# Around the words

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*"Falsehood is never in words; it is in things"* (Italo Calvino - Invisible Cities)

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# Collaborators

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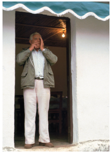
# Silbo Gomero

Article Talk

From Wikipedia, the free encyclopedia

**Silbo Gomero** (Spanish: *silbo gomero* [ˈsilβo ɣoˈmero], "Gomeran whistle"), also known as ***el silbo*** ("the whistle"), is a whistled register of Spanish that is used by inhabitants of La Gomera, in the Canary Islands. It was historically used to communicate across the deep ravines and narrow valleys that radiate through the island and enabled messages to be exchanged over a distance of up to five kilometres.[1] Its loudness causes Silbo Gomero to be generally used for public communication. Messages that are conveyed range from event invitations to public information advisories.[2] A speaker of Silbo Gomero is sometimes called a *silbador* ("whistler").

Silbo Gomero is a transposition of Spanish from speech to whistling. The oral phoneme-whistled phoneme substitution emulates Spanish phonology through a reduced set of whistled phonemes.[3] In 2009, UNESCO declared it a Masterpiece of the Oral and Intangible Heritage of Humanity.[4]

# What is a language?

## Philosophical

- For the pre-Socratics and Eastern theogonies there is identity between being and language ( Enūma eliš, Brhadāranyaka Upanisad, Rgveda- Heraclitus).
- Sophists conventionalism - Plato proponent of realism *Cratylus* - Aristotle *De Interpretatione* logic, categories, and moderate realism.
- The Scholastics of the high medieval period (Abelardus, Scotus, Roscellinus, Thomas Aquinas) *Quaestio de Universalibus*
- Biolinguistic-Psycholinguistics-Mentalism-Behaviorism.

## Scientific

- Aristotle *Organon*
- For the Stoics, logic was a wide field of knowledge that included the study of language, grammar, rhetoric and epistemology.
- Rationalism Descarts - Leibniz *De arte combinatoria*
- Contemporary Philosophy: Wittgenstein, Frege, Russell, de Saussure, Peano, Putnam, Chomsky (decline of linguistic behaviorism).

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Modern technology and necessity of programming → Formal languages:
Chomsky grouped logically possible phrase-structure grammar types into a series of four nested subsets (Chomsky hierarchy).
Relevant to theoretical computer science, programming language theory, compilers, and automata theory (Pānini , Hjelmslev).


PA PA MA MA
PA MA ....

PA = I want that

MA = I got it / I am happy

PA PA MA MA
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PA PA MA MA

PA MA . . . .

PA-> PA @ -> PA-PA@->.......->PA-PA-MA-MA-PA
@->PA@
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@->

**$\mathcal{L}$= PA{PA,MA}***

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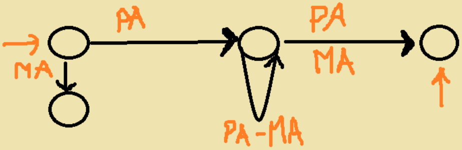

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A **language** $\mathscr{L}$ is a subset of $A^*$ or $A^\omega$.

# Languages-Grammars-Automata

A grammar is a set of rules we can use to generate a language

**Grammar**

- $\mathscr{M}$ finite set of *metasymbols*;
- $S \in \mathscr{M}$ the *start symbol*;
- $A$ finite set (disjoint from $\mathscr{M}$) of *symbols*;
- $P$ set of *production rules*

$$(\mathscr{M} \cup A)^* \mathscr{M} (\mathscr{M} \cup A)^* \to (\mathscr{M} \cup A)^*.$$

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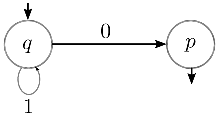

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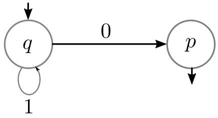


More complicated languages correspond to more sophisticated grammar rules and more complex machines.

