Cohomology and Extensions of Relative Rota–Baxter Groups

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Relative Rota–Baxter Group

Definition

A relative Rota–Baxter group is a quadruple (H, G, ϕ, R), where H and G are groups, $\phi : G \to \mathrm{Aut}(H)$ a group homomorphism (where $\phi(g)$ is denoted by ϕ_g) and $R : H \to G$ is a map satisfying the condition

$$R(h_1)R(h_2) = R(h_1 \phi_{R(h_1)}(h_2))$$

for all $h_1, h_2 \in H$.
The map R is referred as the relative Rota–Baxter operator on H.

We say that the relative Rota–Baxter group (H, G, ϕ, R) is trivial if $\phi : G \to \mathrm{Aut}(H)$ is the trivial homomorphism.

Descendent group of relative Rota–Baxter operator

Let (H, G, ϕ, R) be a relative Rota–Baxter group. Then the binary operation

$$h_1 \circ_R h_2 = h_1 \phi_{R(h_1)}(h_2)$$

defines a group operation on H. The group $H^{(\circ_R)}$ is called the descendent group of R. Moreover, the map $R : H^{(\circ_R)} \to G$ is a group homomorphism.
It follows that if (H, G, ϕ, R) is a relative Rota–Baxter group, then the image $R(H)$ of H under R is a subgroup of G.

Skew Left Brace

Definition

A skew left brace is a triple $(H, \cdot, \circ)$, where $(H, \cdot)$ and $(H, \circ)$ are groups such that

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$$

holds for all $a, b, c \in H$, where a^{-1} denotes the inverse of a in $(H, \cdot)$.

Given a relative Rota–Baxter group (H, G, ϕ, R), we obtain a skew brace $(H, \cdot, \circ_R)$. This is called the induced skew brace of the relative Rota–Baxter group (H, G, ϕ, R) and is denoted by H_R.

Relative Rota–Baxter Subgroup

Definition

Let (H, G, ϕ, R) be a relative Rota–Baxter group, and $K \leq H$ and $L \leq G$ be subgroups. Suppose that $\phi_\ell(K) \subseteq K$ for all $\ell \in L$ and $R(K) \subseteq L$. Then $(K, L, \phi|, R|)$ is a relative Rota–Baxter group, which we refer as a relative Rota–Baxter subgroup of (H, G, ϕ, R) and write $(K, L, \phi|, R|) \leq (H, G, \phi, R)$.

Ideal of relative Rota–Baxter group

Definition

Let (H, G, ϕ, R) be a relative Rota–Baxter group and $(K, L, \phi|, R|) \leq (H, G, \phi, R)$ its relative Rota–Baxter subgroup. We say that $(K, L, \phi|, R|)$ is an ideal of (H, G, ϕ, R) if

$$K \trianglelefteq H \quad \text{and} \quad L \trianglelefteq G,$$
$$\phi_g(K) \subseteq K \text{ for all } g \in G,$$
$$\phi_\ell(h)h^{-1} \in K \text{ for all } h \in H \text{ and } \ell \in L$$

We write $(K, L, \phi|, R|) \trianglelefteq (H, G, \phi, R)$ to denote an ideal of a relative Rota–Baxter group.

Quotient of relative Rota–Baxter group

Let (H, G, ϕ, R) be a relative Rota–Baxter group and $(K, L, \phi|, R|)$ an ideal of (H, G, ϕ, R). Then there are maps $\overline{\phi} : G/L \to \mathrm{Aut}(H/K)$ and $\overline{R} : H/K \to G/L$ defined by

$$\overline{\phi}_{\overline{g}}(\overline{h}) = \overline{\phi_g(h)} \quad \text{and} \quad \overline{R}(\overline{h}) = \overline{R(h)}$$

for all $\overline{g} \in G/L$ and $\overline{h} \in H/K$, such that $(H/K, G/L, \overline{\phi}, \overline{R})$ is a relative Rota–Baxter group.

We write $(H, G, \phi, R)/(K, L, \phi|, R|)$ to denote the quotient relative Rota–Baxter group $(H/K, G/L, \overline{\phi}, \overline{R})$.

Homomorphism of relative Rota–Baxter Groups

Definition

Let (H, G, ϕ, R) and (K, L, φ, S) be two relative Rota–Baxter groups.

