Rational Representations and Rational Group Algebras of Metacyclic p-groups

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Outline of the Presentation

- Semisimple Rings and Wedderburn-Artin Theorem
- Studies on the Wedderburn Decomposition of Rational Group Algebras
- Rational Group Algebras of Split and Ordinary Metacyclic p-Groups
- Rational Matrix Representations of p-Groups
- Rational Representations of Ordinary Metacyclic p-Groups

Group Algebra

- **Group algebra.** Let G be a group and let $\mathbb{F}$ be field. Then the group algebra (ring) of G over $\mathbb{F}$ is given by

$$\mathbb{F}G = \left\{ \sum a_g g : a_g \in \mathbb{F} \right\}.$$

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- **Semisimple ring.** A ring R is said to be semisimple if it is semisimple as left R-module.
- **Maschke's theorem.** Let G be a finite group and $\mathbb{F}$ a field whose characteristic does not divide the order of G. Then $\mathbb{F}G$ is semisimple.
- In particular, the rational group algebra of a finite group G, that is, $\mathbb{Q}G$ is semisimple.

Theorem (Wedderburn-Artin theorem)

Let R be an (Artinian) semisimple ring. Then

$$R \cong \bigoplus_i M_{n_i}(D_i),$$

where each $M_{n_i}(D_i)$ is a full matrix ring over a division ring D_i, with n_i and D_i uniquely determined up to permutation.

[1] J. H. M. Wedderburn, On hypercomplex numbers, *Proc. London Math. Soc.* **6** (1908), 77–118.

[2] E. Artin, Zur Theorie der hyperkomplexen Zahlen, *Abh. Math. Sem. Univ. Hamburg* **5** (1927), 251–260.

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- Joseph Wedderburn[1] proved the theorem for finite-dimensional algebras in 1908. Nineteen years later, Emil Artin[2] extended it to rings satisfying chain conditions.
- In particular, any simple left or right Artinian ring R is isomorphic to $M_n(D)$ for a unique n and division ring D.

[1] J. H. M. Wedderburn, On hypercomplex numbers, *Proc. London Math. Soc.* **6** (1908), 77–118.

[2] E. Artin, Zur Theorie der hyperkomplexen Zahlen, *Abh. Math. Sem. Univ. Hamburg* **5** (1927), 251–260.

- **Perlis-Walker's Theorem**[3]**.** Let G be a finite abelian group of order n, and let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \nmid n$. Then

$$\mathbb{F}G \cong \bigoplus_{d|n} a_d \, \mathbb{F}(\zeta_d),$$

where ζ_d is a primitive d-th root of unity, $a_d = \frac{n_d}{[\mathbb{F}(\zeta_d):\mathbb{F}]}$ and n_d is the number of elements of order d in G.

[3] S. Perlis and G. L. Walker, Abelian group algebras of finite order, *Trans. Amer. Math. Soc.* **68**(3) (1950), 420–426.

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- $\mathbb{C}D_8 \cong \mathbb{C}Q_8 \cong 4\mathbb{C} \oplus M_2(\mathbb{C})$.

- $\mathbb{Q}D_8 \cong 4\mathbb{Q} \oplus M_2(\mathbb{Q})$.

- $\mathbb{Q}Q_8 \cong 4\mathbb{Q} \oplus \mathbb{H}(\mathbb{Q})$.

[3] S. Perlis and G. L. Walker, Abelian group algebras of finite order, *Trans. Amer. Math. Soc.* **68**(3) (1950), 420–426.

Why Is the Study of the Wedderburn Decomposition of Rational Group Algebras Important?

The Wedderburn decomposition of the rational group algebra of a finite group G has gained significant attention due to its implications for algebraic structures. For example:

- It helps describe the automorphism group of $\mathbb{Q}G$ (see Herman[4] and Olivieri et al.[5]).

[4] A. Herman, On the automorphism group of rational group algebras of metacyclic groups, *Comm. Algebra* **25**(7) (1997), 2085–2097.

[5] A. Olivieri, Á. del Río and J. J. Simón, The group of automorphisms of a rational group algebra of a finite metacyclic group, *Comm. Algebra* **34**(10) (2006), 3543–3567.

[6] E. Jespers and Á. del Río, A structure theorem for the unit group of the integral group ring of some finite groups, *J. Reine Angew. Math.* **521** (2000), 99–117.