# Regular languages

$$A = \{0, 1\} \qquad \mathscr{M} = \{S, M\}$$

$$S \to 1M \mid M \to 1M \mid M \to 0$$

$$S \longmapsto 1M \longmapsto 11M \longmapsto 111M \longmapsto 1110 \longmapsto 1110 \qquad \mathscr{L} = \{1^n 0 \mid n \geq 0\}$$

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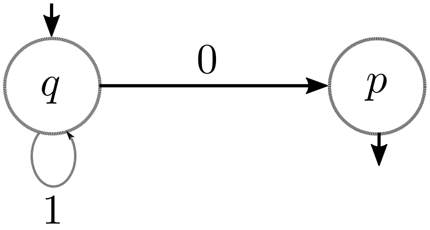
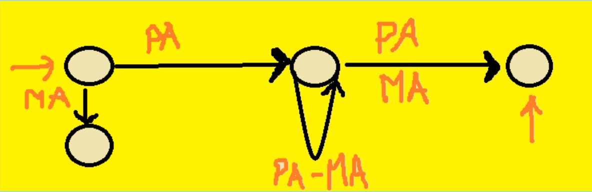
Regular languages generated by regular grammars or *equivalently* by finite state automata.






*If we want to construct a Language of words with (a part the first PA) the SAME number of PA & MA*

*like PA-MA-MA-PA-PA, PA-PA-PA-PA-MA-MA-MA etc*

*the structure of the automaton seems to be useless….*

*we need to count*

# Context-free languages

$$A = \{a, b\} \qquad \mathscr{M} = \{S, M\}$$

$$S \to aMb \mid M \to aMb \mid M \to \varepsilon$$

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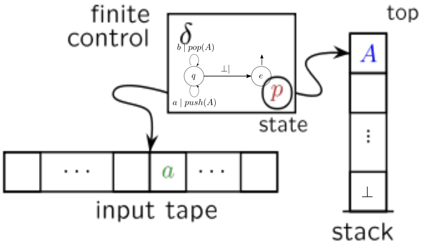
Context-free languages generated by context-free grammars or *equivalently* by pushdown automata



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→ *The word problem*
(Max Dehn. Über unendliche diskontinuierliche Gruppen. *Math. Ann.*, 71:116- 144, 1911.)
Is there an algorithm that determines whether a given $w$ in the generators of G represents the identity of $G$?

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To a finitely generated group $G = \langle A \rangle$ we associated the usual coding map

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Associated to our group we have the so-called co-**word-problem language**

$$coWP(G, A) := \{w \in A^* \mid w\pi \neq 1_G\}.$$

We say that a group is co-$\mathscr{C}$ if a co-word problem is $\mathscr{C}$ as a language.

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**Question**

*In general, can the structure of the group be captured by the complexity of its WP?*

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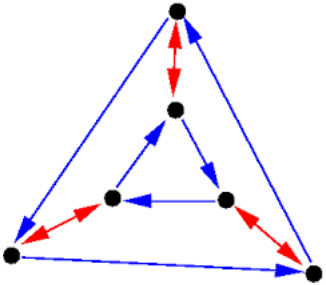
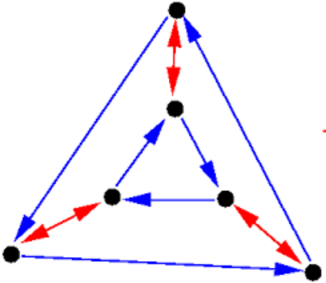
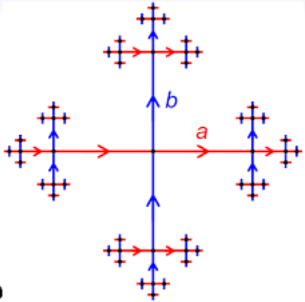
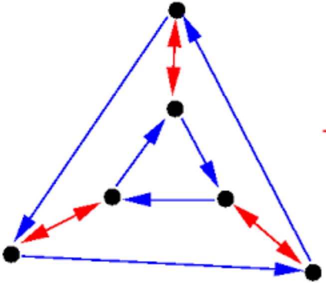
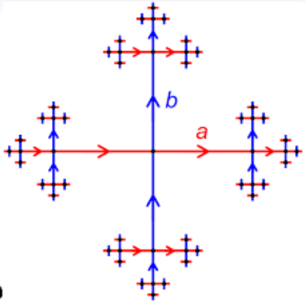
In terms of Cayley graphs of $G$ wrt the symmetric gen. set $S$ $g \to gs$ :
finite $\leftrightarrow$ regular, quasi-isometric to a tree $\leftrightarrow$ context-free.






