On a Simultaneous Ping-Pong game

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Joint work with Doryan Temmerman and François Thilmany

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$\rightsquigarrow$ a group theoretical aspect of why repr.th. of finite groups works

A Belgian contribution

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Let $\Gamma \leq GL_n(F)$ finitely generated. Then one of following:

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$\rightsquigarrow$ yield many dichotomies (word growth, Day conjecture, ...)

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Some natural questions arise :

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Margulis-Soifer

$\exists$ free subgroup in $\Gamma \Leftrightarrow \exists$ max. subgroup of infinite index

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Uniform Tits Alternative (Breuillard, 2008)

If $\Gamma = \langle S \rangle \leq GL_n(F)$, then

$$\exists d(n) \in \mathbb{N} \text{ and } a, b \in S^{d(n)}\colon\ \langle a, b \rangle \cong F_2$$

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$$GL_n(F) = \{x \in F^{n^2} \mid \underbrace{det(x)}_{polynomial} \neq 0\}$$

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Yet another question: a fixed generator

Let S a *finite* subset of Γ. An element $\gamma \in \Gamma$ is *a simultaneous ping-pong partner* for S if for all $g \in S$:

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Conjecture (Bekka–Cowling–de la Harpe, 1994)

Let Γ be a Zariski-dense subgroup of a connected semisimple real Lie group $\mathbf{G}$ without compact factors.

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Example: $\mathbf{G} = \prod_{i=1}^{q} SL_{n_i}(\mathbb{R})$ and Γ finite index in $\prod_{i=1}^{q} SL_{n_i}(\mathbb{Z})$.

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Historical motivation: obtain simplicity of associated C^*-algebra

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Obstruction: 'Almost faithful embedding'

Let

- $\mathcal{S} := \{G \mid \text{no free subgroup in } G\}$
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$$\implies \exists\, i : \ker(\langle A, B\rangle \to G_i) \subseteq A \cap B.$$

Some positive results

Take $x \in S \subset \Gamma$ and Chevalley decompose:

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Known results

1. If $\mathbf{G}$ simple of real rank 1 (e.g. $SL_2(\mathbb{R})$) and all S (BCH, 1994)
2. If $\mathbf{G}$ simple not of type A_n, D_{2n+1}, E_6 and x_{ss} 'very proximal' (Poznansky, 2009)
3. If $\Gamma = SL_n(\mathbb{Z})$ and x_{ss} torsion or 'very proximal' (Soifer-Vishkautsan, 2010)

Our setting:

$$\mathbf{G} = \prod_i GL_{n_i}(D_i) \times \prod_j SL_{n_j}(D_j)$$

$$\Gamma = \text{ any Zariski-dense in } \mathbf{G}$$

with D_ℓ f.d. division algebra over any field F.

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Example: Γ commensurable with $\prod_i GL_{n_i}(\mathcal{O}_i) \times \prod_j SL_{n_j}(\mathcal{O}_j)$ where $\mathcal{O}_\ell$ is an order in D_ℓ.

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Spoiler

For any reductive group $\mathbf{G}$, Zariski-dense Γ and S as above we reduce the (modified) conjecture to 2 concrete, and of independent interest to verify, properties.

At dawn of group theory...

Ping-Pong lemma (Klein, Schottky $\sim$ 1880)

Let $G_1, G_2 \leq \Gamma$ s.t. $|G_1| > 2$ and

- Γ acts on the set P
- $P_1, P_2 \subset P$ distinct and non-empty
- $g(P_i) \subseteq P_j$ for every $1 \neq g \in G_i$ and $\{i, j\} = \{1, 2\}$

$$\Longrightarrow \langle G_1, G_2 \rangle \cong G_1 \star G_2.$$

Prototypical example

Let $\Gamma = \langle a, b \rangle$ with $a = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

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Consider

$$P_a = \{z \in \mathbb{C} \mid |z| < 1\} \text{ and } P_b = \{z \in \mathbb{C} \mid |z| > 1\}$$

$$\rightsquigarrow \varphi_a(P_a) \subseteq P_b \text{ and } \varphi_b(P_b) \subseteq P_a$$

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Remark: φ_a is 'propulsing' and φ_b is'attracting'

Doing projective dynamics since 1972

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and write $V = A(x) \oplus A^{co}(x)$.

