Tensor products of Representations of $\mathrm{GL}_2(\mathfrak{o}_\ell)$

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Introduction

- Let G be a finite group. A **representation** of G is a homomorphism $\phi : G \mapsto GL(V)$, where V is a finite dimensional vector space over complex field.
- A sub-representation of a representation is a subspace W of V which is invariant under G-action.
- A representation V of a group G is irreducible if the only G-invariant subspaces of V are $\{0\}$ and V itself.
- **Question:** What are all irreducible representations of G?

  Frobenius initiated the study of character theory (1896–1903), and Schur advanced representation theory in the 1920s.

J.-P. Serre (1977) Linear Representations of Finite Groups. Springer-Verlag, Graduate Texts in Mathematics, Vol. 42.

- All irreducible representations of an abelian group are one dimensional.
- S_n : Alfred Young (1930's)[1]
- Dihedral Groups : Classical group theory [2]
- $GL_n(\mathbb{F}_q)$: James A. Green (1955) [3]
- Coxeter Groups : R. Brauer, I. schur[4], N. Bourbaki (1920-60) [5].
- Finite groups of Lie type : P. Deligne and G. Lusztig (1960-70)[6].

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[1] Young (1900). On quantitative substitutional analysis. Proc. Lond. Math. Soc.
[2] Burnside (1911). Theory of Groups of Finite Order.
[3] Green (1955). The characters of the finite general linear groups. Trans. AMS.
[4] Brauer , Schur (1937). On irreducible representations of groups. Ann. Math.
[5] Bourbaki (1968). Groupes et algèbres de Lie.
[6] Deligne Lusztig (1976). Representations of reductive groups over finite fields. Annals of Math.

The group

- $GL_n(R)$, R is a finite commutative ring.

Structure theorem for finite commutative rings (Analogue of chinese remainder theorem)[7]

Any finite commutative ring with identity is isomorphic to a direct sum of finite local rings (rings with unique maximal ideal).

- An important class of local rings are **discrete valuation rings** , which are local principal ideal domains that are not fields.
- $\mathbb{Z}_p$, the rings of p-adic integers for any prime p, the valuation assigns to each p-adic integer x the largest integer k such that p^k divides x.
- Examples: For $\ell \geq 1$, $\mathbb{Z}/p^\ell\mathbb{Z}$, $F_p[t]/(t^\ell)$, $\mathbb{Z}/p^\ell\mathbb{Z}[t]/(t^\ell)$.

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[7] M. F. Atiyah I. G. Macdonald (1969). *Introduction to Commutative Algebra*, Addison-Wesley, Chapter 8.

Another approach

- F be a non Archimedean local field with finite residue field $\mathbb{F}_q$ of odd characteristic p. Example : $\mathbb{Q}_p$, $\mathbb{F}_q((t))$.
- $\mathfrak{o}$ be the ring of integers of F. Example : p-adic integers $\mathbb{Z}_p$, $F_p[[t]]$.
- $\mathfrak{o}$ is a DVR.
- $\mathfrak{p}$ be the unique maximal ideal of $\mathfrak{o}$ and π be a fixed generator of $\mathfrak{p}$.
- For $\ell \geq 1$, $\mathfrak{o}_\ell := \mathfrak{o}/\mathfrak{p}^\ell$ ($\mathfrak{o}_1 = \mathbb{F}_q$).
- By profiniteness of $GL_n(\mathfrak{o})$, every complex continuous irreducible representation of $GL_n(\mathfrak{o})$ factors through some group $GL_n(\mathfrak{o}_\ell)$ [8].

Question

How to construct irreducible representations of $GL_n(\mathbb{Z}/p^\ell\mathbb{Z})$ and $GL_n(F_p[t]/(t^\ell))$ uniformly?

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[8] A. Jaikin-Zapirain (2014). "Smooth representations of $GL_n(\mathfrak{o})$", *Journal of Algebra*, 414, 37–51.

