this result.)

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For $|\varphi|$ divisible by at least **three** primes, the idea is to use the Busetto–Jabara factorization result. But for non-coprime automorphism φ there may not be φ-invariant Sylow or Hall subgroups.

(In Jabara's paper, although for non-coprime, the existence of φ-invariant Hall subgroups was guaranteed by $C_G(\varphi) = 1$.)

A crucial step in the proof of the Theorem is a reduction to the situation where φ-invariant Hall subgroups do exist.

Centralizers and commutators

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Proof: We have $|G : C_G(\varphi)| = |\{[g,\varphi] \mid g \in G\}|$.

Clearly, $[\bar{x},\varphi]$ is the image of $[x,\varphi]$ under $G \to G/N$.

Hence, $|\{[g,\varphi] \mid g \in G\}| \leqslant |N| \cdot |\{[\bar{g},\varphi] \mid g \in G/N\}|$, whence

$\frac{|G|}{|C_G(\varphi)|} \leqslant |N| \cdot \frac{|G/N|}{|C_{G/N}(\varphi)|} = \frac{|G|}{|C_{G/N}(\varphi)|}$, so that $|C_G(\varphi)| \geqslant |C_{G/N}(\varphi)|$.

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$\frac{|G|}{|C_G(\varphi)|} \leqslant |N| \cdot \frac{|G/N|}{|C_{G/N}(\varphi)|} = \frac{|G|}{|C_{G/N}(\varphi)|}$, so that $|C_G(\varphi)| \geqslant |C_{G/N}(\varphi)|$.

For $|C_G(\varphi)| = |C_{G/N}(\varphi)|$, we must have "$=$" instead of "$\leqslant$" throughout. Then the full inverse image of every $[\bar{x},\varphi]$ consists only of elements $[x',\varphi]$.

In particular, so does the inverse image of $1 = [1,\varphi]$, which is N. □

φ-Invariant Hall subgroups

Lemma 2

Suppose that N is a normal φ-invariant subgroup such that $|C_{G/N}(\varphi)| = |C_G(\varphi)|$. Then for every set of primes $\pi \subseteq \pi(N)$ the group N has a φ-invariant Hall π-subgroup.

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Proof: By Lemma 1, we have $N \subseteq \{g^{-1}g^{\varphi} \mid g \in G\}$.

Let S be a Hall π-subgroup of N.

If $S^{\varphi} \neq S$, then $S^{\varphi} = S^{a}$ for some $a \in N$.

There is $g \in G$ such that $a = g^{-1}g^{\varphi}$.

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If $S^{\varphi} \neq S$, then $S^{\varphi} = S^{a}$ for some $a \in N$.

There is $g \in G$ such that $a = g^{-1}g^{\varphi}$.

Then $S^{g^{-1}}$ is also a Hall π-subgroup of N, which is φ-invariant:

$$\left(S^{g^{-1}}\right)^{\varphi} = S^{a(g^{-1})^{\varphi}} = S^{g^{-1}g^{\varphi}(g^{-1})^{\varphi}} = S^{g^{-1}}. \qquad \square$$

Key lemma

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If $|\varphi|$ is divisible by at most two primes, the result follows by Hartley (unpublished 1994) and Khukhro (2015).

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By Lemma 2, N has φ-invariant Hall p'-, q'-, r'-subgroups U, V, W.

On the Hall p'-subgroup U, the Sylow p-subgroup $\langle\varphi_p\rangle$ induces a **coprime** automorphism. Its centralizer $C_U(\varphi_p)$ admits the cyclic automorphism group $\langle\varphi\rangle/\langle\varphi_p\rangle$ of smaller order, with centralizer equal to $C_U(\varphi)$. By induction, the Fitting height of $C_U(\varphi_p)$ is $(m, n-1)$-bounded.

Key lemma (continued)

Recall: φ_p is a **coprime** automorphism of the Hall p'-subgroup U, and we know that the Fitting height of $C_U(\varphi_p)$ is $(m, n-1)$-bounded.

Now by Thompson's theorem (or Turull's with better bound) the Fitting height of U is $(m, n-1)$-bounded. The same argument applies to the Hall subgroups V, W.

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Since $N = UV = UW = VW$, by the Busetto–Jabara theorem

$$h(N) \leqslant h(U) + h(V) + h(W) - 2. \qquad \square$$

Proof of the Theorem

Recall: G is a finite **soluble** group with an automorphism φ of order n having $m = |C_G(\varphi)|$ fixed points; we need show that the Fitting height of G is (m, n)-bounded.

Induction on $|C_G(\varphi)|$.

When $C_G(\varphi) = 1$, the result follows by Dade's theorem
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Let $g(m, n)$ be the function furnished by Lemma 3.

If $F_{g(m,n)+1}(G) = G$, then we are done.

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By induction, $h(G/F_{g(m,n)+1}(G)) \leqslant f(m-1, n)$. As a result,

$$h(G) \leqslant g(m, n) + 1 + f(m-1, n). \qquad \square$$

Open problems on bounding Fitting height

By analogy with the coprime theorem of Hartley and Isaacs (1990):

Conjecture 1

If a finite group G admits an automorphism φ of order n having $m = |C_G(\varphi)|$ fixed points, then G has a soluble subgroup of (m, n)-bounded index whose Fitting height is n-bounded, or even $\alpha(\langle\varphi\rangle)$-bounded.

So far this result is known (even in the 'strong' version with $\alpha(\langle\varphi\rangle)$-bounded Fitting height) for automorphisms of prime-power order due to Hartley and Turau (1987).

Open problems in terms of rank

By analogy with the coprime theorems of Mazurov and Khukhro (2006):

Conjecture 2

If a finite soluble group G admits an automorphism φ of order n such that the fixed point subgroup $C_G(\varphi)$ has (Prüfer) rank r, then the Fitting height of G is (n, r)-bounded,

or even there is a normal subgroup N such that the rank of G/N is (n, r)-bounded, while the Fitting height of N is n-bounded.

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The affirmative answer to even the 'weak' version of Conjecture 2 would provide an affirmative solution to Hartley's problem 13.8(b) in Kourovka Notebook about a locally soluble locally finite group containing an element with Chernikov centralizer. So far even this weak rank version is only known for $|\varphi| = p^k q^l$ due to Hartley (unpublished, 1994).

Open problems for nilpotent groups

Bounds for the Fitting height reduce further studies to the case of nilpotent groups. Perhaps the most important problem is to obtain a group-theoretic analogue of Kreknin's theorem on Lie algebras.

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If a (finite) nilpotent group G admits an automorphism φ of order n with $C_G(\varphi) = 1$, then the derived length of G is n-bounded.

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Definitive general results have also been obtained for almost fixed-point-free p-automorphisms of finite p-groups by Alperin (1961), Jaikin-Zapirain (2000), Khukhro (1985, 1993), Medvedev (1999), Shalev (1993).

Thank you!