- A homomorphism $(\psi, \eta) : (H, G, \phi, R) \to (K, L, \varphi, S)$ of relative Rota–Baxter groups is a pair (ψ, η), where $\psi : H \to K$ and $\eta : G \to L$ are group homomorphisms such that

$$\eta\, R = S\, \psi \quad \text{and} \quad \psi\, \phi_g = \varphi_{\eta(g)}\, \psi$$

  for all $g \in G$.

Extensions of RRB Groups

Definition

Let (K, L, α, S) and (A, B, β, T) be relative Rota–Baxter groups. An extension of (A, B, β, T) by (K, L, α, S) is a relative Rota–Baxter group (H, G, ϕ, R) that fits into the sequence

$$\mathcal{E}: \quad \mathbf{1} \longrightarrow (K, L, \alpha, S) \xrightarrow{(i_1, i_2)} (H, G, \phi, R) \xrightarrow{(\pi_1, \pi_2)} (A, B, \beta, T) \longrightarrow \mathbf{1},$$

where (i_1, i_2) and (π_1, π_2) are morphisms of relative Rota–Baxter groups such that (i_1, i_2) is an embedding, (π_1, π_2) is an epimorphism of relative Rota-Baxter groups and $(\mathrm{im}(i_1), \mathrm{im}(i_2), \phi|, R|) = (\ker(\pi_1), \ker(\pi_2), \phi|, R|)$.

Group Extensions from RRB Group Extensions

Proposition

An extension

$$\mathcal{E}: \quad \mathbf{1} \longrightarrow (K, L, \alpha, S) \xrightarrow{(i_1, i_2)} (H, G, \phi, R) \xrightarrow{(\pi_1, \pi_2)} (A, B, \beta, T) \longrightarrow \mathbf{1}$$

of relative Rota–Baxter groups induces extensions of groups

$$\mathcal{E}_1: \quad 1 \longrightarrow K \xrightarrow{i_1} H \xrightarrow{\pi_1} A \longrightarrow 1 \quad \text{and} \quad \mathcal{E}_2: \quad 1 \longrightarrow L \xrightarrow{i_2} G \xrightarrow{\pi_2} B \longrightarrow 1.$$

Furthermore, (K, L, α, S) is an ideal of (H, G, ϕ, R) and the quotient relative Rota–Baxter group $(H, G, \phi, R)/(K, L, \alpha, S)$ is isomorphic to (A, B, β, T).

Cohomology of RRB Groups

Let s_H and s_G be set-theoretic sections to $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. The ordered pair (s_H, s_G) will be referred as a set-theoretic section to the extension $\mathcal{E}$. Then each element $h \in H$ and $g \in G$ can be uniquely written as $h = s_H(a)\, k$ and $g = s_G(b)\, l$ for some $k \in K$, $l \in L$ and $a \in A$, $b \in B$.

Proposition [–, Rathee, Singh]

Let $a \in A$, $b \in B$, $k \in K$, and $l \in L$. Then, the following statements hold:

- The action ϕ is characterized by the equation

$$\phi_{s_G(b)l}(s_H(a)k) = s_H(\beta_b(a))\, \rho(a, b)\, \phi_{s_G(b)}(f(l, a)k),$$

  where $f : L \times A \to K$ is defined as $f(l, a) = s_H(a)^{-1}\phi_l(s_H(a))$, and $\rho : A \times B \to K$ is given by $\rho(a, b) := (s_H(\beta_b(a)))^{-1}\phi_{s_G(b)}(s_H(a))$.

Cohomology of RRB Groups

Proposition...

- The relative Rota–Baxter operator R is expressed as

$$R(s_H(a)k) = s_G(T(a))\,\chi(a)\,S(\phi^{-1}_{s_G(T(a))}(k)),$$

where $\chi : A \to K$ is defined by, $\chi(a) := s_G(T(a))^{-1}R(s_H(a))$.

Cohomology of RRB groups

Proposition [–, Rathee, Singh]

Consider the abelian extension

$$\mathcal{E}: \quad \mathbf{1} \longrightarrow (K, L, \alpha, S) \xrightarrow{(i_1, i_2)} (H, G, \phi, R) \xrightarrow{(\pi_1, \pi_2)} (A, B, \beta, T) \longrightarrow \mathbf{1}$$

of relative Rota–Baxter groups. Then the following hold:

- The map $\nu : B \to \mathrm{Aut}(K)$ defined by

$$\nu_b(k) = \phi_{s_G(b)}(k)$$

  for $b \in B$ and $k \in K$, is a homomorphism of groups.

