Hopf–Galois structures on parallel extensions.

Advances in Group Theory and Applications

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Galois theory

L/K Galois, $G := \mathrm{Gal}(L/K)$, $H_1, H_2 \leq G$:

$$\begin{tikzcd}
L \arrow[d, dash] \arrow[dr, dash] \arrow[dd, bend right=50, dash, "G"'] & \\
L^{H_1} \arrow[d, dash] & L^{H_2} \arrow[dl, dash] \\
K &
\end{tikzcd}$$

$$\{\text{subgroups of } G\} \longleftrightarrow \{\text{intermediate fields } K \leq F \leq L\}$$

and

$$|G : H| = [L^H : K]$$

What about non-Galois extensions?

Hopf–Galois structure

Idea: replace G with $K[G]$...

Hopf–Galois structures

$K[G]$ has the structure of a K-Hopf algebra: $\forall g, h \in G$

- $\mu(g \otimes h) = gh$
- $\iota(1_K) = 1_G$
- $\Delta(g) = g \otimes g$
- $\varepsilon(g) = 1_K$
- $S(g) = g^{-1}$

and acts on L/K in a 'nice' way (via usual Galois action).

But there may be more than one K-Hopf algebra acting on L in a very similar way...

Hopf–Galois structures

A **Hopf–Galois structure** on L/K is a pair $(H, \cdot)$, where:

- H is a K-Hopf algebra
- $\cdot$ is an action of H on L
- & compatibility (so it behaves like $K[G]$)

But this can make sense for non-Galois extensions too!

Hopf–Galois structures

L/K separable

$$\begin{tikzcd}
E \arrow[d, dash] \arrow[d, bend left, dash, "G'"] \arrow[dd, bend right=60, dash, "G"'] \\
L \arrow[d, dash] \\
K
\end{tikzcd}$$

Byott (1996):

$$\{\text{HGS (of type } N\text{) on } L/K\} \longleftrightarrow M \overset{\text{trans.}}{\leq} N \rtimes \mathrm{Aut}(N) =: \mathrm{Hol}(N)$$

$$\text{s.t. } G \overset{\phi}{\cong} M \text{ with } \phi(G') = \mathrm{Stab}_M(1_N). \text{ Then } H = (E[N])^G.$$

Question

Let $H \leq G$, $|G : H| = n$.

$$\begin{tikzcd}
E \arrow[d, dash] \arrow[dr, dash] & \\
L \arrow[d, dash, "n"'] & L' = E^H \arrow[dl, dash, "n"] \\
K &
\end{tikzcd}$$

We say L'/K is a **parallel** extension to L/K.

If L/K admits a HGS, what about L'/K?

Example

$$
\begin{tikzcd}
E \arrow[d, dash, "2"'] \arrow[dr, dash, "2"] \arrow[dd, dash, bend right=60, "D_8"'] & \\
L \arrow[d, dash, "4"'] & L' = L^H \arrow[dl, dash, "4"] \arrow[dl, dash, bend left=60, "D_8/H"] \\
K &
\end{tikzcd}
$$

(e.g. $L/K = \mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$)

$\rightarrow$ L/K has

1 HGS of type C_4

1 HGS of type $C_2 \times C_2$

$\rightarrow$ L'/K has

3 HGSs of type C_4

1 HGS of type $C_2 \times C_2$

A problem in group theory!

$C = \mathrm{Core}_G(H) = \bigcap_{g \in G} gHg^{-1} \implies E^C/K$ is the Galois closure of L'/K:

$$\begin{tikzcd}
E \arrow[dr, dash] \arrow[dd, dash] & \\
& E^C \arrow[d, dash, "H/C"'] \arrow[ddl, dash, bend left, "G/C"] \\
L \arrow[d, dash, "n"'] & L' \arrow[dl, dash, "n"] \\
K &
\end{tikzcd}$$

So we can play the same game!

A problem in group theory!

Given

- $M \overset{\text{trans.}}{\leq} \mathrm{Hol}(N)$, $|N| = n = [L : K]$
- $H \leq M$ with $|M : H| = n$,

Does there exist a $J \overset{\text{trans.}}{\leq} \mathrm{Hol}(N)$ and an isom.

$$\phi : M/C \to J$$

with

$$\phi(H/C) = \mathrm{Stab}_J(1_N)?$$

Observation

Suppose $\exists\, \phi \in \mathrm{Aut}(G)$ s.t. $\phi(G') = H$ (e.g. H & G' are conjugate in G).

Then L/K admits a HGS (of type N) iff L'/K does.

(can also play the same game with any pair $H_1, H_2 \leq G$ with $[G : H_i] = n$)

Sub-question

Do there exist L/K admitting HGSs, with parallel extensions L'/K s.t. L'/K admits no HGS?

If "yes", we will say L/K (& corresponding G) admit the **parallel no-HGS property**.

Some results

- (A.D. 2025) $[L : K] = pq \implies$ all parallel ext's of L/K admit at least one HGS (of same type)
- Computer:

Degree	#Trans. sbgps	#Parallel no-HGS
8	148	8
12	134	23
24	4752	396
27	739	163

- (A.D. 2025) G (odd degree) has parallel no-HGS $\implies$ can fit into infinite family.

Future?

- Need to see more examples and build techniques
- Is there a relationship with how close L/K is to being Galois (& other properties)?
- Related structures (e.g. skew bracoids)?

Thank You!

Questions?