b
a

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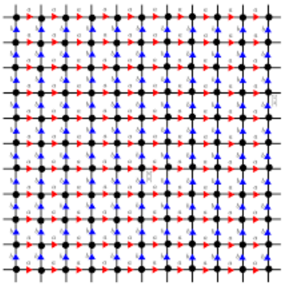

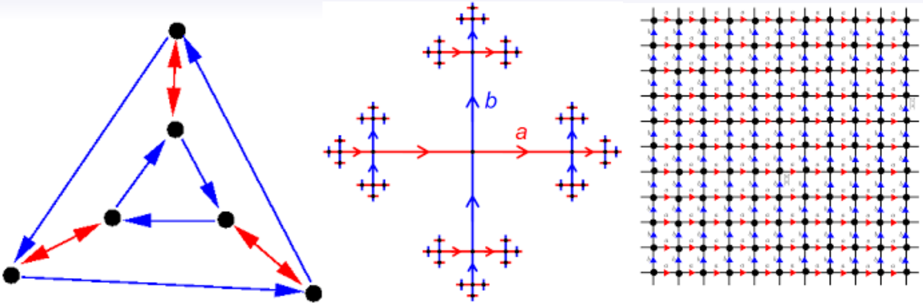


*Loops in the Cayley graph correspond to WP language.*

# ET0L languages

Inspired by the biologist Lindenmayer to model plant growth

In the case of **Extended Tabled 0-interaction Lindenmayer**(ET0L), we have the following:

**Alphabet:** $A$
**Alphabet of non-terminals:** $P$
**Starting symbol:** $S$
**Tables:** $T = \{\tau_1, \tau_2, \ldots, \tau_k\}$
**Rational control:** $\mathscr{R} \subseteq T^*$ is a regular language of tables

A table $\tau$ is a set of rules one applies simultaneously. A rule maps a non-terminal to an element in $(A \cup P)^*$.

# ET0L Example

$$A = \{a, b\} \qquad P = \{S, p\}$$

Our tables are $T = \{\tau_1, \tau_2, \tau_3\}$ where

$$\tau_1 : \begin{cases} S \mapsto Sap \\ p \mapsto p \end{cases} \qquad \tau_2 : \begin{cases} S \mapsto S \\ p \mapsto ap \end{cases} \qquad \tau_3 : \begin{cases} S \mapsto \varepsilon \\ p \mapsto b \end{cases} \qquad \mathscr{R} = T^*$$

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Then, the language is

$$L = \{a^{n_1} b a^{n_2} b a^{n_3} b \cdots b a^{n_k} b \mid 1 \leq n_1 \leq n_2 \leq n_3 \leq \cdots \leq n_k\}.$$

The above is called the **language of partitions**.

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$$S \xrightarrow{\tau_1} Sap \xrightarrow{\tau_2} Saap \xrightarrow{\tau_1} Sapaap \xrightarrow{\tau_1} Sapapaap \xrightarrow{\tau_3} ababaab$$

We know that

$$\text{Context-free} \subset ET0L \subset \textit{indexed} \subset \text{Context-sensitive}$$