Conjecture (Onn, 2018)[9]

Let $\mathfrak{o}$ and $\mathfrak{o}'$ be the rings of integers of F and F' such that $\mathfrak{o}/p \cong \mathfrak{o}'/p'$. Then

$$\mathbb{C}[GL_n(\mathfrak{o}_\ell)] \cong \mathbb{C}[GL_n(\mathfrak{o}'_\ell)].$$

- $\mathrm{Irr}(GL_n(\mathfrak{o}_\ell)) \quad \xleftrightarrow{\text{dim-preserving bijection}} \quad \mathrm{Irr}(GL_n(\mathfrak{o}'_\ell))$

n	ℓ	Authors
2		Stasinski (2006), Onn (2008)
	2	Singla (2010)
3		Char$(F) = 0$: Avni, Klopsch, Onn, Voll (2015), Char$(F) = p$: Prasad, Onn, Singla (2024) ($p > 3$).

Table: History of construction of irreducible representations for $GL_n(\mathfrak{o}_\ell)$

- Conjecture status: True for $(n, 2)$, $(2, \ell)$, $(3, \ell)$ and open otherwise.

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[9] Uri Onn, Representations of automorphism groups of finite O-modules of rank two, Adv. Math. 219 (2008), no. 6, 2058-2085.

Clifford Theory [10]

Let N be a normal subgroup of G. For an irreducible representation φ of N, let $C_G(\varphi) = \{g \in G \mid \varphi^g \cong \varphi\}$. Then the following hold.

1. If ρ is an irreducible representation of G such that $\langle \rho|_N, \varphi \rangle \neq 0$, then $\rho|_N = e(\bigoplus_{g \in \Omega} \varphi^g)$ where Ω is the orbit containing φ under the action of G on $\mathrm{Irr}(N)$ and $e = \langle \rho|_N, \varphi \rangle$.
2. Suppose φ is an irreducible representation of N. Let $A = \{\rho \in \mathrm{Irr}(G) \mid \langle \rho|_N, \varphi \rangle \neq 0\}$ and $B = \{\chi \in \mathrm{Irr}(C_G(\varphi)) \mid \langle \chi|_N, \varphi \rangle \neq 0\}$. Then the map $\chi \mapsto \mathrm{Ind}_{C_G(\varphi)}^G(\chi)$ is a bijection from A to B.
3. Let φ be an irreducible representation of N such that it extends to G, say $\tilde{\varphi}$ (i.e. $\tilde{\varphi}|_N = \varphi$). Then the set of irreducible representations of G lying above φ is given by $\mathrm{Irr}(G \mid \varphi) = \{\tilde{\varphi} \otimes \mu \mid \mu \in \mathrm{Irr}(G/N)\}$.

[10] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, 1976.

Construction

- $\mathfrak{g}_\ell := M_2(\mathfrak{o}_\ell)$.
- For any integer i such that $1 \leq i \leq \ell$, let

  $$\rho_{\ell,i} \colon GL_2(\mathfrak{o}_\ell) \to GL_2(\mathfrak{o}_i)$$

  be the homomorphism induced by the canonical map $\mathfrak{o}_\ell \to \mathfrak{o}_i$.
- The i^{th} principle congruence subgroup $Ker(\rho_{\ell,i}) = K^i = I + \mathfrak{p}^i\mathfrak{g}_\ell$.
- Let $i \geq \ell/2$. Fix an additive character $\psi : \mathfrak{o}_\ell \longrightarrow \mathbb{C}^\times$ such that $\pi^{\ell-1}\mathfrak{o}_\ell \not\subseteq Ker(\psi)$ Let $\beta \in M_2(\mathfrak{o}_\ell)$. Define a homomorphism $\psi_\beta : K^{\lceil \frac{\ell}{2} \rceil} \longrightarrow \mathbb{C}^\times$ by,

  $$\psi_\beta(1 + \pi^{\lceil \frac{\ell}{2} \rceil}x) = \psi(\pi^{\lceil \frac{\ell}{2} \rceil} tr(\beta x)),\ x \in \mathfrak{p}^i\mathfrak{g}_\ell.$$

- Let $\beta \in M_2(\mathfrak{o}_\ell)$. Let $\bar{\beta}$ denote the image of β in $M_2(\mathbb{F}_q)$.