[7] J. Ritter and S. K. Sehgal, Construction of units in integral group rings of finite nilpotent groups, *Trans. Amer. Math. Soc.* **324**(2) (1991), 603–621.

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- It helps describe the automorphism group of $\mathbb{Q}G$ (see Herman[4] and Olivieri et al.[5]).
- It contributes to analyzing the unit group of $\mathbb{Z}G$ (see Jespers and del Río[6] and Ritter and Sehgal[7]).

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- The Wedderburn decompositions of rational group algebras for various families of groups have been extensively studied by Bakshi et al.[8]; Bakshi and Maheshwary[9]; Jespers et al.[10]; Jespers et al.[11]; Olteanu[12]; and others.

[8] G. K. Bakshi, J. Garg and G. Olteanu, Rational group algebras of generalized strongly monomial groups: primitive idempotents and units, *Math. Comp.* **93**(350) (2024), 3027–3058.

[9] G. K. Bakshi and S. Maheshwary, The rational group algebra of a normally monomial group, *J. Pure Appl. Algebra* **218**(9) (2014), 1583–1593.

[10] E. Jespers, G. Leal and A. Paques, Central idempotents in rational group algebras of finite nilpotent groups, *J. Algebra Appl.* **2**(1) (2003), 57–62.

[11] E. Jespers, G. Olteanu and Á. del Río, Rational group algebras of finite groups: from idempotents to units of integral group rings, *Algebr. Represent. Theory* **15**(2) (2012), 359–377.

[12] G. Olteanu, Computing the Wedderburn decomposition of group algebras by the Brauer-Witt theorem, *Math. Comp.* **76**(258) (2007), 1073–1087.

- The Wedderburn decompositions of rational group algebras for various families of groups have been extensively studied by Bakshi et al.[8]; Bakshi and Maheshwary[9]; Jespers et al.[10]; Jespers et al.[11]; Olteanu[12]; and others.
- These studies employ Shoda pair theory to identify the simple components of rational group algebras.
- Although the WEDDERGA package is developed in GAP based on these studies, exact computations remain challenging in practice, especially for groups of large order.

[8] G. K. Bakshi, J. Garg and G. Olteanu, Rational group algebras of generalized strongly monomial groups: primitive idempotents and units, *Math. Comp.* **93**(350) (2024), 3027–3058.

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Theorem (Choudhary and Prajapati[13])

Let p be an odd prime and let ζ_d be a primitive d-th root of unity. Consider a finite non-abelian split metacyclic p-group

$$G = \langle a, b \mid a^{p^n} = b^{p^m} = 1, bab^{-1} = a^{1+p^{n-s}} \rangle,$$

where $n \geq 2, m \geq 1$ and $1 \leq s \leq \min\{n-1, m\}$. Then we have the following.

1. **Case $(n - s \geq m)$.** *In this case,*

$$\mathbb{Q}G \cong \mathbb{Q} \bigoplus_{\lambda=1}^{m} (p^{\lambda} + p^{\lambda-1}) \mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{\lambda=m+1}^{n-s} p^{m} \mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{t=1}^{s} p^{m-t} M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}})).$$

[13] R. K. Choudhary and S. K. Prajapati, A combinatorial formula for the Wedderburn decomposition of rational group algebras of split metacyclic p-groups, *J. Algebra Appl.* (2026) 2650068 (17 pages). 10.1142/S0219498826500684

2. **Case $(n-s<m)$.** Suppose $m=(n-s)+k$. Then we have following two sub-cases.

   2.1. **Sub-case $(k\le s)$.** In this sub-case,

$$\mathbb{Q}G \cong \mathbb{Q}\bigoplus_{\lambda=1}^{n-s}(p^{\lambda}+p^{\lambda-1})\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{\lambda=n-s+1}^{m} p^{n-s}\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{t=1}^{k-1} p^{n-s}M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}}))$$
$$\bigoplus_{t=1}^{k-1}\bigoplus_{\lambda=n-s+1}^{m-t}(p^{n-s}-p^{n-s-1})M_{p^t}(\mathbb{Q}(\zeta_{p^{\lambda}}))\bigoplus_{t=k}^{s}p^{m-t}M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}})).$$

   2.2. **Sub-case $(k>s)$.** In this sub-case,

$$\mathbb{Q}G \cong \mathbb{Q}\bigoplus_{\lambda=1}^{n-s}(p^{\lambda}+p^{\lambda-1})\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{\lambda=n-s+1}^{m} p^{n-s}\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{t=1}^{s} p^{n-s}M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}}))$$
$$\bigoplus_{t=1}^{s}\bigoplus_{\lambda=n-s+1}^{m-t}(p^{n-s}-p^{n-s-1})M_{p^t}(\mathbb{Q}(\zeta_{p^{\lambda}})).$$

Finite metacyclic groups were classified by Hempel[14], while earlier, the classification of finite metacyclic p-groups was addressed by several authors, including King[15]; Liedahl[16], [17]; Sim[18].

