On Pronormalizers in Finite Groups

(a joint work with M. Brescia and M. Trombetti)

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Definition

Let p be a prime, G a group and P a p-subgroup of G. A subgroup S is said to be a *p-Sylowizer* of P in G if S is maximal in the set of all subgroups of G containing P as a Sylow p-subgroup.

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- P can have more than one p-Sylowizer. If $P = \{1\}$, then every maximal p'-subgroup is a p-Sylowizer of P in G.
- Every p-subgroup of G has only one p-Sylowizer if and only if G has a normal Sylow p'-subgroup.

Theorem (Gaschütz)

Let p be a prime, G a finite soluble group and P a p-subgroup of G. If P is normal in a Sylow p-subgroup of G, then the p-Sylowizers of P in G are conjugate.

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In $G = \mathrm{GL}(2,3)$, $P = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$ has non-conjugate 2-Sylowizers.

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- Normal subgroups are pronormal;
- Maximal subgroups are pronormal;
- Sylow p-subgroups in finite groups are pronormal.

Basic results

Let G be a group and U a pronormal subgroup of G. Then

- If $U \lhd\lhd G$, then $U \lhd G$.

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- If $U \lhd\lhd G$, then $U \lhd G$.
- If $K \lhd G$ and $U \leq K$, then $G = N_G(U)K$.

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- If $U \lhd\lhd G$, then $U \lhd G$.
- If $K \lhd G$ and $U \leq K$, then $G = N_G(U)K$.

Theorem (Gaschütz)

Let G be a group, U a subgroup of G and $K \lhd G$. Then following are equivalent:

(1) U is pronormal in G,

(2) UK is pronormal in G and U is pronormal in $N_G(UK)$.

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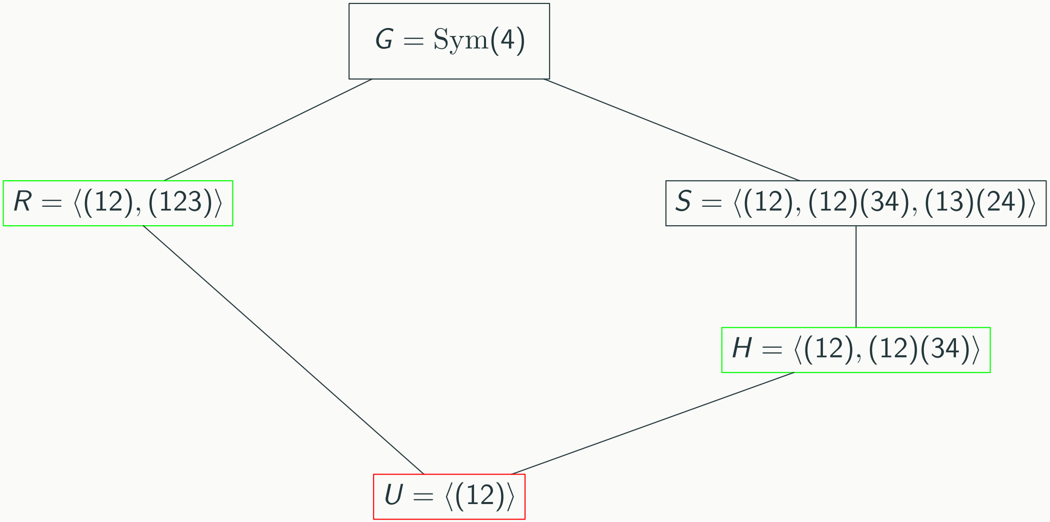
- If U is a p-subgroup, every p-Sylowizer of U in G is contained in a pronormalizer of U in G.

Definition

Let G be a group and U a subgroup of G. A subgroup R of G is said to be a *pronormalizer* of U in G, if R is maximal in set of all subgroups of G containing U as a pronormal subgroup.

- If U is a p-subgroup, every p-Sylowizer of U in G is contained in a pronormalizer of U in G.
- U can have more than one pronormalizer.


$G = \mathrm{Sym}(4)$
$R = \langle (12), (123) \rangle$
$S = \langle (12), (12)(34), (13)(24) \rangle$
$H = \langle (12), (12)(34) \rangle$
$U = \langle (12) \rangle$

Remark

Let p be a prime, G a finite group and U a p-subgroup of G. If N is a normal p'-subgroup of G, then R is a pronormalizer of U in G if and only if R is a pronormalizer of UN in G.

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Let π be a set of primes, G a finite soluble group and U a π-subgroup of G. If N is a normal π'-subgroup of G, then R is a pronormalizer of U in G if and only if R is a pronormalizer of UN in G.

Let G be a group and U a subgroup of G.

- If $U \lhd\lhd G$, then $N_G(U)$ is the only pronormalizer of U in G.

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- If $U \lhd\lhd G$, then $N_G(U)$ is the only pronormalizer of U in G.
- If $U \leq K \lhd G$, and R is the only pronormalizer of U in K, then $N_G(U)R$ is the only pronormalizer of U in G.

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Theorem

Let G be a group, U a subgroup of G and K a normal subgroup of G. Then

(1) If U has only one pronormalizer R in G and U is pronormal in $N_G(UK)$, then R is the only pronormalizer of UK in G.

(2) If UK has only one pronormalizer L in G and U is pronormal in $N_L(UK)$, then L is the only pronormalizer of U in G.

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Recall that G is said to be *metanilpotent* if there exists a normal nilpotent subgroup N of G such that G/N is nilpotent.

- If G is a finite metanilpotent, then every p-subgroup has only one pronormalizer in G for each prime p.

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- If G is a finite metanilpotent, then every p-subgroup has only one pronormalizer in G for each prime p.

Recall that G has a *Sylow tower* if G has a normal series of Sylow subgroup.

- If G is a finite group with a Sylow tower, then every p-subgroup has only one pronormalizer in G for each prime p.

In 1968, J. G. Thompson classified the finite minimal simple groups.

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If G is a finite minimal simple group, then G is isomorphic to one of the following:

(i) $\mathrm{PSL}(2, 2^{\mathrm{p}})$ where p is a prime number;

(ii) $\mathrm{PSL}(2, 3^{\mathrm{p}})$ where p is an odd prime;

(iii) $\mathrm{PSL}(2, \mathrm{p})$ where $p > 3$ is a prime such that 5 divides $p^2 + 1$;

(iv) $\mathrm{PSL}(3, 3)$;

(v) $\mathrm{Sz}(2^{\mathrm{p}})$ where p is an odd prime.

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Corollary

Let G be finite group in which every 2-subgroup has only one pronormalizer, then G is soluble.

Theorem

Let G be a finite metabelian group. Then every subgroup of G has only one pronormalizer.

Thank you