Ordering groups and the Identity Problem

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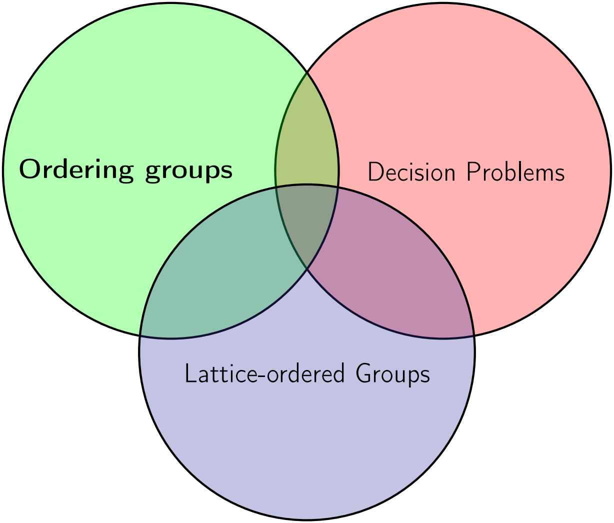
(with Corentin Bodart and George Metcalfe)

AGTA 2025, June 27, Napoli


Ordering groups
Decision Problems
Lattice-ordered Groups

I. Basics on orderability

Left-orderable and biorderable groups

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- A group G is *bi-orderable* if there exists a total order $<$ on G st

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(b) A left-orderable group is torsion-free.

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(b) $\mathbb{Z}^n$, $n \geq 2$, is bi-orderable. It has uncountably many orders.

(c) *Torsion-free nilpotent groups are bi-orderable.*

(d) Free groups and surface groups are bi-orderable.

(e) Braid groups are left-orderable but not bi-orderable.

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Positive cones

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Rmk: The positive cone $P = P_+$ is a *subsemigroup* of G that omits e_G:

$$a, b \in P \text{ implies } ab \in P, \text{ and } e_G \notin P.$$

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- For any subsemigroup $P \subset G \setminus \{e_G\}$, the binary relation $<_P$ on G defined by

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- KEY POINT: can study orders via subsemigroups.

II. Questions on orderability

Extending orders: Left-order and Bi-order Problems

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Def: A group G is *fully bi-orderable* if every partial bi-order on G extends to a bi-order.

- In a fully bi-orderable group Question 2 can be reduced to asking if a set S can be extended to a partial bi-order.

Full orderability in nilpotent groups

Theorem (Malcev, 1951)

Every torsion-free nilpotent group is fully bi-orderable.

Theorem (Rhemtulla, 1972)

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Questions 1 & 2 can be phrased as: *Does a finite set extend to a partial order?*

Groups that are not fully orderable

- (Rhemtulla)

$$BS(1,-1) = \langle a, b \mid a^{-1}ba = b^{-1} \rangle$$

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- Torsion-free groups containing free groups: e.g. surface groups are left-orderable but not fully left-orderable.

III. Membership problems for subsemigroups

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Membership problems for subsemigroups

- Let S^+ be the subsemigroup generated by S.
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Identity Problem. Is there an algorithm that can decide if, for $S \subset G$

$$\boxed{e_G \in S^+?}$$

Normal Identity Problem. Is there an algorithm that can decide if, for $S \subset G$

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Connecting the orderability problems and membership problems

Proposition

Let G be a group.

1. *If G is fully left-orderable, then the Identity Problem and the Left-order Problem for G are complements: a finite subset S of G extends to a left-order on G if and only if $e_G \notin S^+$.*

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2. *If G is fully bi-orderable, then the Normal Identity Problem and the Bi-order Problem for G are complements: a finite subset S of G extends to a bi-order on G if and only if $e_G \notin S^\circ$.*

III. Ordering and membership problems in nilpotent groups

Orderability in nilpotent groups

Theorem (Malcev, 1951)

Every torsion-free nilpotent group is fully bi-orderable.

Theorem (Rhemtulla, 1972)

Every torsion-free nilpotent group is fully left-orderable.

The Left-order Problem (Q1) is equivalent to the Identity Problem.

The Identity Problem

The **Identity Problem** has been studied in

- various matrix groups (Blondel, Cassaigne, Karhumäki, 2004 - 2005, Bell, Hirvensalo, Potapov, 2010 - 2017, Dong 2022, 2023),
- nilpotent groups of class ≤ 10 (Dong, 2022),
- some metabelian groups, wreath products (Dong 2023).

