Investigating some Products of Groups by means of the Tensor Product

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(joint work with Marco Trombetti)

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What do we mean by "Product of Groups"?

Let $(G, \cdot)$ be a group, and A and B its subgroups.

We define the product:

$$A \cdot B = \{x \cdot y \,|\, x \in A, y \in B\} \ .$$

Clearly we will omit the multiplication symbol $\cdot$ as usual.

Then if

$$G = AB$$

we will say that G is the Product of A and B.

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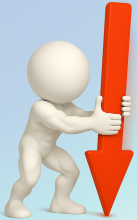

+ additional conditions

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Itô's theorem

In this regard, the first result was a theorem due to Itô, where $\mathfrak{X}$ is the property of being abelian.

Itô's theorem [1955]

Let G be a group such that:

$$G = AB$$

for some subgroups A and B.
If the factors A and B are abelian, then the product G is metabelian (the derived subgroup G' is abelian).

Usually, to transfer a property from the structure of the factors to that of the product, we need additional conditions.

Particular types of Products

So, we will consider two stronger definitions of product.

Let G be a group such that:

$$G = AB \ .$$

- If $A, B \trianglelefteq G$,
  then we'll say that G is the Normal Product of A and B.
- If $A, B \trianglelefteq G$ and $A \cap B = \{1\}$,
  then we'll say that G is the Direct Product of A and B.

From now on, we focus our attention on products with supersoluble factors.

Supersoluble group

A group H is said to be supersoluble iff there exists a finite series from $\{1\}$ to H:

$$\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_k = H$$

with every factor H_{i+1}/H_i cyclic, and every term $H_i \trianglelefteq H$.

Products of supersoluble groups

How does the supersolubility behave with respect to the Direct Product and the Normal Product of subgroups?

Exercise - Direct product of two supersoluble groups

Let $G = AB$ be a group that is the Direct Product of A and B. If the factors A and B are supersoluble, then the product G itself is supersoluble.

Theorem - Normal Product of two supersoluble groups [R. Baer; 1957]

Let $G = AB$ be a finite group that is the Normal Product of A and B. If the factors A and B are supersoluble, then the product G itself is supersoluble, **provided that G' is nilpotent**.

Given its foundational role, over the years numerous attempts have been made to weaken the hypotheses of this theorem.

Weakening product hypotheses

A way to weaken the hypotheses of Direct Products and Normal Products consists in dropping the normality condition on one factor in favor of a weaker permutability$^{(*)}$ condition.

In other words we could consider products of the type:

$$G = XN$$

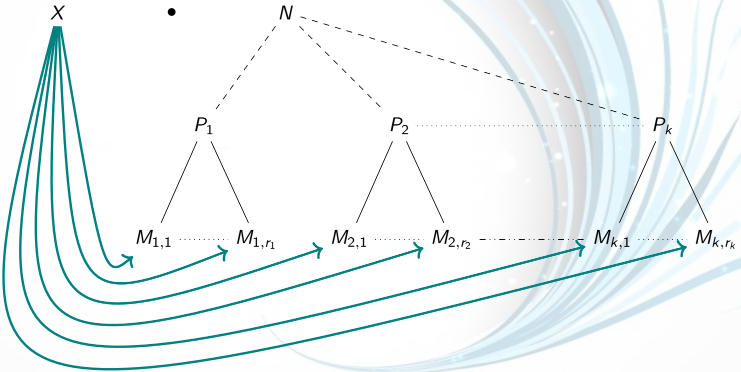
with N a normal subgroup of G and X a subgroup that permutes with *some* other subgroups of G (**...the fewer, the better**).

Then we can consider or not the hypothesis $X \cap N = \{1\}$.

$(*)$ Two subgroups A and B permute iff $AB = BA$;
a normal subgroup permutes with *all* the other subgroups.

A recent result

In recent times, Ballester-Bolinches, Madanha, Pedraza-Aguilera and Wu have studied products of this type where X permutes only with maximal subgroups of the Sylow subgroups of the normal factor N,
in the context of finite groups.



A recent result

They give the following definitions of products...

Weak Normal Product (WNP)

A Weak Normal Product (WNP) is a product of the type:

$$G = XN$$

where G is a finite group, $N \trianglelefteq G$ and X permutes with the maximal subgroups of the Sylow subgroups of N.

Weak Direct Product (WDP)

A Weak Direct Product is a Weak Normal Product with factors that have trivial intersection (in the previous notation $X \cap N = \{1\}$).
In this case, instead of the notation $G = XN$, we could also write:

$$G = X \ltimes N$$

(that is the notation of *Semidirect Product*).

