Skew braces of finite Morley rank

Moreno Invitti

Joint work with M.Ferrara, M.Trombetti and F.O. Wagner

24/06/25

Partially supported by Grant No. AP19677451 of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan

Introduction

Definition

A **skew brace** is a set B equipped with two operations $+$ and $\circ$ such that

- $(B,+)$ and $(B,\circ)$ are groups;
- B satisfies the skew left distributivity property i.e. for any $a,b,c \in B$

$$a \circ (b+c) = a \circ b - a + a \circ c$$

A skew brace is trivial if $+$ equals $\circ$, it is almost trivial if $\circ$ equals $+^{op}$.
Any group can be equipped with the structure of a trivial, or an almost trivial, skew brace. We put $\lambda_a(b) = -a + a \circ b$. Then $\lambda : a \mapsto \lambda_a$ is an homorphism from $(B,\circ)$ to $\mathrm{Aut}(B,+)$.

Skew braces have been introduced by Guarnieri and Vendramin [3] in order to study the set-theoretic solution of the Yang-Baxter equation i.e., couples (X, r) where X is a set and

$$r : X \times X \mapsto X \times X$$

is a bijective map such that

$$(r \times \mathrm{Id})(\mathrm{Id} \times r)(r \times \mathrm{Id}) = (\mathrm{Id} \times r)(r \times \mathrm{Id})(\mathrm{Id} \times r)$$

Given a skew brace B, define a map $r : B \times B \mapsto B \times B$ by

$$r : (a, b) \mapsto (\lambda_a(b), \lambda^{-1}_{\lambda_a(b)}((a+b)^{-1} \circ a \circ (a+b))).$$

Then the couple (B, r) is a set-theoretic solution of the Yang-Baxter equation.

Structures of finite Morley rank

Rather than study the interactions with the Yang-Baxter equation, we will analyze the algebraic structure of infinite skew-braces from a model theoretic point of view. In particular, we will assume that the skew braces have finite Morley rank.

Definition

Given $\mathfrak{M}$ a structure and $\phi(x)$ a formula in $\mathscr{L}(M)$, we say

- $\mathrm{RM}(\phi(x)) \geq 0$ *if $\phi(x)$ has a realisation;*
- $\mathrm{RM}(\phi(x)) \geq n+1$ *if there exist disjoint formulas $\{\phi_i(x)\}_{i<\omega}$ such that $\phi_i(M) \subseteq \phi(M)$ and $\mathrm{RM}(\phi_i(x)) \geq n$.*

A structure is said to have *finite Morley rank* if $\mathrm{RM}(x = x)$ is finite.

Definition

Given a structure $\mathfrak{M}$, a subset $X \subseteq \mathfrak{M}$ is definable if it coincides with the set of realizations of a formula $\phi(x) \in \mathscr{L}(M)$.

If the set X is defined by $\phi(x)$, we put $\mathrm{RM}(\phi(x)) = \mathrm{RM}(X)$.

Groups of finite Morley rank

Groups of finite Morley rank have been extensively studied by many model theorists (see [2],[1]). They include, for example, algebraic groups (over algebraically closed field).

Proposition

Given G a definable group of finite Morley rank and a definable subgroup H of G, we have $\mathrm{RM}(H) = \mathrm{RM}(G)$ *iff* $|G : H|$ *is finite.*

Groups of finite Morley rank

Groups of finite Morley rank have been extensively studied by many model theorists (see [2],[1]). They include, for example, algebraic groups (over algebraically closed field).

Proposition

Given G a definable group of finite Morley rank and a definable subgroup H of G, we have $\mathrm{RM}(H) = \mathrm{RM}(G)$ *iff* $|G : H|$ *is finite.*

In particular we will be interested in connected groups of finite Morley rank.

Definition

Given a definable group G, the connected component of G, denoted G^0, is the intersection of all definable subgroup of finite index. A group G is said to be connected if $G = G^0$.

Proposition

Let G be a definable group of finite Morley rank. Then G^0 is a definable definably characteristic subgroup of finite index.

We have a characterization of connected groups of small Morley rank.

Cherlin-Frecon's Theorem

Let G be a definable connected group of finite Morley rank. Then

- *If* $\mathrm{RM}(G) = 1$, *then G is abelian.*

Characterization of groups of small Morley rank

We have a characterization of connected groups of small Morley rank.