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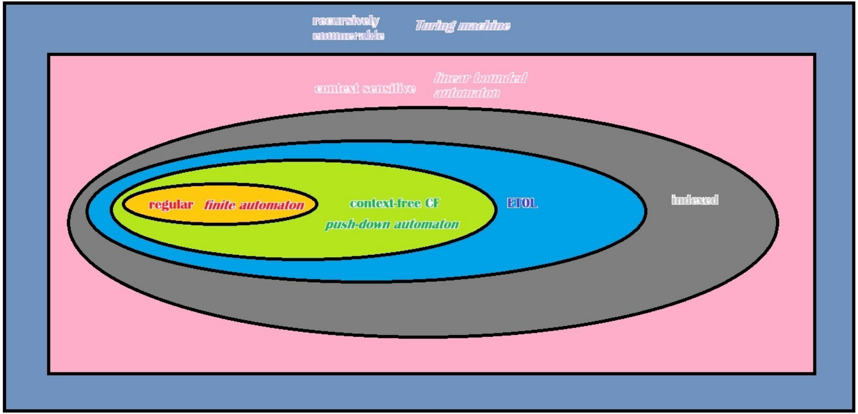
Context-free ⊂ $\textcolor{blue}{ET0L}$ ⊂ *indexed* ⊂ Context-sensitive



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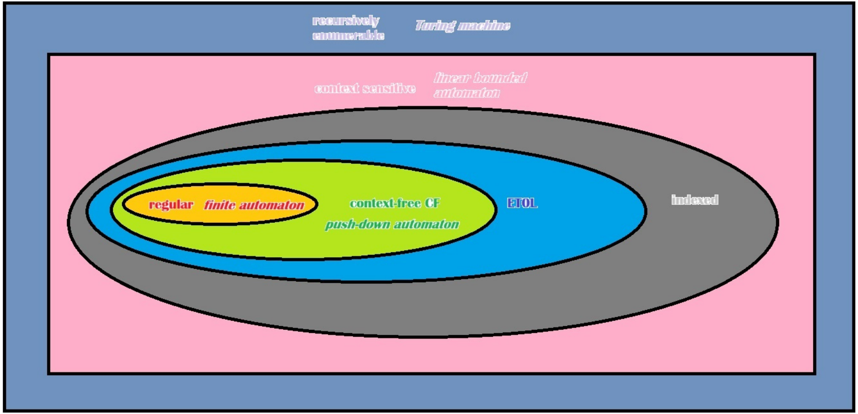


So far, only regular and CF languages we have an analog group characterization (finite-virtually free).

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- Characterizations in terms of co-WP ($\rightarrow$ Lehnert's Conjecture).

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**Conjecture (Ciobanu, Elder, Ferov)**

*A group with ET0L WP is virtually free.*

**Conjecture (Holt, Rees)**

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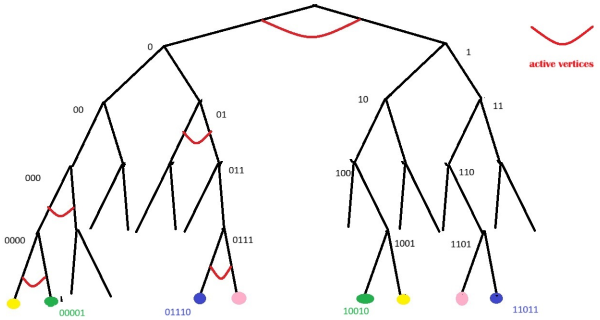