[14] C. E. Hempel, Metacyclic groups, *Comm. Algebra* **28**(8) (2000), 3865–3897.

[15] B. W. King, Presentations of metacyclic groups, *Bull. Aust. Math. Soc.* **8** (1973), 103–131.

[16] S. Liedahl, Presentations of metacyclic p-groups with applications to K-admissibility questions, *J. Algebra* **169**(3) (1994), 965–983.

[17] S. Liedahl, Enumeration of metacyclic p-groups, *J. Algebra* **186**(2) (1996), 436–446.

[18] H. S. Sim, Metacyclic groups of odd order, *Proc. London Math. Soc.* (3) **69**(1) (1994), 47–71.

Finite metacyclic groups were classified by Hempel[14], while earlier, the classification of finite metacyclic p-groups was addressed by several authors, including King[15]; Liedahl[16], [17]; Sim[18]. We adopt the classification of metacyclic p-groups as given by King.

- King showed that each metacyclic p-group admits a uniquely reduced presentation (up to isomorphism).
- He classified them into **ordinary** and **exceptional** types.
- Ordinary type includes all except a special class of exceptional metacyclic 2-groups.

[14] C. E. Hempel, Metacyclic groups, *Comm. Algebra* **28**(8) (2000), 3865–3897.
[15] B. W. King, Presentations of metacyclic groups, *Bull. Aust. Math. Soc.* **8** (1973), 103–131.
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(A) **Ordinary metacyclic groups:**

$$G = \langle a, b \ : \ a^{p^n} = 1, \ b^{p^m} = a^{p^{n-r}}, \ bab^{-1} = a^{1+p^{n-s}} \rangle$$

where $p \geq 3$, or $p = 2$ and $s < n - 1$ for $n \geq 2$.

(a) **Split:** $0 = r \leq s < \min\{m+1, n\}$

(b) **Non-split:** $\max\{1, n-m+1\} \leq r < \min\{s, n-s+1\}$

Presentations of Metacyclic p-groups (continued)

(A) **Ordinary metacyclic groups:**

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where $p \geq 3$, or $p = 2$ and $s < n - 1$ for $n \geq 2$.

(a) **Split:** $0 = r \leq s < \min\{m+1, n\}$

(b) **Non-split:** $\max\{1, n-m+1\} \leq r < \min\{s, n-s+1\}$

(B) **Exceptional metacyclic groups:**

$$G = \langle a, b \ : \ a^{2^n} = 1,\ b^{2^m} = a^{2^{n-r}},\ bab^{-1} = a^{-1+2^{n-s}} \rangle$$

for certain integers m, n, r, and s.

(a) **Split:** $0 = r \leq s < \min\{m+1, n-1\}$

(b) **Non-split:**

- $r = 1$, $\max\{1, n-m+1\} \leq s < \min\{m, n-1\}$, or
- $r = 1$, $s = 0$, $m = 1 < n$ (gives generalized quaternion group)

Theorem (Choudhary and Prajapati[19])

Let p be a prime and ζ_d a primitive d-th root of unity for some positive integer d. Consider an ordinary, finite, non-cyclic metacyclic p-group G, with a unique reduced presentation:

$$G = \langle a, b \ : \ a^{p^n} = 1,\, b^{p^m} = a^{p^{n-r}},\, bab^{-1} = a^{1+p^{n-s}} \rangle,$$

for certain integers m, n, r and s. Then we have the following.