Cohomology of RRB groups

Proposition...

- The $\mu : A \to \mathrm{Aut}(K)$ defined by
$$\mu_a(k) = s_H(a)^{-1}\, k\, s_H(a)$$
for $a \in A$ and $k \in K$, is an anti-homomorphism of groups.
- The map $\sigma : B \to \mathrm{Aut}(L)$ defined by
$$\sigma_b(l) = s_G(b)^{-1}\, l\, s_G(b)$$
for $b \in B$ and $l \in K$, is an anti-homomorphism of groups.

Cohomology of RRB groups

Proposition...

Further, all the maps are independent of the choice of a section to $\mathcal{E}$.

- The map $\tau_1 : A \times A \to K$ given by
$$\tau_1(a_1, a_2) := s_H(a_1 a_2)^{-1} s_H(a_1) s_H(a_2)$$
for $a_1, a_2 \in A$ is a group 2-cocycle with respect to the action μ.
- The map $\tau_2 : B \times B \to L$ given by
$$\tau_2(b_1, b_2) := s_G(b_1 b_2)^{-1} s_G(b_1) s_G(b_2)$$
for $b_1, b_2 \in B$ is a group 2-cocycle with respect to the action σ.

Cohomology of RRB Groups

Definition

A module over a relative Rota–Baxter group (A, B, β, T) is a trivial relative Rota-Baxter group (K, L, α, S) such that there exists a quadruple (ν, μ, σ, f) (called action) of maps satisfying the following conditions:

- The group K is a left B-module and a right A-module with respect to the actions $\nu : B \to \mathrm{Aut}(K)$ and $\mu : A \to \mathrm{Aut}(K)$, respectively.
- The group L is a right B-module with respect to the action $\sigma : B \to \mathrm{Aut}(L)$.

Cohomology of RRB Groups

Definition...

- The map $f : L \times A \to K$ has the property that $f(-, a) : L \to K$ is a homomorphism for all $a \in A$ and $f(l, -) : A \to K$ is a derivation with respect to the action μ for all $l \in L$.
- $S\big(\nu_{T(a)}^{-1}(\mu_a(k))\,\nu_{T(a)}^{-1}(f(S(k), a))\big) = \sigma_{T(a)}(S(k))$ for all $a \in A$ and $k \in K$.
- $\nu_b(\mu_a(k)) = \mu_{\beta_b(a)}(\nu_b(k))$ for all $a \in A$, $b \in B$ and $k \in K$.

Cohomology of RRB Groups

Let us set

$$\begin{aligned}
C^0_{RRB} &= K \times_{(\nu,\mu,\sigma,f)} L, \\
C^1_{RRB} &= C^1(A,K) \oplus C^1(B,L), \\
C^2_{RRB} &= C^2(A,K) \oplus C^2(B,L) \oplus C(A \times B, K) \oplus C(A,L), \\
C^3_{RRB} &= C^3(A,K) \oplus C^3(B,L) \oplus C(A \times B^2, K) \oplus C(A^2 \times B, K) \oplus C^2(A,L),
\end{aligned}$$

where

$$\begin{aligned}
K \times_{(\nu,\mu,\sigma,f)} L = \Big\{ &(k,l) \in K \times L \mid f(\sigma_b(l), a) = f(l,a),\ \nu_b(k) = k, \\
&\sigma_{T(a)}(l) = l \text{ and } S(\mu_a(k)) = S(k) \text{ for all } a \in A \text{ and } b \in B \Big\}
\end{aligned}$$