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The **Identity Problem** is a specific case of the

- **Monoid Membership Problem**, where for arbitrary $S \subset G$ semigroup and $g \in G$, we ask if
$$g \in S?$$
- **Fixed-target Membership Problem** (Gray & Nyberg-Brodda), where for fixed $g_0 \in G$ and arbitrary semigroup $S \subset G$ we ask if
$$g_0 \in S?$$

The Identity Problem: results

Theorem (BCM, 2024)

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Shafrir: independent proof (2024)

Bodart & Dong: generalisation (2024)

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Subgroup Problem $\implies$ Identity Problem

The main technical result

Proposition (BCM, 2024)

Let G be a finitely generated infinite nilpotent group, and consider the map $\pi\colon G \to G/I_G([G,G]) \simeq \mathbb{Z}^r$ with $r \in \mathbb{N}^+$.

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1. *$Conv(\pi(S)) \subseteq \mathbb{R}^r$ contains a ball $B(\mathbf{0}, \varepsilon)$ for some $\varepsilon > 0$.*
2. *For every non-zero linear form $f : \mathbb{R}^r \to \mathbb{R}$, there exists $s \in S$ such that $f(\pi(s)) < 0$. If S is finite, we can restrict to rational linear forms.*
3. *For every non-zero homomorphism $\phi\colon G \to \mathbb{R}$, there exists $s \in S$ such that $\phi(s) < 0$.*
4. *The subsemigroup S^+ is a finite-index subgroup of G.*

Compare and contrast

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1. The **Identity Problem** is decidable in nilpotent groups. (BCM, Shafrir, 2024)
2. The **Fixed-target Membership** Problem is undecidable in nilpotent groups. (BCM, 2024)
3. The **Submonoid Membership Problem** is undecidable for nilpotent groups. (Roman'kov, 2022)

V. Bi-orderability

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$\implies$ *The Left-order Problem in tf nilpotent groups is decidable.*

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Theorem (BCM, 2024)

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$\Longrightarrow$ *The Left-order Problem in tf nilpotent groups is decidable.*

Question: What about the Bi-order Problem (Question 2)?

Bi-orderability Problems

Proposition (BCM, 2024)

The Normal Identity and Bi-order Problems are decidable for any finitely generated free nilpotent group of class c.

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The Normal Identity and Bi-order Problems are decidable for any finitely generated free nilpotent group of class c.

- This follows from results (Kopytov '82, Colacito & Metcalfe '19, Metcalfe & Paoli &Tsinakis '23) about the equational theory and the Word Problem of class c nilpotent lattice-ordered groups.

Lattice-ordered groups

A *lattice-ordered group* (*ℓ-group*) is an algebraic structure $(L, \wedge, \vee, \cdot, {}^{-1}, e)$ such that

- $(L, \cdot, {}^{-1}, e)$ is a group and
- $(L, \wedge, \vee)$ is a lattice (partially order set where every pair of elements has a join $\vee$ and a meet $\wedge$) with

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Since

$$x(y \wedge z)t = xyt \wedge xzt$$

holds, $a \leq b$ implies $cad \leq cbd$ for all $a, b, c, d \in L$.

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Open Questions:

- Is the Bi-order Problem decidable in all tf nilpotent groups?
- Is the Normal Identity Problem decidable in nilpotent groups?

Vi. Groups that are not fully orderable

The Order Problems in groups that are NOT fully orderable

- Question 1.

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The Order Problems in groups that are NOT fully orderable

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- The Normal Identity Problem in free groups is equivalent to:

> Is the equational theory of totally ordered ℓ-groups decidable?

VII. Fully orderable groups, part 2

Other examples of fully orderable groups

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- However, $BS(1, m)$ are fully left-orderable.

  (Rivas, 2010 + Deroin, Navas, Rivas, 2014)

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- $BS(1, m)$ are a subclass of G_λ, which are fully left-orderable (BCM, 2024)

  For each $\lambda \in \mathbb{Q}_{>1}$, the groups $G_\lambda \leq \mathrm{Aff}_+(\mathbb{Q})$, are defined as

$$G_\lambda = \left\{ x \mapsto \lambda^n x + c \,\middle|\, n \in \mathbb{Z},\ c \in \mathbb{Z}\left[\lambda, \lambda^{-1}\right] \right\} \simeq \mathbb{Z}\left[\lambda, \lambda^{-1}\right] \rtimes_\lambda \mathbb{Z}.$$

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- Are Questions 1 and 2 decidable in metabelian groups?

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BUT the Normal Identity Problem is not known in any of the groups above.

Grazie!