1. **Case ($n - s \geq m$).** *In this case,*

$$\mathbb{Q}G \cong \mathbb{Q} \bigoplus_{\lambda=1}^{m} (p^{\lambda} + p^{\lambda-1}) \mathbb{Q}(\zeta_{p^\lambda}) \bigoplus_{\lambda=m+1}^{n-s} p^m \mathbb{Q}(\zeta_{p^\lambda}) \bigoplus_{t=1}^{s} p^{m-t} M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}})).$$

[19] R. K. Choudhary and S. K. Prajapati, A combinatorial formula for the Wedderburn decomposition of rational group algebras and the rational representations of ordinary metacyclic p-groups, *arXiv* (2024). arXiv:2410.20933 [math.RT]

2. **Case $(n-s<m)$.** Suppose $m=(n-s)+k$. Then we have following two sub-cases.

   2.1. **Sub-case $(k \leq s-r)$.** In this sub-case,

$$\mathbb{Q}G \cong \mathbb{Q}\bigoplus_{\lambda=1}^{n-s}(p^{\lambda}+p^{\lambda-1})\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{\lambda=n-s+1}^{m} p^{n-s}\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{t=1}^{k-1} p^{n-s}M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}}))$$

$$\bigoplus_{t=1}^{k-1}\bigoplus_{\lambda=n-s+1}^{m-t}(p^{n-s}-p^{n-s-1})M_{p^t}(\mathbb{Q}(\zeta_{p^{\lambda}})) \bigoplus_{t=k}^{s} p^{m-t}M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}})).$$

   2.2. **Sub-case $(k > s-r)$.** In this sub-case,

$$\mathbb{Q}G \cong \mathbb{Q}\bigoplus_{\lambda=1}^{n-s}(p^{\lambda}+p^{\lambda-1})\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{\lambda=n-s+1}^{m} p^{n-s}\mathbb{Q}(\zeta_{p^{\lambda}}) \bigoplus_{t=1}^{s-r} p^{n-s}M_{p^t}(\mathbb{Q}(\zeta_{p^{n-s}}))$$

$$\bigoplus_{t=1}^{s-r}\bigoplus_{\lambda=n-s+1}^{m-t}(p^{n-s}-p^{n-s-1})M_{p^t}(\mathbb{Q}(\zeta_{p^{\lambda}}))$$

$$\bigoplus_{t=s-r+1}^{s} p^{n-r-t}M_{p^t}(\mathbb{Q}(\zeta_{p^{m+r-s}})).$$

Corollary

For ordinary metacyclic p-groups G and H, $\mathbb{Q}G \cong \mathbb{Q}H \iff G \cong H$.

1. Consider $G_4 = \langle a, b \mid a^{16} = 1, b^{16} = a^8, bab^{-1} = a^5 \rangle$. Here, $n = 4$, $m = 4$, $r = 1$ and $s = 2$.

$$\mathbb{Q}G_4 \cong 4\mathbb{Q} \oplus 6\mathbb{Q}(\zeta_4) \oplus 4\mathbb{Q}(\zeta_8) \oplus 4\mathbb{Q}(\zeta_{16}) \oplus 4M_2(\mathbb{Q}(\zeta_4)) \\ \oplus 2M_2(\mathbb{Q}(\zeta_8)) \oplus 2M_4(\mathbb{Q}(\zeta_8)).$$

G_4 is SmallGroup$(256, 320)$ in GAP library of groups.

Examples

1. Consider $G_4 = \langle a, b \mid a^{16} = 1, b^{16} = a^8, bab^{-1} = a^5 \rangle$. Here, $n = 4$, $m = 4$, $r = 1$ and $s = 2$.

$$\begin{aligned}\mathbb{Q}G_4 \cong & 4\mathbb{Q} \oplus 6\mathbb{Q}(\zeta_4) \oplus 4\mathbb{Q}(\zeta_8) \oplus 4\mathbb{Q}(\zeta_{16}) \oplus 4M_2(\mathbb{Q}(\zeta_4)) \\ & \oplus 2M_2(\mathbb{Q}(\zeta_8)) \oplus 2M_4(\mathbb{Q}(\zeta_8)).\end{aligned}$$

   G_4 is SmallGroup$(256, 320)$ in GAP library of groups.

2. Consider $G_5 = \langle a, b \mid a^{27} = 1, b^{27} = a^9, bab^{-1} = a^4 \rangle$. Here, $n = 3$, $m = 3$, $r = 1$ and $s = 2$.

$$\begin{aligned}\mathbb{Q}G_5 \cong & \mathbb{Q} \oplus 4\mathbb{Q}(\zeta_3) \oplus 3\mathbb{Q}(\zeta_9) \oplus 3\mathbb{Q}(\zeta_{27}) \oplus 3M_3(\mathbb{Q}(\zeta_3)) \\ & \oplus 2M_3(\mathbb{Q}(\zeta_9)) \oplus M_9(\mathbb{Q}(\zeta_9)).\end{aligned}$$

   G_5 is SmallGroup$(729, 92)$ in GAP library of groups.

