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On PI-theory for algebras graded by a group

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General setting

- $F :=$ ground field (characteristic zero);
- $A :=$ associative algebra over F;
- $M_n(F) \rightsquigarrow$ algebra of $n \times n$ matrices over F.

Polynomial identities

$X = \{x_1, x_2, \ldots\}$ countable set of non-commuting variables.

$F\langle X\rangle$ is the free algebra of polynomials on X over F.

Definition

A polynomial $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a polynomial identity of the algebra A, and we write $f \equiv 0$, if, for any $a_1, \ldots, a_n \in A$,

$$f(a_1, \ldots, a_n) = 0.$$

PI-algebras: they satisfy at least one non-trivial polynomial identity.

Example

The commutator $[x_1, x_2] \equiv 0$ on any commutative algebra C.

Standard polynomial: $St_m(x_1,\dots,x_m)=\sum_{\sigma\in S_m}(-1)^\sigma x_{\sigma(1)}\cdots x_{\sigma(m)}.$

Theorem (Amitsur, Levitzki - 1950)

$St_{2n}(x_1,\dots,x_{2n})\equiv 0$ *on* $M_n(F)$.

Combinatorial approach in PI-theory.

Identities of an algebra

Problem: finding all identities satisfied by a given algebra A.

$$\mathrm{Id}(A) = \{f \in F\langle X\rangle : f \equiv 0 \text{ on } A\}.$$

Example (Drensky - 1981)

$\mathrm{Id}(M_2(F)) = \langle St_4, [[x_1, x_2]^2, x_3]\rangle$.

$\mathrm{Id}(M_3(F)) \rightsquigarrow$ no idea !!!

Theorem (Kemer - 1987)

If $\mathrm{char} F = 0$, $\mathrm{Id}(A)$ *is finitely generated.*

The codimension sequence

Multilinear polynomials: every variables at degree 1 in any monomial.

$$P_n = \mathrm{span}_F \left\{ x_{\sigma(1)} \dots x_{\sigma(n)} \mid \sigma \in S_n \right\}.$$

Definition

The n-th codimension of an algebra A is the non-negative integer:

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)} \leq n!$$

Analytical approach in PI-theory

- **Regev - 1972**: If A is a PI-algebra, $c_n(A)$ is exponentially bounded.
- **Kemer - 1979**: characterizes algebras with polynomial growth of $c_n(A)$.
- **Giambruno, Zaicev - 1999**: existence of $\exp(A) = \lim \sqrt[n]{c_n(A)} \in \mathbb{N}$.

Algebras graded by a group

Graded algebras

Let $G = \{1, g_2, \dots, g_m\}$ be a finite group.

Definition

An algebra A is said G-graded if it can be decomposed as

$$A = \bigoplus_{g \in G} A_g, \quad \text{with } A_g A_h \subseteq A_{gh}.$$

Example (trivial grading)

$A_1 = A$ and $A_h = \{0\}$, for any $h \in G \setminus \{1\}$.

Superalgebras

Superalgebra: algebra graded by $\mathbb{Z}_2 := \{1, g\}$.

$$A = \underbrace{A_1}_{\text{bosons}} \oplus \underbrace{A_g}_{\text{fermions}}$$

Example

$$M_2(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in F \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in F \right\}.$$

Theorem (**Kemer - 1988**)

For any algebra A there exists a finite dimensional superalgebra B such that

$$\mathrm{Id}(A) = \mathrm{Id}(G(B)).$$

- $F\langle X\rangle :=$ free algebra on $X = \{x_1, x_2, \ldots\}$
- $X = \bigcup_{g\in G} X_g$, where $X_g = \left\{x_1^{(g)}, x_2^{(g)}, \ldots\right\}$

$$F\langle X, G\rangle := \text{free } G\text{-graded algebra}$$

Definition

A graded polynomial $f = f\left(x_1^{(g_1)}, \ldots, x_{t_1}^{(g_1)}, \ldots, x_1^{(g_m)}, \ldots, x_{t_m}^{(g_m)}\right) \in F\langle X, G\rangle$ is a graded identity of the G-graded algebra $A = \bigoplus_{g\in G} A_g$ if, for all $a_i^{(g_j)} \in A_{g_j}$

$$f\left(a_1^{(g_1)}, \ldots, a_{t_1}^{(g_1)}, \ldots, a_1^{(g_m)}, \ldots, a_{t_m}^{(g_m)}\right) = 0.$$

Upper triangular matrices

$$UT_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$$

Theorem (Malcev - 1981)

$\mathrm{Id}(UT_2) = \langle [x_1, x_2][x_3, x_4] \rangle.$

***G*-grading:** $UT_2 = \underbrace{\left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in F \right\}}_{A_1} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\}}_{A_h}$

$$x_1^{(h)} x_2^{(h)} \equiv 0.$$

Graded codimensions

- $P_n^G :=$ multilinear graded polynomials
- $\mathrm{Id}^G(A) :=$ graded identities

Definition

$$c_n^G(A) = \dim_F \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)}$$

Some results

- **Polynomial growth**:
  - **Giambruno, Mishchenko, Zaicev - 2001** (superalgebras)
  - **Valenti - 2011** (general case)

- **Exponent**:
  - **Benanti, Giambruno, Pipitone - 2003** (superalgebras)
  - **Aljadeff, Giambruno, La Mattina - 2010/13** (graded algebras)
  - **Gordienko - 2013** (algebras with generalized Hopf action)

- **Exponent** ≤ 2: **I., Martino - 2019**

- **Colenghts**: **Cota, I., Martino, Vieira - 2024**

- **Fundamental structures**: **Pascucci - 2025**


Thank You for Your Attention