

Hopf–Galois structures on parallel extensions.

Advances in Group Theory and Applications

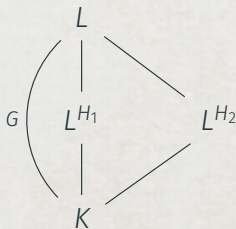
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Thursday 26th June 2025



Galois theory

L/K Galois, $G := \text{Gal}(L/K)$, $H_1, H_2 \leq G$:



$\{\text{subgroups of } G\} \longleftrightarrow \{\text{intermediate fields } K \leq F \leq L\}$

and

$$|G : H| = [L^H : K]$$

What about non-Galois extensions?

Hopf-Galois structure

Idea: replace G with $K[G]$...

Hopf-Galois structures

$K[G]$ has the structure of a K -Hopf algebra: $\forall g, h \in G$

➤ $\mu(g \otimes h) = gh$

➤ $\iota(1_K) = 1_G$

➤ $\Delta(g) = g \otimes g$

➤ $\varepsilon(g) = 1_K$

➤ $S(g) = g^{-1}$

and acts on L/K in a ‘nice’ way (via usual Galois action).

But there may be more than one K -Hopf algebra acting on L in a very similar way...

Hopf–Galois structures

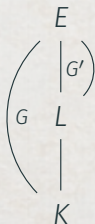
A **Hopf–Galois structure** on L/K is a pair $(H, \cdot)$, where:

- H is a K -Hopf algebra
- $\cdot$ is an action of H on L
- & compatibility (so it behaves like $K[G]$)

But this can make sense for non-Galois extensions too!

Hopf-Galois structures

L/K separable



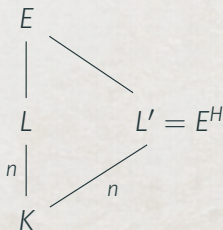
Byott (1996):

$$\{\text{HGS (of type } N) \text{ on } L/K\} \longleftrightarrow M \stackrel{\text{trans.}}{\leq} N \rtimes \text{Aut}(N) =: \text{Hol}(N)$$

s.t. $G \stackrel{\phi}{\cong} M$ with $\phi(G') = \text{Stab}_M(1_N)$. Then $H = (E[N])^G$.

Question

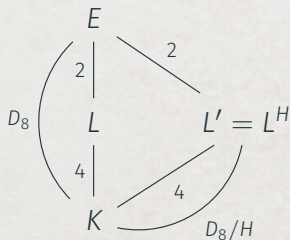
Let $H \leq G$, $|G : H| = n$.



We say L'/K is a **parallel** extension to L/K .

If L/K admits a HGS, what about L'/K ?

Example



(e.g. $L/K = \mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$)

→ L/K has

1 HGS of type C_4

1 HGS of type $C_2 \times C_2$

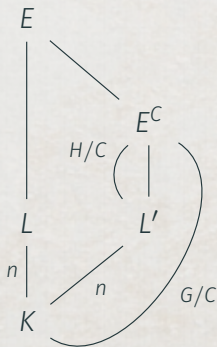
→ L'/K has

3 HGSs of type C_4

1 HGS of type $C_2 \times C_2$

A problem in group theory!

$C = \text{Core}_G(H) = \bigcap_{g \in G} gHg^{-1} \implies E^C/K$ is the Galois closure of L'/K :



So we can play the same game!

A problem in group theory!

Given

- $M \stackrel{\text{trans.}}{\leq} \text{Hol}(N)$, $|N| = n = [L : K]$
- $H \leq M$ with $|M : H| = n$,

Does there exist a $J \stackrel{\text{trans.}}{\leq} \text{Hol}(N)$ and an isom.

$$\phi : M/C \rightarrow J$$

with

$$\phi(H/C) = \text{Stab}_J(1_N)?$$

Observation

Suppose $\exists \phi \in \text{Aut}(G)$ s.t. $\phi(G') = H$ (e.g. H & G' are conjugate in G).

Then L/K admits a HGS (of type N) iff L'/K does.

(can also play the same game with any pair $H_1, H_2 \leq G$ with $[G : H_i] = n$)

Sub-question

Do there exist L/K admitting HGSs, with parallel extensions L'/K s.t. L'/K admits no HGS?

If “yes”, we will say L/K (& corresponding G) admit the **parallel no-HGS property**.

Some results

- (A.D. 2025) $[L : K] = pq \implies$ all parallel ext's of L/K admit at least one HGS (of same type)
- Computer:

Degree	#Trans. sbgps	#Parallel no-HGS
8	148	8
12	134	23
24	4752	396
27	739	163

- (A.D. 2025) G (odd degree) has parallel no-HGS $\implies$ can fit into infinite family.

Future?

- Need to see more examples and build techniques
- Is there a relationship with how close L/K is to being Galois (& other properties)?
- Related structures (e.g. skew bracoids)?

Thank You!

Questions?