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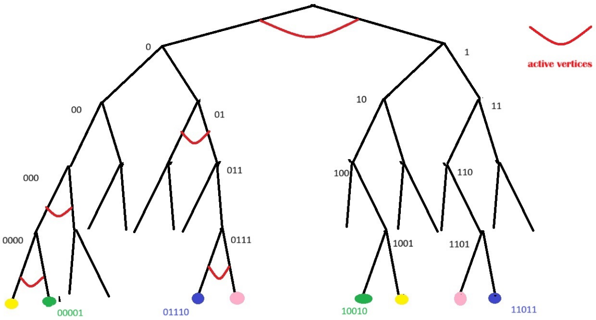
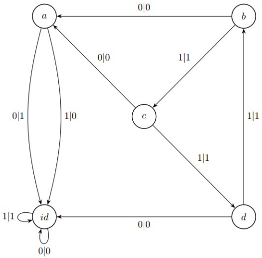
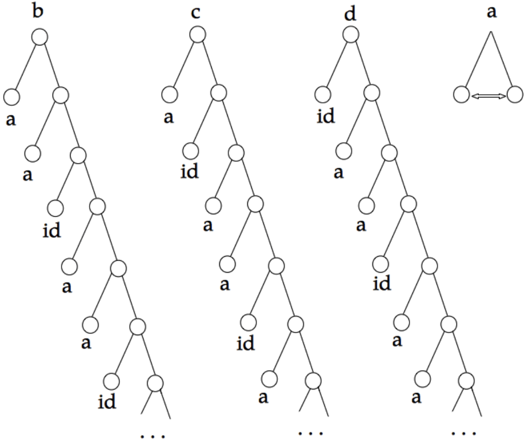



To be a bounded automaton group, *the number of active vertces is bounded* for every level.

The prototype is the Grigorchuk group.




The automaton and the automorphisms generating the Grigorchuk group

## Properties

The Grigorchuk group $G$ has the following properties:

- It is finitely generated, infinite, non-finitely presented, residually finite, and torsion (*BURNSIDE PROBLEM*)
- $G$ admits an $L-$presentation (*LYSIONOK*)
- $G$ has intermediate growth (*MILNOR PROBLEM*)
- $G$ is amenable but non-elementary amenable (*DAY PROBLEM*)
- $G$ is just infinite
- $G$ has solvable word problem and solvable coniugacy problem
- $G$ contains a finite index subgroup $H$ such that $H \times H \leq H$ (*BRANCH GROUP*)
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## Properties

The Grigorchuk group $G$ has the following properties:

- It is finitely generated, infinite, non-finitely presented, residually finite, and torsion (*BURNSIDE PROBLEM*)
- $G$ admits an $L$–presentation (*LYSIONOK*)
- $G$ has intermediate growth (*MILNOR PROBLEM*)
- $G$ is amenable but non-elementary amenable (*DAY PROBLEM*)
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  V is an example of finitely presented, infinite, simple group

# The generalized WP

The action of a bounded automaton group on the boundary of the tree gives rise to the language of the stabilizers of boundary points. I.e. Given an element of the boundary of the tree (an infinite word), among all possible strings in the generators of a bounded automaton group we consider only those that fix such a vertex.

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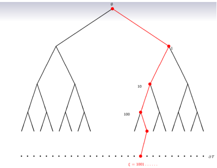

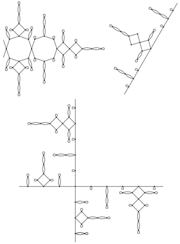

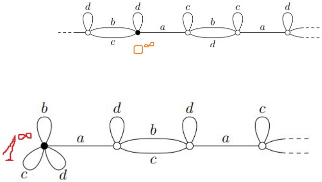

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## Theorem (B-D-M-N-P-R)

*Languages of the boundary stabilizers of groups generated by* **bounded automata** *are (co-)ET0L.*

A. Bishop, D. D'Angeli, F. Matucci, T. Nagnibeda, D. Perego, E. Rodaro, On the subgroup membership problem in bounded automata groups, submitted

# Characterizations in terms of co-WP (Lehnert's Conjecture)

- Grigorchuk group is co-ET0L (and $\pi M$ with $M$ compact 3-manifold admits a normal form which is ET0L) *(Ciobanu, Elder, Ferov)*.
- In general, finitely generated bounded automata groups are co-ET0L *(Bishop, Elder)*.

- Set of solutions of a system of equations in many classes of groups E(D)T0L

$$\begin{cases} f(x) = 1_G \\ g(x) = 1_G \end{cases} \qquad g_1, g_2 \in G \text{ are solutions} \Rightarrow g_1 \# g_2 \in L.$$

**Authors: Ciobanu, Diekert, Duncan, Elder, Evetts, Holt, Jez, Kufleitner, Levine, Rees*

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Very important examples, since....