Cohomology of RRB Groups

Define $\partial^0_{RRB} : C^0_{RRB} \to C^1_{RRB}$ by

$$\partial^0_{RRB}(k, l) = (\kappa_k, \omega_l),$$

where $\kappa_k : A \to K$ and $\omega_l : B \to L$ are defined by

$$\begin{aligned} \kappa_k(a) &= \mu_a(k)k^{-1}, \\ \omega_l(b) &= \sigma_b(l)l^{-1}, \end{aligned}$$

for $k \in K$, $l \in L$, $a \in A$ and $b \in B$. Define $\partial^1_{RRB} : C^1_{RRB} \to C^2_{RRB}$ by

$$\partial^1_{RRB}(\theta_1, \theta_2) = (\partial^1_\mu(\theta_1), \partial^1_\sigma(\theta_2), \lambda_1, \lambda_2),$$

$$\begin{aligned} \lambda_1(a, b) &= \nu_b\big(f(\theta_2(b), a)\theta_1(a)\big)\,\big(\theta_1(\beta_b(a))\big)^{-1}, \\ \lambda_2(a) &= S\big(\nu^{-1}_{T(a)}(\theta_1(a))\big)\,\big(\theta_2(T(a))\big)^{-1}, \end{aligned}$$

Cohomology of RRB Groups

Define $\partial^2_{RRB} : C^2_{RRB} \to C^3_{RRB}$ by

$$\partial^2_{RRB}(\tau_1, \tau_2, \rho, \chi) = (\partial^2_{\mu}(\tau_1), \partial^2_{\sigma}(\tau_2), \gamma_1, \gamma_2, \gamma_3),$$

$$\begin{aligned}
\gamma_1(a, b_1, b_2) &= \rho(a, b_1 b_2)\, \nu_{b_1 b_2}(f(\tau_2(b_1, b_2), a))\, (\rho(\beta_{b_2}(a), b_1))^{-1}\, (\nu_{b_1}(\rho(a, b_2)))^{-1}, \\
\gamma_2(a_1, a_2, b) &= \rho(a_1 a_2, b)\, \nu_b(\tau_1(a_1, a_2))\, (\mu_{\beta_b(a_2)}(\rho(a_1, b)))^{-1}\, (\rho(a_2, b))^{-1}\, (\tau_1(\beta_b(a_1), \beta_b(a_2)))^{-1}, \\
\gamma_3(a_1, a_2) &= S\big(\nu^{-1}_{T(a_1 \circ_T a_2)}(\rho(a_2, T(a_1))\, \tau_1(a_1, \beta_{T(a_1)}(a_2))\, \nu_{T(a_1)}(f(\chi(a_1), a_2)))\big) \\
&\quad (\tau_2(T(a_1), T(a_2)))^{-1}\, (\delta^1_{\sigma}(\chi)(a_1, a_2))^{-1}
\end{aligned}$$

for $\tau_1 \in C^2(A, K)$, $\tau_2 \in C^2(B, L)$, $\rho \in C(A \times B, K)$, $\chi \in C(A, L)$, $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$.

Cohomology of RRB Groups

Theorem(—,N. Rathee, M. Singh)

$\mathrm{im}(\partial^0_{RRB}) \subseteq \ker(\partial^1_{RRB})$ and $\mathrm{im}(\partial^1_{RRB}) \subseteq \ker(\partial^2_{RRB})$.

We define the first and the second cohomology group of $\mathcal{A} = (A, B, \beta, T)$ with coefficients in $\mathcal{K} = (K, L, \alpha, S)$ by

$$\mathrm{H}^1_{RRB}(\mathcal{A}, \mathcal{K}) = \ker(\partial^1_{RRB}) / \mathrm{im}(\partial^0_{RRB})$$

and

$$\mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K}) = \ker(\partial^2_{RRB}) / \mathrm{im}(\partial^1_{RRB}).$$

Equivalent Extensions of RRB Groups

Theorem(—,N. Rathee, M. Singh)

Let $\mathcal{A} = (A, B, \beta, T)$ be a relative Rota–Baxter group and $\mathcal{K} = (K, L, \alpha, S)$ a trivial relative Rota-Baxter group, where K and L are abelian groups. Let (ν, μ, σ, f) be the quadruple of actions that makes $\mathcal{K}$ into an $\mathcal{A}$-module. Then there is a bijection between $\mathrm{Ext}_{(\nu,\mu,\sigma,f)}(\mathcal{A}, \mathcal{K})$ and $\mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K})$.

Some Applications

- Equivalence between the category of bijective relative Rota–Baxter groups and the category of skew left braces [6].
- Homomorphism from $\mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K}) \to \mathrm{H}^2_N(A_T, K_S)$ [7, 1].
- Schur Multiplier and Schur Cover for relative Rota–Baxter groups [2].
- Relation between the Schur covers of the relative Rota–Baxter group and the Schur covers of the skew braces [2].

Thank You.

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