Cherlin-Frecon's Theorem

Let G be a definable connected group of finite Morley rank. Then

- *If* $\mathrm{RM}(G) = 1$*, then G is abelian.*
- *If* $\mathrm{RM}(G) = 2$*, then G is soluble. If G is not nilpotent, $G/Z(G) \simeq K^{+} \rtimes K^{\times}$ for a definable field K of Morley rank* 1*.*

Characterization of groups of small Morley rank

We have a characterization of connected groups of small Morley rank.

Cherlin-Frecon's Theorem

Let G be a definable connected group of finite Morley rank. Then

- *If* $\mathrm{RM}(G) = 1$, *then G is abelian.*
- *If* $\mathrm{RM}(G) = 2$, *then G is soluble. If G is not nilpotent, $G/Z(G) \simeq K^{+} \rtimes K^{\times}$ for a definable field K of Morley rank* 1.
- *If* $\mathrm{RM}(G) = 3$ *and G is simple non soluble, then $G \simeq \mathrm{PSL}_2(K)$ for some field K of Morley rank* 1.

Definition

We define two homomorphisms $\lambda, \mu : (B, \circ)$ *to* $\mathrm{Aut}(B, +)$ *as follows*

$$\lambda_b : a \mapsto -b + b \circ a \quad \mu_b : a \mapsto b \circ a - b.$$

For $a, b \in B$*, we define*

$$a * b = -a + a \circ b + b = \lambda_a(b) - b.$$

It is not difficult to see that $\mu = \lambda^{op}$ where λ^{op} is the λ function for the skew-brace $B^{op} := (B, +^{op}, \circ)$.

Strong left ideals and ideals

Definition

An additive normal subgroup L of B is a ***strong left ideal*** *if $\lambda_b(L) = L$ for every $b \in B$. Observe that, by λ-invariance, a strong left ideal is always a multiplicative subgroup. A strong left ideal L is an* ***ideal*** *if $(L, \circ)$ is normal in $(B, \circ)$. Equivalently $\lambda_\ell(b) - b \leq L$ for $\ell \in L$ and $b \in B$.*

Let L, L_1 be skew sub-braces in B. We denote by $L_1 * L$ the additive subgroup generated by $\{\ell_1 * \ell\}_{\ell_1 \in L_1, \ell \in L}$.

Definition

*B is **weakly soluble** if there exists $n < \omega$ and a chain of skew sub-braces*

$$B = L_0 \geq L_1 \geq ... \geq L_n = \{0\}$$

such that $L_i * L_i, [L_i, L_i]_+ \leq L_{i+1}$.
*B is **strongly left nilpotent** if there exists $n < \omega$ and a chain of strong left ideals*

$$B = L_0 \geq L_1 \geq ... \geq L_n = \{0\}$$

such that $L * L_i, [L, L_i]_+ \leq L_{i+1}$.

Definition

*B is **bi-soluble** if $(B, +)$ and $(B, \circ)$ are soluble as groups. B is **bi-nilpotent** if $(B, +)$ and $(B, \circ)$ are nilpotent as groups.*

We have the following easy lemma.

Lemma

If B is strongly left nilpotent, B is bi-nilpotent.

Finally, we state an easy Lemma that can be found in [4].

Lemma

Let $(B+,\circ)$ be a definable skew brace of finite Morley rank. Then $(B,+)^0 = (B,\circ)^0$ is an ideal in B.

Left chief series

Definition

Let L be a definable strong left ideal in B. Then the series of definable strong left ideals in B

$$L = L_0 \geq L_1 \geq L_2 \geq \dots \geq L_n = \{0\}$$

is a ***(definable) left chief series*** *for L if every quotient is infinite and definably minimal as strong B-left ideal i.e., for every $i \leq n-1$ and definable strong left ideal L with $L_i \geq L \geq L_{i+1}$, either $(L/L_{i+1},+)$ or $(L_i/L,+)$ is finite.*

If we want to emphasize B, we call it a B-left chief series.

Jordan-Hölder theorem

We have the following Jordan-Hölder theorem for left chief series.