$\rightsquigarrow$ x is *proximal* if $\dim A(x) = 1$ and *very proximal* if $x^{\pm 1}$ proximal.

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Proximal = contractive: if x is proximal, then

$$\forall p \in \mathbb{P}(V) \setminus A^{co}(x) : (x^n.p)_n \to A(x)$$

Some important remarks

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3. Definition proximal is for V a vector space, but generalised the classical theory to modules over division algebras

$\mathbf{G}$ a connected algebraic group, Γ a Zariski-dense subgroup of $\mathbf{G}$
$(H_i)_{i\in I}$ be a finite collection of finite subgroups of $\mathbf{G}$.

General reduction theorem (J.-Temmerman-Thilmany 2025)

If $\forall i \in I$, $\exists \rho_i : \mathbf{G} \to PGL(V_i)$, with V_i a f.d. D_i-module, s.t.

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$$\{\gamma \in \Gamma \mid \langle H_i, \gamma\rangle \cong H_i *_{C_i} (\langle\gamma\rangle \times C_i) \text{ for all } i\}$$

is dense in Γ for the join of the profinite topology and the Zariski topology.

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Remark: ρ_i with (1) always exists, but condition (2) is poorly understood !

Existence for product of SL and GL

$\mathbf{G} = \prod_i GL_{n_i}(D_i) \times \prod_j SL_{n_j}(D_j)$
$H_\ell \leq \mathbf{G}$ finite subgroups having an almost-embedding in a simple component GL_{n_ℓ} with $n_\ell \geq 2$ (!)

$$\rightsquigarrow \rho_{st} : GL_{n_\ell} \to PGL(D_{n_\ell}^{\times n_\ell}) : \text{ left multiplication}$$

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Remark: ρ_{st} satisfies a strong (2):

$$\text{span}\{\rho_{st}(xhx^{-1}), 1 \mid x \in \Gamma\} = M_{n_\ell}(D_{n_\ell})$$

When an algebra is hidden behind $\mathbf{G}$

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Theorem (Siegel, $\sim$ 1950)

Let $\mathcal{O}$ be an order in A (e.g $M_n(\mathbb{Z})$ in $M_n(\mathbb{Q})$). Then $\mathcal{U}(\mathcal{O})$ is a finitely presented group which is Zariski-dense in $\mathcal{U}(A)$.

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$\rightsquigarrow$ almost embedding in GL_n with $n \geq 2$ can be optimized in that case

Reunderstanding the $\prod_i GL_{n_i}$ case

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Theorem (JTT, 2025)

Let $\mathcal{O}$ an order in A, Γ Zariski-dense in $\mathcal{U}(\mathcal{O})$ and $H \leq \mathcal{U}(A)$. Then H has ping-pong partner $\gamma \in \Gamma$ iff H almost embeds in simple component of A which is

- neither a field
- nor a totally definite quaternion algebra

In that case, profinite densely many partners.

Reunderstanding the $\prod_i GL_{n_i}$ case

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What theorem doesn't tell:

(i) How construct γ?

(ii) When H has such almost-embedding?

Question (i) in too brief

Idea:

take another finite subgroup $K \leq \mathcal{U}(A)$ and change K by a conjugate K^{γ} such that H and K^{γ} are 'very proximal to each other'

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Conclusion

Via 1^{st} order deformations there exists a down to earth linear algebra method

Question (ii) in too brief

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Theorem (JTT, 2025)

If H has a faithful irreducible representation, then the required strong almost embedding exists.

Example: H has cyclic Sylow subgroups.

Thank you Francesco !!