Regular representations for $GL_2(\mathfrak{o}_\ell)$

Definiton [11]

An element $\beta \in M_2(\mathfrak{o}_\ell)$ is said to be **regular** if the characteristic and minimal polynomials of $\bar{\beta}$ are the same.

Definition

An irreducible representation ρ of $GL_2(\mathfrak{o}_\ell)$ is called **regular** if $\rho\,|_{K^{\lceil \ell/2 \rceil}}$ contains ψ_β with $\beta \in \mathfrak{g}_\ell$ regular.

Consider $\beta \in M_2(\mathfrak{o}_\ell)$ such that :

- $\bar{\beta}$ is scalar.
- $\bar{\beta}$ is regular.
- $\bar{\beta}$ is non scalar non regular.

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[11] G. Hill, Regular elements and regular characters of $\mathrm{GL}_n(\mathfrak{o})$,(1995)

- Notice that for $\bar{\beta} \in M_2(\mathbb{F}_q)$, any $\bar{\beta}$ is either scalar or regular.
- For scalar $\bar{\beta}$, $\mathrm{Irr}(\mathrm{GL}_2(\mathfrak{o}_\ell) \mid \psi_\beta)$ is obtained by $\mathrm{GL}_2(\mathfrak{o}_{\ell-1})$ and these are called **lower level representations** of $\mathrm{GL}_2(\mathfrak{o}_\ell)$.
- For regular $\bar{\beta}$, the construction is known via clifford's method [12].
- For $n \geq 3$, $\bar{\beta}$ can be of non scalar non regular type. **For example :** For $n = 3$, the matrix
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
is neither a scalar nor a regular matrix since its chararcterstic polynomial is x^3 and minimal polynomial is x^2.
- For $n = 3$, the construction for these type of representations is known.[13].
- For $n \geq 4$ it is open.

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[12] Roi Krakovski, Uri Onn and Pooja Singla, Regular characters of groups of type An over discrete valuation rings, J. Algebra 496 (2018), 116–137.

[13] Uri Onn, Amritanshu Prasad and Pooja Singla,Representation Zeta Functions of Groups of Type A2 in Positive Characteristic, IMRN, Volume 2025, Issue 2, January 2025,

Types of regular representations of $\mathrm{GL}_2(\mathfrak{o}_\ell)$

The regular representations of $G := \mathrm{GL}_2(\mathfrak{o}_\ell)$ lying above ψ_β looks like $\mathrm{Ind}_{C_G(\psi_\beta)}^{G}\phi$ for some representation ϕ of $C_G(\psi_\beta)$ lying above ψ_β.

By Clifford theory, all regular representations are of this form.
Let ρ_β be a regular representation of $GL_2(\mathfrak{o}_\ell)$ such that $\rho\,\big|_{K^{\lceil \ell/2 \rceil}}$ contains ψ_β. Then we call ρ_β

- **Split semisimple** if $\bar{\beta}$ is conjugate to $\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)$ where $u \neq v$, $u, v \in \mathbb{F}_q$.
- **Split non-semisimple** if $\bar{\beta}$ is conjugate to $\left(\begin{smallmatrix} u & 1 \\ 0 & u \end{smallmatrix}\right)$, $u \in \mathbb{F}_q$.
- **Cuspidal** if characteristic polynomial of $\bar{\beta}$ is irreducible in $\mathbb{F}_q$.

Tensor products of representations

- If V, W are representations of a group G, then their tensor product is tensor product of vector spaces $V \otimes W$ with the linear action of G defined by,
$$g.(v \otimes w) = (g.v) \otimes (g.w),$$
for all $v \in V$ and $w \in W$.
- Tensor products of irreducible representations are not irreducible mostly: $V \otimes V$ is irreducible if and only if V is one-dimensional.
- The Clebsch–Gordan coefficients tell us how many times each irreducible representation appears when we break down the tensor product of two irreducible representations into a direct sum of irreducible parts. In quantum mechanics, these numbers for the groups $SO(3), SU(3)$ appear in angular momentum coupling.
- The tensor products together with the Clebsch–Gordan procedure, can be used to generate additional irreducible representations if some are known already.