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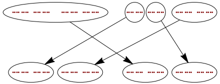

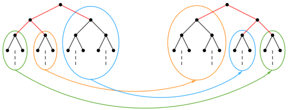

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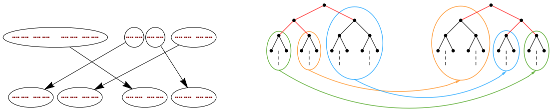

## Remark

- Grigorchuk group is f.g. infinite torsion and any f.g. torsion subgroup of Thompson $V$ is finite (Röver);
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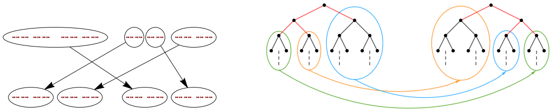

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**Goal:** prove that Grigorchuk or $\mathbb{Z} * \mathbb{Z}^2$ is co-CF.

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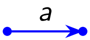


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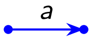


**Involutive** every $e \in E$ has an opposite
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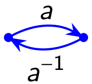


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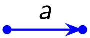


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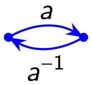


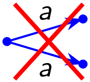


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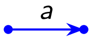


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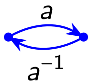
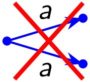
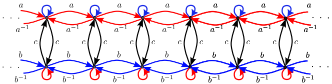



Deterministic at most one for each label

**Inverse graph** = all of the above + connected



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# Transition groups

An inverse graph $\Gamma$ is **complete** when for every vertex $x$ and every $a \in A$ there exists an edge $x \xrightarrow{a} y$.

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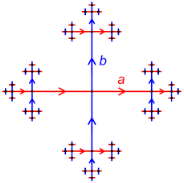
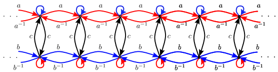
In this way, any $a \in A$ induces a permutation $\sigma_a$ on the vertices. The **transition group** of $\Gamma$ is

$$G(\Gamma) := \langle \sigma_a \mid a \in A \rangle.$$

**Examples:**

1. The transition group of a Cayley graph is the group itself.
2. $G($$)$ is isomorphic to $\mathbb{Z}^2 \rtimes C_2$.



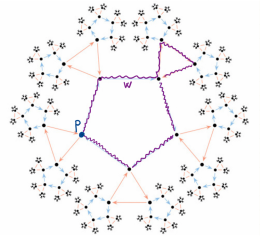



# Context-free inverse graphs

## Definition

An inverse graph $\Gamma$ on the symmetric alphabet $A$ is called context-free if the language of the closed walks on some root $p$ is a context-free language:

$$L(\Gamma, p) = \{w \in A^* : p \xrightarrow{w} p\} \text{ is a context-free language}$$



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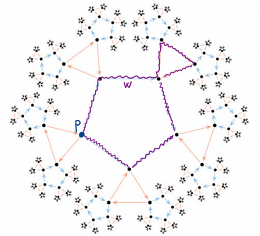
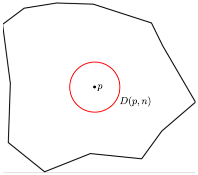
- For a Cayley graph $L(\Gamma, p) = WP(G; A)$, so this definition extends the group case.



Erasing from $\Gamma$ a disk $D(p, n)$ centered at $p$ of radius $n$ we obtain some connected components called end-cones.

