CENTRO DE CIENCIAS MATEMÁTICAS

Indecomposable components of fibered Burnside rings

Advances in Group Theory and Applications 2025

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Fibered permutation representations

G = finite group, A = abelian group.

Definition

An A**-fibered** G**-set** is a $G \times A$-set with finitely many orbits on which the action of A is free. A **morphism** of A-fibered G-sets is a map $f : X \longrightarrow Y$ commuting with the actions.

$$_G\mathrm{set}^A := \text{category of } A\text{-fibered } G\text{-sets.}$$

Monoidal operations. For all $X, Y \in {}_G\mathrm{set}^A$,

- **Disjoint union**: $X \sqcup Y \in {}_G\mathrm{set}^A$.
- **Tensor product**: $A \curvearrowright X \times Y$, $a \cdot (x, y) := (a^{-1}x, ay)$. Then

$$X \otimes_A Y := X \times Y/A \in {}_G\mathrm{set}^A.$$

Distributive laws: For all $X, Y, Z \in {}_G\mathrm{set}^A$, there are natural isomorphisms

$$X \otimes_A (Y \sqcup Z) \cong (X \otimes_A Y) \sqcup (X \otimes_A Z),$$

$$(Y \sqcup Z) \otimes_A X \cong (Y \otimes_A X) \sqcup (Z \otimes_A X).$$

The A-fibered Burnside ring

Definition

The A**-fibered Burnside ring** of G, denoted by $B^A(G)$, is defined as the Grothendieck ring of ${}_G\mathrm{set}^A$ with respect to $\sqcup$ and product induced by $\otimes_A$. More precisely,

$$B^A(G) := \mathbb{Z}^{({}_G\mathrm{set}^A/\cong)}/\langle [X] + [Y] - [X \sqcup Y] \mid X, Y \in {}_G\mathrm{set}^A \rangle.$$

Let

$$\mathcal{M}_G^A := \{(K, \phi) \mid K \leq G, \phi \in \mathrm{Hom}(K, A)\}$$

be the set of A**-monomial pairs** of G, on which G acts by conjugation. Then

$$B^A(G) \cong \bigoplus_{(K,\phi) \in [\mathcal{M}_G^A]} \mathbb{Z}[K, \phi]_G, \tag{1}$$

as rings, where $[K, \phi]_G := G \cdot (K, \phi)$, and

$$[K, \phi]_G [L, \psi]_G := \sum_{g \in [K \backslash G / L]} [K \cap {}^g L, \phi|_{K \cap {}^g L} {}^g\psi|_{K \cap {}^g L}]_G. \tag{2}$$

JOURNAL OF ALGEBRA 18, 137–157 (1971)

The Ring of Monomial Representations I. Structure Theory

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Communicated by B. Huppert

1. INTRODUCTION

Let G be a finite group and A a finite abelian group, on which G acts from the right: $(a, g) \mapsto a^g$ ($a \in A$, $g \in G$). A(G, A)-set S is a finite set, on which the semidirect product $G \times A$ acts by permutations from the left, such that $A \subseteq G \times A$ acts freely. (We assume A and G to be imbedded in $G \times A$.

Examples:

- $B^{\{1\}}(G) \cong B(G)$;
- $B^{\mathbb{C}^\times}(G) \cong D(G)$;
- $B^A(G)$ has finite rank $\iff \mathrm{Tor}_{\exp(G)}(A) = \{a \in A \mid a^{\exp(G)} = 1\}$ is finite.

Species

From now on, we assume that $\mathrm{Tor}_{\exp(G)}(A)$ is finite. Let

$$\mathcal{D}_G^A := \left\{(K,\Phi) \mid K \leq G, \Phi \in \mathrm{Hom}(K,A)^\star\right\}$$

on which G acts by conjugation. Each pair (K,Φ) induces a map

$$s_{K,\Phi}^{G,A} : B^A(G) \longrightarrow \mathbb{C}, [L,\psi]_G \mapsto \sum_{\substack{g\in[G/L],\\ K\leq {}^g L}} \Phi\left({}^g\phi|_K\right). \tag{3}$$

The following holds:

- $s_{K,\Phi}^{G,A}$ is a ring homomorphism,
- $\mathrm{Im}\left(s_{K,\Phi}^{G,A}\right) \subseteq \mathbb{Z}[\zeta]$ for $\zeta = e^{\frac{2\pi}{n}i}$, $n = \exp(\mathrm{Tor}_{\exp(G)}(A))$,
- $s_{K,\Phi}^{G,A} = s_{L,\Psi}^{G,A}$ if and only if $(K,\Phi) =_G (L,\Psi)$.

Scalar extension and splitting fields

If R is a commutative ring, we set $B_R^A(G) := R \otimes B^A(G)$.

Theorem (Boltje - Yılmaz, 2021)

Let $\mathbb{K} \subseteq \mathbb{C}$ be a subfield containing a root of 1 of order $\exp\left(Tor_{\exp(G)}(A)\right)$. Then the map

$$\prod_{(K,\Phi)\in[\mathcal{D}_G^A]} s_{K,\Phi}^{G,A} : B_{\mathbb{K}}^A(G) \longrightarrow \prod_{(K,\Phi)\in[\mathcal{D}_G^A]} \mathbb{K} \tag{4}$$

obtained by scalar extension is an isomorphism of $\mathbb{K}$-algebras. Moreover, for each $(K,\Phi) \in \mathcal{D}_G^A$, there is a primitive idempotent $e_{K,\Phi}^{G,A} \in B_{\mathbb{K}}^A(G)$ satisfying

$$s_{L,\Psi}^{G,A}\left(e_{K.\Phi}^{G,A}\right) = \begin{cases} 1 & \text{if } (L,\Psi) =_G (K,\Phi), \\ 0 & \text{else.} \end{cases}$$

Blocks of $B^A(G)$

The primitive idempotents of $B(G)$ are in bijection with conjugacy classes of perfect subgroups of G (Dress 1969, Yoshida 1983). We have a commutative diagram

$$\begin{array}{ccc} B(G) & \hookrightarrow & B^A(G) \\ \downarrow & & \downarrow \\ B_{\mathbb{Q}[\zeta]}(G) & \hookrightarrow & B^A_{\mathbb{Q}[\zeta]}(G) \end{array}$$

Proposition (G. - Raggi-Cárdenas, 2025)

Let $H \leq G$ be a perfect subgroup. Then

$$e^G_{[H]} = \sum_{\substack{(K,\Phi)\in[\mathcal{D}^A_G], \\ O^s(K)=_G H}} e^{G,A}_{K,\Phi} \in B^A_{\mathbb{Q}[\zeta]}(G) \tag{5}$$

lies in $B^A(G)$. Furthermore, these idempotents account for all blocks of $B^A(G)$.

Theorem (G. - Raggi-Cárdenas, 2025)

If $H \leq G$ is a perfect subgroup, we have

$$B^A(G)e^G_{[H]} = \bigoplus_{\substack{(K,\phi)\in[\mathcal{M}^A_G], \\ O^s(K)=_G H}} \mathbb{Z}[K,\phi]_G e^G_{[H]}. \tag{6}$$

In particular,

$$B^A(G)e^G_{[1]} = \bigoplus_{\substack{(K,\phi)\in[\mathcal{M}^A_G], \\ K \text{ is solvable}}} \mathbb{Z}[K,\phi]_G. \tag{7}$$

Corollary

G is solvable if and only if 1 is primitive in $B^A(G)$.

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Grazie per l'attenzione!