Lemma (Yamada[20])

Let $\psi \in \mathrm{lin}(G)$, *and let* $N = \ker(\psi)$ *with* $n = [G : N]$. *Suppose* $G = \cup_{i=0}^{n-1} Ny^i$. *Then*

$$\psi(xy^i) = \zeta_n^i \quad (0 \leq i < n,\ x \in N).$$

Now, let $f(X) = X^s - a_{s-1}X^{s-1} - \cdots - a_1 X - a_0$ *be the minimal polynomial of* ζ_n *over* $\mathbb{Q}$, *where* $s = \phi(n)$. *Define*

$$\Psi(xy^i) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & \cdots & \cdots & a_{s-1} \end{pmatrix}^i \qquad (0 \leq i < n,\ x \in N).$$

Then Ψ *is an irreducible* $\mathbb{Q}$*-representation of* G *affording* $\Omega(\psi)$.

[20] T. Yamada, Remarks on rational representations of a finite group, *SUT J. Math.* **29**(1) (1993), 71–77.

Algorithm (Choudhary and Prajapati[21])

Input: An irreducible complex character χ of a finite p-group G, where p is an odd prime.

1. *Find a pair (H, ψ), where $H \leq G$ and $\psi \in \mathrm{lin}(H)$, such that $\psi^G = \chi$ and $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)$.*
2. *Find an irreducible $\mathbb{Q}$-representation Ψ of H that affords the character $\Omega(\psi)$.*
3. *Induce Ψ to G.*

Output: Ψ^G, an irreducible $\mathbb{Q}$-representation of G whose character is $\Omega(\chi)$.

[21] R. K. Choudhary and S. K. Prajapati, Rational representations and rational group algebra of VZ p-groups, *J. Aust. Math. Soc.* **118**(1) (2025), 1-30.

[22] C. E. Ford, Characters of p-groups, *Proc. Amer. Math. Soc.* 101(4) (1987), 595-601.

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Output: Ψ^G, an irreducible $\mathbb{Q}$-representation of G whose character is $\Omega(\chi)$.

- The algorithm heavily relies on the work of Ford[22] and Yamada[23].

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Properties of Ordinary Metacyclic p-Groups

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Lemma

Let G be a faithful, ordinary, finite, non-cyclic metacyclic p-group. Then G has, up to isomorphism, exactly one uniquely reduced presentation, which is one of the following two types:

1. $G_1 = \langle a, b \ : \ a^{p^n} = 1, \ b^{p^m} = a^{p^s}, \ bab^{-1} = a^{1+p^{n-s}} \rangle$ *for certain integers* n, m *and* s, *where* $\max\{1, n-m+1\} \leq n-s < s$ *for* $p \geq 3$, *and* $\max\{2, n-m+1\} \leq n-s < s$ *for* $p = 2$.
2. $G_2 = \langle a, b \ : \ a^{p^n} = b^{p^s} = 1, \ bab^{-1} = a^{1+p^{n-s}} \rangle$, *where* $1 \leq s \leq n-1$ *for* $p \geq 3$, *and* $1 \leq s < n-1$ *for* $p = 2$.

Theorem (Choudhary and Prajapati[24])

Let p be a prime. Consider a faithful ordinary, finite, non-cyclic metacyclic p-group G. Let $\chi \in \mathrm{FIrr}(G)$. Then there exists an abelian subgroup H of G with $\psi \in \mathrm{lin}(H)$ such that $\psi^G = \chi$ and $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)$. Moreover, we have the following.

