The Pseudocentre of a Group

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Pseudocentre

Definition

Let G be a group. The *pseudocentre* of G is defined as the following subgroup:

$$\bigcap_{g \in G} C_G(g)^G$$

- This subgroup was first introduced by J.Weigold in 1973
- J. Weigold, "Pseudonilpotent Groups", University College, Cardiff, (1973).

Finite

- The only result Weigold proved in his paper is the following:

Theorem (J.Weigold, 1973)

Every non-trivial finite group has a non-trivial pseudocentre.

- In our work we have improved it:

Theorem

Let G be a non-trivial group. If N is a minimal normal subgroup of G, then $N \leq P(G)$.

Min-n

Corollary

Let G be a group. If G satisfies the minimal condition on normal subgroups, then $P(G)$ is not trivial.

Can this result be extended to classes of groups satisfying other finiteness conditions?

Max

Example

Let A be a free abelian group of rank 2 and consider the following matrix:

$$x = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Let G be the natural semidirect product of x by A. It's easy to see that $P(G) \leq A$. We observe also that x is self-centralizing.
Let $f(n)$ be the Fibonacci sequence. For the action of x and for properties of $f(n)$ we have that $[A, \langle x^{12n} \rangle] \leq A^{f(6n)}$. So we obtain that $P(G) \leq \bigcap_{n \in \mathbb{N}} [A, \langle x^{12n} \rangle] \leq \bigcap_{n \in \mathbb{N}} A^{f(6n)} = \{1\}$

- The following result proves that the previous counterexample is one of smallest Hirsch length.

Theorem

Let G be a group. If N is a non-trivial paracentral subgroup of G, then $N \cap P(G) \neq 1$. More precisely, if N is non-periodic, then $N^2 \leq P(G)$, while if N is periodic, then $Soc(N) \leq P(G)$.

A subgroup N of a group G is said *paracentral* if every subgroup of N is normal in G.

Nilpotency

Of course, the center of a group is contained in the pseudocenter. Moreover, if G is a nilpotent group of class 2, then the centralizers of elements of G are normal, so $P(G) = Z(G)$.

One might think that in groups of higher nilpotency class the pseudocenter always coincides with one of the terms of the upper central series. However, this is not the case.

Example

There exists a finite (metabelian) nilpotent group G of class 3 in which the pseudocentre does not coincide with any term of the upper and lower series of G.

Lemma

Let G be a group. If N is a normal subgroup of a group G, then $P(G)N/N \leq P(G/N)$.

The Lemma easily implies that $P(G) \leq Z_{n-1}(G)$ whenever G is a nilpotent group of class n.

In certain special cases the pseudocenter of a nilpotent group coincides with a term of the upper central series.

Unitriangular matrix groups

Theorem

Let p a prime and $\mathbb{F}_q$ the finite field of order p^m for some natural number m. Let $U := UT(n, \mathbb{F}_q)$ be the group of upper unitriangular matrix over $\mathbb{F}_q$ of degree n for some natural number n. Then $P(U) = Z_\rho(U)$, where $\rho = \lfloor n/2 \rfloor$.

Relations with central series

Theorem

Let G be a group. The followings hold:

- $P(G)' \leq G''$;
- if H/K is a section of G such that $[H, G'] \leq K$ then $[H, P(G)] \leq K$. In particular, we have that $C_G(G') \leq C_G(P(G))$ and $P(G) \leq C_G(Z_2(G))$.

By means of the previous theorem, we are able to derive the following results:

- If G' is hypercentral (resp. nilpotent), then $P(G)$ is hypercentral (resp. nilpotent).
- If G is hypercyclic (resp. supersoluble), then $P(G)$ is hypercentral (resp. nilpotent).

Let $\mathcal{X}$ be a subgroup theoretical property.

- A group G is said to be *hyper*-$\mathcal{X}$ if it has an ascending normal series with factors belonging to $\mathcal{X}$.
- A group G is said to be *hypo*-$\mathcal{X}$ if it has an descending normal series with factors belonging to $\mathcal{X}$.

Thanks to these properties, we expect that groups that coincides with their pseudocentre, may yield extremal cases under suitable nilpotency and solvability conditions.

Pseudocentral groups

Definition

A group G is said to be *pseudocentral* if coincides with its pseudocenter.

The class of pseudocentral groups is closed under homomorphic images and direct products.

Clearly, by considering simple groups, we can conclude that it is not closed under taking subgroups.

Corollary

Let G be a group. If G is pseudocentral, then $G' = G''$ and $Z_2(G) = Z(G)$.

Corollary

Within the class of pseudocentral groups, the notions of hypocentrality, hypoabelianity, hypercentrality and hypercyclicity are all equivalent to abelianity.

Of course every group can be embedded in a pseudocentral group.

Can any group be embedded in a group with trivial pseudocenter?

Free groups

Lemma

Free groups have trivial pseudocentre.

What can be stated in general about free products of groups?

Theorem

Let $G = H * K$ be the free product of the non-trivial groups H and K with $|H| > 2$. Then $P(G) = 1$.

Thanks to these results we can say that every group can also be embedded in a group with trivial pseudocenter.

Weigold in his work also defined pseudonilpotent groups starting from the pseudocentre in the same way as nilpotent groups are defined starting from the centre.

Definition

Let G be a group. Define $P^0(G) := G$ and $P^1(G) := P(G)$.Then for all $n \in \mathbb{N}$ then $P^{n+1}(G)$ is defined by:

$$P^{n+1}(G)/P^n(G) := P(G/P^n(G))$$

This is the *upper pseudocentral series* of G.
Then G is said to be *pseudonilpotent* if $G = P^n(G)$ for some natural number n.

Clearly every finite group is pseudonilpotent.

We have extended this result to the class of Černikov groups.

There is no relation between the pseudonilpotent class and the length of the upper/lower central series. In fact, for every positive integer $n \geq 2$, the dihedral group of order 2^{n+1} has nilpotency class n and pseudonilpotency class 2.

By means of suitable wreath products we built finite groups of arbitrarily pseudonilpotent length.

However, the class of pseudonilpotent groups is so vast that it is very difficult to say something about them.

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