Uncountable groups whose proper large subgroups have a permutability transitive relation

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Motivations

- Let G be an **infinite** group
- Let Σ be a collection of **proper subgroups** of G having a **given structure**

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- Let Σ be a collection of **proper subgroups** of G having a **given structure**

Can we obtain information about the **whole group** G ?
Do **all subgroups** of G share the **same structure** as Σ ?

From Small to Large

Definition (R. Baer, 1962)

A group class $\mathfrak{X}$ is said to be ***countably recognizable*** if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group.

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- **Examples of countably recognizable classes:**
  Nilpotent groups, soluble groups, FC-groups [1]
- **Examples of <u>not</u> countably recognizable classes:**
  Gruenberg groups, countable groups

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This kind of approach is very classical: starting from *small* subgroups, it allows to get information about the whole group and its *large* subgroups.

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What does ***large*** mean for a subgroup?

1. Let G be a group. A subgroup H of G is said ***large*** if it has **infinite rank**.

Definition

A group G is said to have ***finite (Prüfer) rank*** $r = r(G)$ if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property; if such an r does not exist, we say that the group G has ***infinite rank***.

From Large to Small

A dual approach explores the structure of a group G and its ***small*** subgroups by assuming that only its ***large*** subgroups belong to a specific group class $\mathfrak{X}$.

What does ***large*** mean for a subgroup?

1. Let G be a group. A subgroup H of G is said ***large*** if it has **infinite rank**.
2. Let G be an **uncountable** group of **cardinality** $\aleph$. A subgroup H of G is said ***large*** if it has the **same cardinality** as G, and ***small*** otherwise.

$\aleph$-recognizability

- Let G be an **uncountable group of cardinality** $\aleph$
- Assume that **all proper large subgroups** of G lie in some **group class** $\mathfrak{X}$

$\aleph$-recognizability

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- Assume that **all proper large subgroups** of G lie in some **group class** $\mathfrak{X}$

Can we conclude that **G lies in $\mathfrak{X}$** as well ?

If so, we say that $\mathfrak{X}$ is ***$\aleph$-recognizable***

Jónsson groups

There exist infinite groups G in which all proper subgroups have cardinality strictly smaller than that of G, the so-called ***Jónsson groups***.

- Prüfer groups and Tarski groups are countable Jónsson groups.
- Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by S. Shelah (1980) and V.N. Obraztsov (1990).

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How to **avoid Jónsson** groups?

Imposing a **weak solubility condition**: in any uncountable Jónsson group G, the quotient $G/Z(G)$ is a simple large group.

Some $\aleph$-recognizable classes

Definition

Let $\aleph$ be a cardinal number. $\aleph$ is said ***regular*** if it cannot be expressed as a sum of $\aleph' < \aleph$ smaller cardinals.

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The following group classes are **$\aleph$-recognizable** in the universe of **(generalized) soluble groups** (with $\aleph$ an uncountable **regular** cardinal number):

- quasihamiltonian groups [2] (**F. de Giovanni and M. Trombetti, 2016**);
- groups having modular subgroup lattice (**F. de Giovanni and M. Trombetti, 2016**);
- FC-groups (**F. de Giovanni and M. Trombetti, 2016**);
- Gruenberg groups (**F. de Giovanni and M. Trombetti, 2022**).

[2] i.e. groups in which all subgroups are permutable

Permutable subgroups

Definition

Let G be a group. A subgroup H of G is called ***permutable in G*** (***H per G***) if $HK = KH$ for all subgroups K of G.

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- Normal subgroups are permutable
- The dihedral group D_8 is the minimal example which shows that **normality** and **permutability are not transitive relations**

PT-groups and $\overline{PT}$-groups

Definition

A group G is called

- a ***T*-group** if, whenever $H \trianglelefteq K$ and $K \trianglelefteq G$, then $H \trianglelefteq G$
- a ***PT*-group** if, whenever H **per** K and K **per** G, then H **per** G

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The *PT*-property (T-property) is **inherited** by permutable subgroups (subnormal subgroups), but <u>**not**</u> **by arbitrary subgroups**.

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- a ***PT*-group** if, whenever $\boldsymbol{H}$ **per** $\boldsymbol{K}$ and $\boldsymbol{K}$ **per** $\boldsymbol{G}$, then $\boldsymbol{H}$ **per** $\boldsymbol{G}$
- a ***$\overline{T}$*-group** if all its subgroups have the T-property
- a ***$\overline{PT}$*-group** if all its subgroups have the PT-property

Soluble finite PT-groups

- **W. Gaschütz (1957)** and **D. J. S. Robinson (1964)** have completely described **soluble T-groups** in the finite and infinite case, respectively.
- **M. De Falco, F. de Giovanni, C. Musella** and **M. Trombetti** analyzed, in the same universe, the T-property both in the infinite rank **(2013)** and uncountable cases **(2017)**.