1. *Let $G = G_1$ and $\chi \in \mathrm{FIrr}(G)$. Then we have the following two cases.*
   1.1 **Case ($m \geq 2s$).** *In this case, $H = \langle a, b^{p^s} \rangle$ and*
   $$\psi(x) = \begin{cases} \zeta_{p^n} & \text{if } x = a, \\ \zeta_{p^{n+m-2s}} & \text{if } x = b^{p^s}. \end{cases}$$

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1.2 **Case ($m < 2s$).** In this case, $H = H_\mu = \langle a^{p^{2s-m}}, a^\mu b^{p^{m-s}} \rangle$ and $\psi \in \mathrm{lin}(H_\mu)$ is defined as follows:

$$\psi(x) = \begin{cases} \zeta_{p^{n+m-2s}} & \text{if } x = a^{p^{2s-m}}, \\ 1 & \text{if } x = a^\mu b^{p^{m-s}}, \end{cases}$$

where μ is an integer such that $\mu k \equiv -1 \pmod{p^{n-s}}$, $-\mu l \equiv 1 \pmod{p^{n+m-2s}}$, $k = p^{-s} \frac{(1+p^{n-s})^{p^m}-1}{(1+p^{n-s})^{p^{m-s}}-1}$ and $l = p^{m-2s} \frac{(1+p^{n-s})^{p^s}-1}{(1+p^{n-s})^{p^{m-s}}-1}$.

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2. Let $G = G_2$ and $\chi \in \text{FIrr}(G)$. Then $H = \langle a^{p^s}, b \rangle$ and

$$\psi(x) = \begin{cases} \zeta_{p^{n-s}} & \text{if } x = a^{p^s}, \\ 1 & \text{if } x = b. \end{cases}$$

Example: Rational Representation of a Metacyclic p-group

Consider a faithful metacyclic 2-group of order 512

$$G = \langle a^{32} = 1,\, b^{16} = a^8,\, bab^{-1} = a^5 \rangle$$

G is of type G_1 with $n = 5$, $m = 4$, $s = 3$. Let $\chi \in \mathrm{FIrr}(G)$ with $\chi(1) = 8$, defined as:

$$\chi(a^i b^j) = \begin{cases} 8\zeta_4^{i_1}\zeta_8^{j_1} & \text{if } i = 4i_1,\ j = 4j_1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } \zeta_8^2 = \zeta_4.$$

Since $m < 2s$, the theorem applies to give (H_μ, ψ) for the rational matrix representation corresponding to $\Omega(\chi)$, where

$$\mu k \equiv -1 \pmod 4, \quad k = \frac{1}{8}\left(\frac{5^{16}-1}{5^2-1}\right) = 794728597$$

$$-\mu l \equiv 1 \pmod 8, \quad l = \frac{1}{4}\left(\frac{5^8-1}{5^2-1}\right) = 4069$$

This implies that $\mu = 3$.

Set $\mu = 3$. Then

$$H_\mu = \langle a^4,\, a^3b^2 \rangle, \quad \psi(a^4) = \zeta_8,\ \psi(a^3b^2) = 1$$

We have: $\ker(\psi) = \langle a^3b^2 \rangle = N$ and $H_\mu = \bigcup_{i=0}^{7} Na^{4i}$ with $[H_\mu : N] = 8$.

$\Psi : H \to GL_4(\mathbb{Q})$ is an irreducible $\mathbb{Q}$-representation of H that affords $\Omega(\psi)$.

$$\Psi(a^4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi(a^3b^2) = I_4$$

Let $P = \Psi(a^4)$ and O be the 4×4 zero matrix. Since $[G : H_\mu] = 8$, Ψ^G is an irreducible $\mathbb{Q}$-representation of G of degree 32 that affords $\Omega(\chi)$.

$\Psi^G : G \to GL_{32}(\mathbb{Q})$ is defined on generators a and b as follows:

$$\Psi^G(a) = \begin{pmatrix} O & O & P^7 & O & O & O & O & O \\ O & O & O & I_4 & O & O & O & O \\ O & O & O & O & P^5 & O & O & O \\ O & O & O & O & O & P^6 & O & O \\ O & O & O & O & O & O & P^3 & O \\ O & O & O & O & O & O & O & P^4 \\ P^2 & O & O & O & O & O & O & O \\ O & P^3 & O & O & O & O & O & O \end{pmatrix}$$

and

$$\Psi^G(b) = \begin{pmatrix} O & I_4 & O & O & O & O & O & O \\ O & O & I_4 & O & O & O & O & O \\ O & O & O & I_4 & O & O & O & O \\ O & O & O & O & I_4 & O & O & O \\ O & O & O & O & O & I_4 & O & O \\ O & O & O & O & O & O & I_4 & O \\ O & O & O & O & O & O & O & I_4 \\ P & O & O & O & O & O & O & O \end{pmatrix}.$$

Thank You!