Theorem

*Let L be a strong left ideal in B and $(L_i)_{i\leq m}, (L'_i)_{i\leq n}$ two left chief series for L. Then $n = m$. Therefore the **left chief length** of L, defined as the length of a left chief series for L, is well-defined.*

Theorem

A bi-soluble definable connected skew brace of finite Morley rank with left chief length at most 3 *is weakly soluble.*

Theorem

A bi-nilpotent definable connected skew-brace of finite Morley rank is strongly left nilpotent.

We now characterize skew braces of small Morley rank.

Proposition

Let $(B, +, \circ)$ be a definable connected skew brace of Morley rank 1. Then B is trivial and abelian i.e. $(B, +) = (B, \circ)$ is an abelian group.

Therefore, in this case, the characterization restricts to the characterization of connected abelian group of Morley rank 1 (see for example [1]).

We first introduce a family of skew braces of Morley rank 2. Let K be a definable field of Morley rank 1 and ϕ^+, ϕ° two homorphisms from $K^\times$ to $\mathrm{Aut}(K^+)$ such that

$$\phi_+(k) \circ \phi_\circ(k') = \phi_\circ(k') \circ \phi_+(k)$$

as elements in $\mathrm{Aut}(K^+)$.
Now put

$$B_{K,\phi^+,\phi^\circ} := (K \times K^\times, +, \circ)$$

with

$$(K \times K^\times, +) = K \rtimes_{\phi^+} K^\times$$

and

$$(K \times K^\times, \circ) = K \rtimes_{\phi^\circ} K^\times.$$

It is easy to verify that this is a skew brace.

Theorem

Let B be a connected skew brace of Morley rank 2. Then one of the following alternatives holds:

(1) *B is strong left nilpotent;*

(2) *B is 2-step soluble not strongly left-nilpotent and $\mathrm{Ann}(B)$ is finite. Moreover, either $B/\mathrm{Ann}(B)$ or $B^{op}/\mathrm{Ann}(B)$ is isomorphic to a skew-brace of the form $B_{K,\phi_+,\phi_\circ}$.*

Skew braces of Morley rank 3

Given the classification of connected groups of Morley rank 3, we have a priori four possibilities for a connected skew brace of Morley rank 3.

(a) $(B, +)$ and $(B, \circ)$ are soluble, i.e. B is bi-soluble;

(b) $(B, +)/Z(B, +) \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field K of Morley rank 1, and $(B, \circ)$ is soluble;

(c) $(B, +)$ is soluble, and $(B, \circ)/Z(B, \circ) \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field K of Morley rank 1;

(d) $(B, +)/Z(B, +) \simeq \mathrm{PSL}_2(K)$ and $(B, \circ)/Z(B, \circ) \simeq \mathrm{PSL}_2(F)$ for some algebraically closed fields K and F of Morley rank 1.

Given the classification of connected groups of Morley rank 3, we have a priori four possibilities for a connected skew brace of Morley rank 3.

(a) $(B,+)$ and $(B,\circ)$ are soluble, i.e. B is bi-soluble;

(b) $(B,+)/Z(B,+) \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field K of Morley rank 1, and $(B,\circ)$ is soluble;

(c) $(B,+)$ is soluble, and $(B,\circ)/Z(B,\circ) \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field K of Morley rank 1;

(d) $(B,+)/Z(B,+) \simeq \mathrm{PSL}_2(K)$ and $(B,\circ)/Z(B,\circ) \simeq \mathrm{PSL}_2(F)$ for some algebraically closed fields K and F of Morley rank 1.

Theorem

In case (a), *B is weakly soluble. The cases* (b) *and* (c) *are impossible. In case* (d) $Z(B,\circ) = Z(B,+) = \mathrm{Ann}(B)$ *and* $(B,+,\circ)$ *is trivial or almost trivial. In particular* $F = K$.

Bibliography

[1] A. Borovik – A. Nesin. *Groups of finite Morley rank*. Oxford university press, 1994.

[2] B. Poizat. *Stable groups*. Vol. 87. American Mathematical Soc., 2001.

[3] L. Guarnieri – L. Vendramin. “Skew braces and the Yang–Baxter equation”. In: *Mathematics of Computation* 86.307 (2017), pp. 2519–2534.

[4] M. Ferrara – M. Invitti – M. Trombetti – F.O. Wagner. *Stabilty and ω-categoricity of skew braces*.