A recent result

...and they were able to prove the following generalizations of the previously stated theorems.

Theorem - WDP and WNP of two supersoluble groups [Ballester-Bolinches, Madahna, Pedraza-Aguilera, Wu; 2023]

(I) Let $G = XN$ be a Weak Direct Product with X and N supersoluble factors. Then the product G itself is supersoluble.

(II) Let $G = XN$ be a Weak Normal Product with X and N supersoluble factors. If G' is nilpotent, then the product G itself is supersoluble.

Therefore, the theorems we have seen earlier about direct and normal products still hold in the case of these types of weak products.

Ballester-Bolinches, A., Madanha, S.Y., Pedraza-Aguilera, M.C., and Wu, X.
"On Some Products of Finite Groups."
Proceedings of the Edinburgh Mathematical Society, 66(1), 89–99 (2023).

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  - from WNP to VWNP (Very Weak Normal Product);
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In the last year with a change of perspective on this problem we were able...

- ...to prove these results within a more versatile framework.
- ...to further weaken the hypotheses, involving in the permutability condition only the **non-cyclic** Sylow subgroups; so we made a slight change to our terminology:
  - from WNP to VWNP (Very Weak Normal Product);
  - from WDP to VWDP (Very Weak Direct Product).
- ...to extend *all* these results to infinite groups, in a certain sense. Furthermore, in the context of periodic linear groups, it was possible to replace the property of supersolubility with its best studied generalizations in infinite groups (paranilpotency, hypercyclicity, local supersolubility).

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N supersoluble

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with the factors N_{i+1}/N_i cyclic, and terms N_i normal **in N**.

N G-supersoluble

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The change of perspective arose from the understanding that under the assumptions of a VWDP, denoted as $G = XN$, we have:

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...and from this we could easily conclude that the product G itself is supersoluble, by attaching a G-supersoluble series of N:

$$\{1\} = N_0 \lhd N_1 \lhd \cdots \lhd N_k = N$$

with the numerators of a supersoluble series of $G/N \simeq X$:

$$\{1\} = \frac{N}{N} = \frac{N_{k+1}}{N} \lhd \frac{N_{k+2}}{N} \lhd \cdots \lhd \frac{N_m}{N} = \frac{G}{N} .$$

The Magic Wand

In order to carry out the passage

supersolubility $\rightarrow$ G-supersolubility

we made use of a fundamental Theorem, due to Robinson and Hall, which establishes a deep connection between:

THE TENSOR PRODUCT (of abelian groups)

$\Updownarrow$

THE LOWER CENTRAL SERIES (LCS)

The Robinson-Hall Theorem

Let G be a group, and P a normal subgroup of G.

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Now, let us just point out, without details, that we can endow

- P
- the factors $\gamma_i(P)/\gamma_{i+1}(P)$ of the LCS of P
- and their **TENSOR PRODUCTS**

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In these structures, the submodules are exactly the subgroups that are G-invariant.

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Then, for each $i \in \mathbb{N}$, there exists an epimorphism of $\mathbb{Z}G$-modules:

$$\varphi_i : \frac{P}{P'} \otimes \frac{\gamma_i(P)}{\gamma_{i+1}(P)} \longrightarrow \frac{\gamma_{i+1}(P)}{\gamma_{i+2}(P)} \quad .$$

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As consequence, given a module-theoretic property $\mathfrak{X}$ that is inherited by homomorphic images of tensor products of $\mathbb{Z}G$-modules, we have:

$$\frac{P}{P'} \text{ has } \mathfrak{X} \implies \begin{array}{l} \mathfrak{X} \text{ is passed recursively to } \textit{all} \\ \text{the factors of the LCS of } P \end{array} \quad .$$

The Robinson-Hall Theorem - Application

Let G be a group, and P a **nilpotent** normal subgroup of G.

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So, by the Robinson-Hall theorem, it follows that *all* the factors of the LCS of P

$$\frac{\gamma_i(P)}{\gamma_{i+1}(P)}$$

are G-supersoluble.