Approach

Mackey's tensor product theorem

Let G be a group and H, K be its subgroups. Let ϕ be a representation of H and ψ be a representation of K. Then

$$\mathrm{Ind}_H^G \phi \otimes \mathrm{Ind}_K^G \psi \cong \bigoplus_{g \in H \backslash G / K} \mathrm{Ind}_{H \cap K^g}^G (\phi \otimes \psi^g)$$

To understand the multiplicity of a representation ρ as a constituent of $\rho_1 \otimes \rho_2$, we carry out the following steps:

(A) Understand and simplify the double coset representatives of $H \backslash G / K$

(B) Understand the decomposition of the representation $V(\phi_1, \phi_2^g) := \mathrm{Ind}_{H \cap K^g}^G (\phi_1 \otimes \phi_2^g)$ for every $g \in H \backslash G / K$.

(C) For distinct $g, h \in H \backslash G / K$, understand the intertwiners $\mathrm{Hom}_G(V(\phi_1, \phi_2^g), V(\phi_1, \phi_2^h))$.

Main Results

Let ρ_1, ρ_2, ρ_3 be three regular representations of $\mathrm{GL}_2(\mathfrak{o}_\ell)$. Define $m_{\rho_1\rho_2}^{\rho_3}$ to be the multiplcity of ρ_3 in $\rho_1 \otimes \rho_2$.

Main Theorem

Let $\ell \geq 1$ and ρ_1, ρ_2, ρ_3 be three regular irreducible representations of $\mathrm{GL}_2(\mathfrak{o}_\ell)$.

1. Let **cuspidal** $\in \{type(\rho_1), type(\rho_2), type(\rho_3)\}$. Then
$$m_{\rho_1\rho_2}^{\rho_3} \leq 1.$$
2. Let $|\{type(\rho_1), type(\rho_2), type(\rho_3)\}| = 2$. Then
$$m_{\rho_1\rho_2}^{\rho_3} \leq 2.$$
Further, $m_{\rho_1\rho_2}^{\rho_3} = 2$ only if the tuple $(type(\rho_1), type(\rho_2), type(\rho_3))$ is a permutation of
(**split semisimple**, **split non-semisimple**, **split semisimple**).
3. Let $\{type(\rho_1), type(\rho_2), type(\rho_3)\} = \{$**split semisimple**$\}$. Then
$$m_{\rho_1\rho_2}^{\rho_3} \leq \ell + 1$$

Corollary

Let ρ_1 and ρ_2 be regular irreducible representations of $\mathrm{GL}_2(\mathfrak{o}_\ell)$ such that *type*$(\rho_1) =$ **cuspidal**.

1. For *type*$(\rho_1) \neq$ *type*(ρ_2), the representation $\rho_1 \otimes \rho_2$ is multiplicity free.
2. The regular part of $\rho_1 \otimes \rho_2$ is multiplicity free.

Let $\ell \geq 1$. such that ρ_1, ρ_2, ρ_3 be regular representations of $\mathrm{GL}_2(\mathfrak{o}_\ell)$ such that *type*(ρ_1), *type*(ρ_2), *type*$(\rho_3) = \{$**cuspidal**$\}$.

1. For $\ell = 1$, we have $m_{\rho_1\rho_2}^{\rho_3} \leq 2$.
2. For $\ell \geq 2$, there exists $\rho_1\rho_2, \rho_3$ such that $m_{\rho_1\rho_2}^{\rho_3} \geq (q-2)$.

For $\ell \geq 2$, there exists regular irreducible **split non-semisimple** representations ρ_1, ρ_2 and ρ_3 of $\mathrm{GL}_2(\mathfrak{o}_\ell)$ such that $m_{\rho_1\rho_2}^{\rho_3}$ depends on the cardinality of the residue field $\mathbb{F}_q$.

References

- Archita Gupta and M Hassain, Tensor product of irreducible characters of $\mathrm{GL}_2(\mathbb{F}_q)$, Journal of Algebra and its Applications, 2026.
- Archita Gupta, M Hassain and Pooja Singla, On kronecker Products for the general linear and unitary groups over the principal ideal local rings (in preparation).

Thank You!