Some results about the nilpotent graph of a finite group

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- $V = G$;
- $(g, h) \in E$ if and only if $g \neq h$ and $\langle g, h \rangle$ has the property $\mathcal{P}$.

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Theorem (Morgan, Parker, 2013)

If G is a finite group with trivial center then every connected component of the commuting graph of G has diameter at most 10.

A. Morgan, C. Parker, *The diameter of the commuting graph of a finite group with trivial center*, J. Algebra 393 (2013) 41-59.

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Theorem (Parker, 2013)

If G is a finite soluble group with trivial center then the commuting graph of G is disconnected or its diameter is at most 8.

C. Parker, *The commuting graph of a soluble group*, Bull. Lond. Math. Soc., **45** (4) 2013, pp. 839–848.

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Let G be a group. Then G is an A-group if the Sylow subgroups of G are all abelian.

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Theorem (Carleton, Lewis, 2025)

Let G be a solvable A-group such that $G/Z(G)$ is neither a Frobenius nor 2-Frobenius group. Then, the diameter of the commuting graph of G is at most 6.

R. Carleton, M. L. Lewis, *The commuting graph of a solvable A-group*, J. Group Theory, **28** (1) 2025, pp. 165–178.

Theorem (Burness, Lucchini, Nemmi, 2023)

Let G be a finite insoluble group. Then soluble graph of G is connected and its diameter is at most 5.

T. Burness, A. Lucchini, D. Nemmi, *On the soluble graph of a finite group*, J. Combin. Theory Ser. A, **194** 2023, 105708.

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V. Grazian, C. Monetta, *A conjecture related to the nilpotency of groups with isomorphic non-commuting graphs*, J. Algebra, **633** 2023, pp. 389–402.

H. Shahverdi, *Finite groups with isomorphic non-commuting graphs have the same nilpotency property*, J. Algebra, **642** 2024, pp. 60–64.

Connectivity of the nilpotent graph

Definition

C. Delizia, M. L. Lewis, M. Gaeta, C. Monetta, *Neighborhoods, connectivity, and diameter of the nilpotent graph of a finite group*, arxiv:2502.03308.

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Let G be a finite group. $\Gamma(G) = (V, E)$:

- $V = G \setminus Z_\infty(G)$;

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Let G be a finite group. $\Gamma(G) = (V, E)$:

- $V = G \setminus Z_\infty(G)$;
- $(g, h) \in E$ if and only if $g \neq h$ and $\langle g, h \rangle$ is nilpotent.

Results

Theorem

For any group G the number of connected components of $\Gamma(G)$ equals the number of connected components of $\Gamma(G/Z_\infty(G))$, and there is a correspondence between the connected components of $\Gamma(G)$ and $\Gamma(G/Z_\infty(G))$ that maps connected components of diameter 1 to connected components of diameter 0 or 1 and preserves the diameter of connected components whose diameter is greater than 1.

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Corollary

If G is a group, then $\Gamma(G)$ is connected if and only if $\Gamma(G/Z_\infty(G))$ is connected.

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If G is a solvable group with trivial center then $\Gamma(G)$ is disconnected if and only if G is a Frobenius group or a 2-Frobenius group.

Theorem

Let G be a non-nilpotent solvable group. Then $\Gamma(G)$ is disconnected if and only if $G/Z_\infty(G)$ is a Frobenius group or a 2-Frobenius group.

Diameter of the nilpotent graph

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Let G be a finite group. The commuting graph $\Gamma_{comm}(G) = (V, E)$:

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[Theorem 1.1] R. Carleton, M. L. Lewis, *The commuting graph of a solvable A-group*, J. Group Theory, **28** (1) 2025, pp. 165–178.

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Theorem (Morgan, Parker, 2013)

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Corollary

Let G be a non-nilpotent group. Then the connected components of $\Gamma(G)$ have diameter at most 10. Moreover, if G is solvable and $\Gamma(G)$ is connected, then $\mathrm{diam}(\Gamma(G)) \leq 8$.

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Proposition

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Theorem

Let G be a non-nilpotent $\{p, q\}$-group with trivial center. If $\Gamma(G)$ is connected, then $\mathrm{diam}(\Gamma(G)) \leq 6$.

Thank you!