The Robinson-Hall Theorem - Application

In other words, for every $\gamma_i(P)/\gamma_{i+1}(P)$ we have a series of *G*-supersolubility (cyclic factors and **numerators normal in G**)...

$$\mathbf{i=1}: \quad \{1\} = \frac{\gamma_2(P)}{\gamma_2(P)} = \frac{H_{1,0}}{\gamma_2(P)} \lhd \frac{H_{1,1}}{\gamma_2(P)} \lhd \cdots \lhd \frac{H_{1,k_1}}{\gamma_2(P)} = \frac{\gamma_1(P)}{\gamma_2(P)} = \frac{P}{P'}$$

$$\mathbf{i=2}: \quad \{1\} = \frac{\gamma_3(P)}{\gamma_3(P)} = \frac{H_{2,0}}{\gamma_3(P)} \lhd \frac{H_{2,1}}{\gamma_3(P)} \lhd \cdots \lhd \frac{H_{2,k_2}}{\gamma_3(P)} = \frac{\gamma_2(P)}{\gamma_3(P)}$$

$$\ldots \quad \ldots$$

$$\mathbf{i=c-2}: \quad \{1\} = \frac{\gamma_{c-1}(P)}{\gamma_{c-1}(P)} = \frac{H_{c-2,0}}{\gamma_{c-1}(P)} \lhd \cdots \lhd \frac{H_{c-2,k_{c-2}}}{\gamma_{c-1}(P)} = \frac{\gamma_{c-2}(P)}{\gamma_{c-1}(P)}$$

$$\mathbf{i=c-1}: \quad \{1\} = \frac{\{1\}}{\gamma_c(P)} = \frac{\gamma_c(P)}{\gamma_c(P)} = \frac{H_{c-1,0}}{\gamma_c(P)} \lhd \cdots \lhd \frac{H_{c-1,k_{c-1}}}{\gamma_c(P)} = \frac{\gamma_{c-1}(P)}{\gamma_c(P)}$$

The Robinson-Hall Theorem - Application

...so taking the numerators of these series...

$$\mathbf{i=1}: \quad \{1\} = \frac{\gamma_2(P)}{\gamma_2(P)} = \frac{H_{1,0}}{\gamma_2(P)} \lhd \frac{H_{1,1}}{\gamma_2(P)} \lhd \cdots \lhd \frac{H_{1,k_1}}{\gamma_2(P)} = \frac{\gamma_1(P)}{\gamma_2(P)} = \frac{P}{P'}$$

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The Robinson-Hall Theorem - Application

...and attaching them in a unique series:

$$\{1\} = \gamma_c(P) \lhd \cdots \lhd \gamma_{c-1}(P) \lhd \cdots \lhd \gamma_2(P) \lhd \cdots \lhd \gamma_1(P) = P$$

in the end, we obtain a finite series connecting $\{1\}$ and P, with cyclic factors and **terms normal in G**.

In conclusion P is G-supersoluble.

A Key-Lemma

To summarize, thanks to this argument, it is possible to prove the following crucial Lemma for our work.

Application of Robinson-Hall Theorem to G-supersolubility

Let G be a group, and P a nilpotent normal subgroup of G. Then:

$$\frac{P}{P'} \text{ is } G\text{-supersoluble} \Rightarrow P \text{ is } G\text{-supersoluble}.$$

A hint at the infinite case

Building on these ideas, and with much much much much much work, we were able to prove the analogous theorems in the infinite case - in a certain sense.

To achieve this, we were forced to add some conditions to the definitions of VWDP and VWNP (which are clearly satisfied in the finite case):

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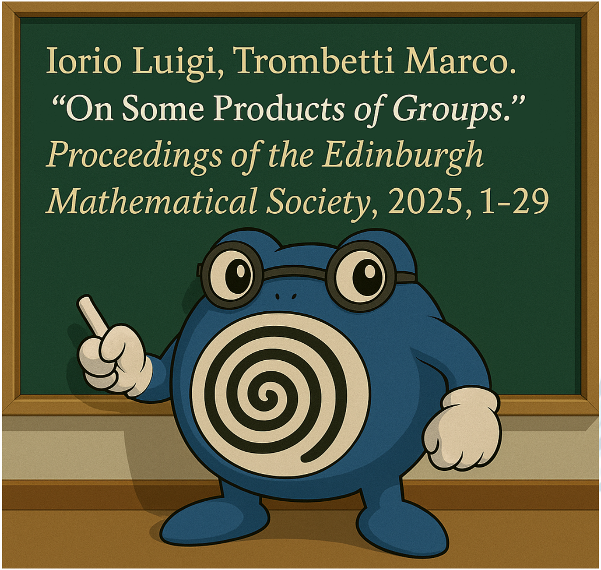
- To deal with the cases in which there are Sylow subgroups with no maximal subgroups (such as the Prüfer groups), we added the requirement that the non-normal factor permutes with the divisible subgroups of N too.

- To enable the same kind of manipulation as in the finite case, we had to add the requirement that passing to quotients groups preserves the assumptions of the very weak products.

Reference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Enjoy Napoli!