## Theorem (Ceccherini Silberstein and Woess, Rodaro)

*$\Gamma$ is a context-free graph iff there are finitely many end-cones up to end-isomorphism: an isomorphism of labeled digraphs $\psi$ preserving the frontier points.*



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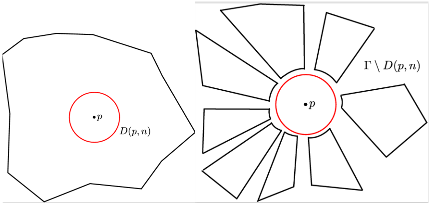

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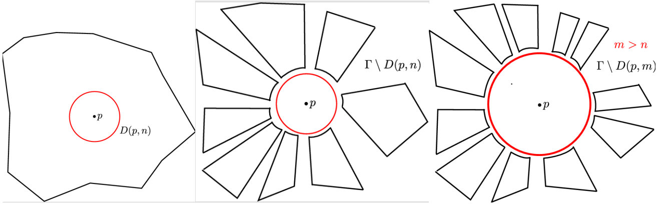

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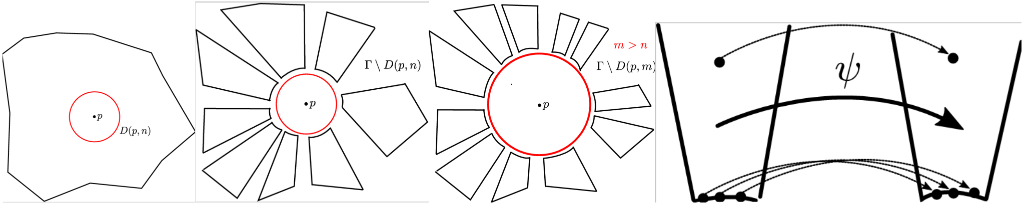
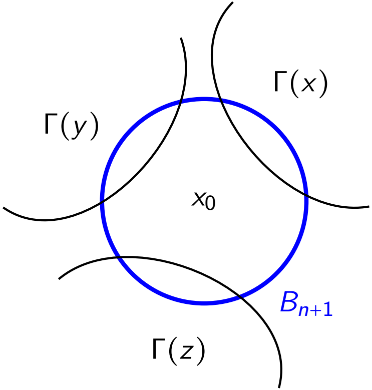



The transition group of a context-free inverse graph is denoted by **CF-TR**

Context-free graphs: recognize CF languages and have finitely many end cone types (Ceccherini Silberstein and Woess, Rodaro)

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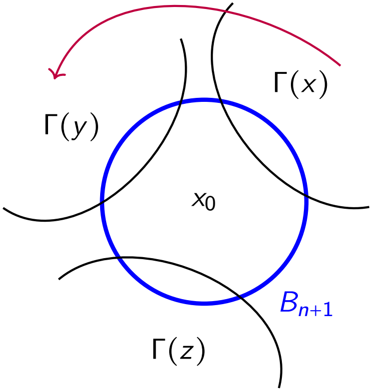


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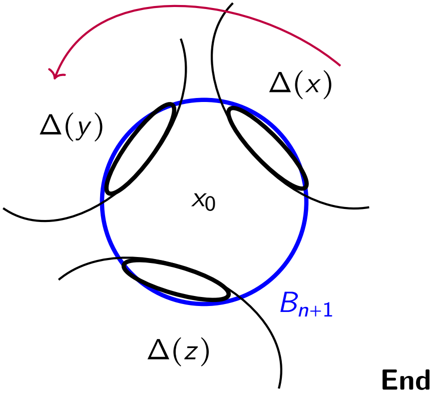


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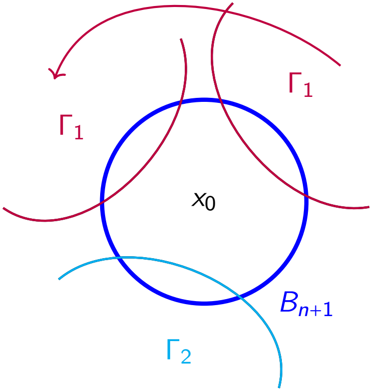


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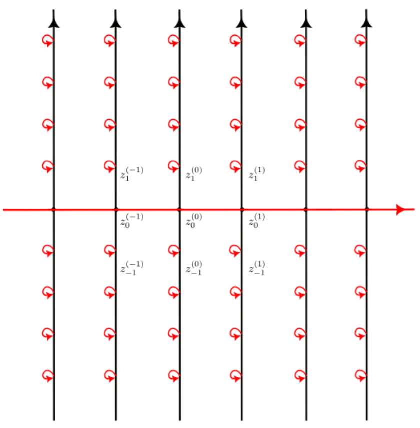
Equiv. classes $\Gamma_0, \Gamma_1, \ldots, \Gamma_m, \ldots$