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Theorem (G. Zacher, 1964)

Let G be a soluble finite group. Then G is a PT-group if and only if there exists a subgroup N such that:

1. *N is an abelian Hall subgroup of odd order whose subgroups are normal in G;*
2. *G/N is a nilpotent group having modular subgroup lattice.*

Soluble infinite *PT*-groups

- F. Menegazzo described **hyperabelian *PT*-groups** also in the **periodic** (1968) and **non-periodic case** (1969) (splitting the separate and non-separate case).

Soluble infinite PT-groups

Definition

A group G is said ***hyperabelian*** if it has an ascending normal series with abelian factors containing $\{1\}$ and G.

- **F. Menegazzo** described **hyperabelian PT-groups** also in the **periodic (1968)** and **non-periodic case (1969)** (splitting the separate and non-separate case).

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A group G is said ***hyperabelian*** if it has an ascending normal series with abelian factors containing $\{1\}$ and G.

- **F. Menegazzo** described **hyperabelian *PT*-groups** also in the **periodic** **(1968)** and **non-periodic case** **(1969)** (splitting the separate and non-separate case).

Theorem (F. Menegazzo, 1968)

Let G be a hyperabelian periodic group. Then, G is a PT-group if and only if there exists a subgroup N such that:

1. *N is an abelian Hall subgroup whose subgroups are normal in G and $2 \notin \pi(N)$;*
2. *G/N is direct product of its Sylow subgroups which have the PT-property.*

Soluble infinite PT-groups

Remark

Let G be a hyperabelian PT-group.

- G is **soluble of derived length at most** 3.
- G contains a **non trivial characteristic abelian subgroup A whose subgroups are normal in G.**

Soluble infinite PT-groups

Remark

Let G be a hyperabelian PT-group. If G is non-periodic non-separate, then:

- G is **soluble of derived length at most** 3.
- G contains a **non trivial characteristic abelian subgroup A whose subgroups are normal in G** and A has finite index in G.

PT-groups of infinite rank

Theorem (A. Ballester Bolinches - M. De Falco - F. de Giovanni - C. Musella, 2024)

Let G be a hyperabelian group of infinite rank whose proper subgroups of infinite rank have the PT-property. Then G is a soluble $\overline{PT}$-group.

PT-groups of infinite rank

Theorem (A. Ballester Bolinches - M. De Falco - F. de Giovanni - C. Musella, 2024)

Let G be a hyperabelian group of infinite rank whose proper subgroups of infinite rank have the PT-property. Then G is a soluble $\overline{PT}$-group.

- If every 2-generated subgroup of a group G has the PT-property, then G is a $\overline{PT}$-group.

PT-property in uncountable groups

Lemma

Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property.

PT-property in uncountable groups

Lemma

Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property.

1. *If G/G' has cardinality $\aleph$, then G is a PT-group.*

PT-property in uncountable groups

Lemma

Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property.

1. *If G/G' has cardinality $\aleph$, then G is a PT-group.*
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PT-property in uncountable groups

Lemma

Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property.

1. *If G/G' has cardinality $\aleph$, then G is a PT-group.*
2. *If there exists a subgroup N of cardinality $\aleph$ whose subgroups are normal in G, then G is a $\overline{PT}$-group.*
3. *If G is abelian-by-cyclic or abelian-by-finite, then G is a $\overline{PT}$-group.*

PT-property in uncountable groups

Is the PT-class $\aleph$-recognizable?

Yes! (in a suitable universe)

Theorem (MC - L. Lancellotti, 2024)

Let G be an uncountable hyperabelian group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property. Then G is a soluble $\overline{PT}$-group.

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Theorem (MC - L. Lancellotti, 2024)

Let G be an uncountable hyperabelian group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property. Then G is a soluble $\overline{PT}$-group.

In these hypotheses:

- G **is soluble of derived length at most** 3.

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In these hypotheses:

- G **is soluble of derived length at most** 3.

First step of the proof: to prove that G is a PT-group by contradiction.

Second step of the proof: to prove that G is a $\overline{PT}$-group.

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- The **periodic** case follows from previous results.

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In these hypotheses:

- G **is soluble of derived length at most** 3.

First step of the proof: to prove that G is a PT-group by contradiction.

- The **periodic** case follows from previous results.
- In the **non-periodic** case, G/G' cannot be large. Then G/G' is finitely generated, so that G contains a proper normal subgroup N of finite index.

  We split the proof in the case that G' is periodic or not.

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Second step of the proof: to prove that G is a $\overline{PT}$-group.

- It is enough to prove that every 2-generated subgroup of G has the PT-property.

PT-property in uncountable groups

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Let G be an uncountable hyperabelian group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property. Then G is a soluble $\overline{PT}$-group.

✓ **First step of the proof:** to prove that G is a PT-group by contradiction.

Second step of the proof: to prove that G is a $\overline{PT}$-group.

- It is enough to prove that every 2-generated subgroup of G has the PT-property.
- By the structure theorems we can assume that the abelian subgroup A whose subgroups are normal in G is not large.
  We split the proof in the case that G is periodic or not.

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Thank you for your attention!