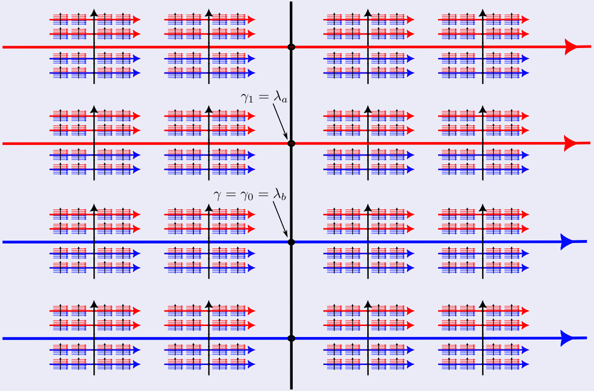
## Theorem

*The following transition group contains $\mathbb{Z}^\infty$, thus in particular it is not* **poly-CF**.



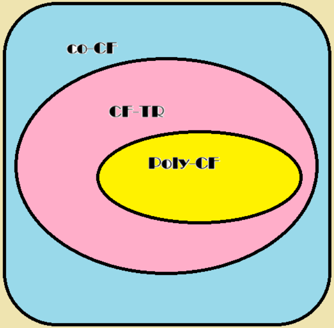
## Theorem

*The following transition group is not residually finite.*



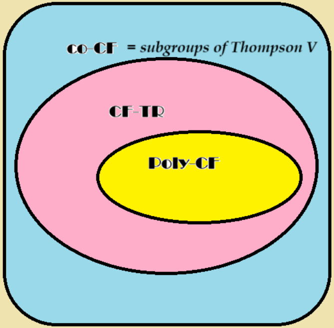
## Theorem (D-M-P-R)

- **CF-TR** *is a subclass of* **co-CF**.
- *A group $G$ belongs to* **CF-TR** *with respect to a connected graph iff it has a core-free subgroup $H$ whose Schreier graph is a context-free. (core-free=trivial normal subgroup in $H$)*
- **CF-TR** *is closed by taking f.g. subgroups, direct products and finite index overgroups. In particular, groups that are virtually subgroups of the direct products of free groups are* **CF-TR**
- *They are never torsion (unless it is finite), in particular, Grigorchuk's group is not* **CF-TR**.
- *Checking if an element has torsion is decidable.*
- *Thomson $\mathbb{F}$ is* **CF-TR**.
- *If Brough conjecture holds (Poly-CF=virtually a finitely generated subgroup of the direct product of free groups) then Poly-CF is properly contained in* **CF-TR**.

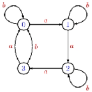
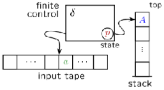



If the conjecture of Tara Brough holds:

*Poly-CF=virtually subgroups direct product of free groups*




If Brough conjecture and Lehnert conjecture both hold

| LANGUAGES | MACHINES | WP GROUPS |
|---|---|---|
| REGULAR | finite automata   | FINITE GROUPS |
| CONTEXT-FREE | push-down automata   | VIRTUALLY FREE |
| POLY-CF | | VIRTUALLY SGR DIRECT PRODUCT FREE GPS ? |
| ET0L | | VIRTUALLY FREE? ? |
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1. Find a CF-TR that is not in $V$.
2. Can we construct a product which group is $\mathbb{Z} * \mathbb{Z}^2$? (Lehnert)
3. Clarify the exact relationship among the classes co-CF, CF-TR, subgroups of V, poly-CF.

D. D'Angeli, F. Matucci, D. Perego, E. Rodaro, Context-free graphs and their transition groups, submitted

*Grazie*

'A bona parola mògne, 'a trista pògne