Vanishing Elements of Prime Power Order

(Joint work with Rahul Kitture)

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Quick Introduction (All groups G considered here are finite groups)

- A $\mathbb{C}$-representation of a group G (of dimension n) is a homomorphism

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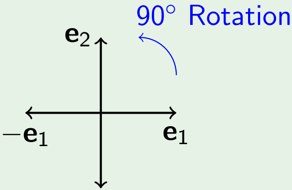
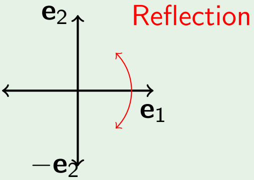
Example ($D_4 = \langle r, s \mid r^4 = s^2 = 1, srs^{-1} = r^{-1} \rangle$)





$$r \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad s \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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$$\rho' : g \mapsto P\rho(g)P^{-1}$$

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- ρ is **reducible** if it is equivalent to ρ' which is in a block-sum (otherwise, it is irreducible):

$$\rho'(x) = \begin{bmatrix} \rho_1(x) & \mathbf{0} \\ \mathbf{0} & \rho_2(x) \end{bmatrix} \qquad \text{(for all } x \in G).$$

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- **Character** of ρ is the map $\chi : G \to \mathbb{C}$, $\quad \chi(x) := \text{Trace}(\rho(x))$.

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- Character χ is said to be **irreducible** if ρ is irreducible.

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Example

Consider character table of group S_3 given below:

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
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- (Malle, Navarro, Olsson, 2000) Every non-abelian finite group contains a vanishing element **of prime power order**.

Influence of Vanishing Elements:

Theorem: (Dolfi, Pacifici, Sanus, 2010)

$$\left\{\begin{array}{c}\text{All vanishing elements have}\\ \text{conjugacy-class size}\\ \text{not divisible by } p\end{array}\right\} \Longrightarrow \left\{\begin{array}{c} G \text{ has normal}\\ p\text{-complement \&}\\ \text{abelian Sylow } p\text{-subgroups}\end{array}\right\}$$

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Example

$C_3 \rtimes C_4$	1	2	3	4A	4B	6
class size	1	1	2	3	3	2
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	1
χ_3	1	-1	1	i	$-i$	-1
χ_4	1	-1	1	$-i$	i	-1
χ_5	2	2	-1	0	0	-1
χ_6	2	-2	-1	0	0	1

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Theorem: (Julian Brough, 2016)

$$\left\{\begin{array}{c}\text{All vanishing elements have}\\ \text{square-free conjugacy-class size}\end{array}\right\} \Longrightarrow \left\{\begin{array}{c}\text{group is}\\ \text{supersolvable}\end{array}\right\}$$

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Theorem: (Julian Brough, Qingjun Kong, 2018)

$$\left\{\begin{array}{c}\text{No vanishing elements}\\ \text{of prime power order have}\\ \text{square-free conjugacy-class size}\end{array}\right\} \Longrightarrow \left\{\begin{array}{c}\text{group is}\\ \text{super-solvable}\end{array}\right\}$$

Influence of Vanishing Elements:

Theorem: (Bianchi, Lewis, Pacifici, 2019)

In a finite group G, all the vanishing elements have conjugacy-class size p if and only if

- either $G =$(p-group with class sizes 1 or p) $\times$ (abelian p'-group)
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Theorem: (Neda Ahanjideh, 2023)

$$\left\{\begin{matrix}\text{All vanishing elements have} \\ \text{same conjugacy-class size}\end{matrix}\right\} \Longrightarrow \left\{\begin{matrix}\text{group is} \\ \text{solvable}\end{matrix}\right\}$$

Theorem A (-, Kitture, 2025)

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- We used **classification of finite simple groups** in proof.

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- G be a non-solvable group
- $\mathrm{Sol}(G)$ =largest solvable normal subgroup of G(**solvable radical**)

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Example: Let $G = A_5 \times S_3$. It has vanishing elements of order 6. And their conjugacy class size is divisible by **three** distinct primes.
But for vanishing elements of **prime-power order**, their size of conjugacy class has at most two distinct prime divisors.

Example: $A_5 \times S_3$

Example

class $\rightarrow$	1a	2a	3a	3b	6a	3c	2b	2c	6b	5a	10a	15a	5b	10b	15b
Size $\rightarrow$	1	3	2	20	60	40	15	45	30	12	36	24	12	36	24
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_3	2	1	-1	-1	0	1	0	-1	0	1	0	-1	0	1	0
χ_4	3	-3	3	3	-1	-1	1	1	-1	-1	A	A	*A	*A	A
χ_5	-3	3	-3	-3	1	1	-1	-1	1	1	-A	-A	-*A	-*A	-A
χ_6	3	3	3	3	-1	-1	-1	-1	-1	-1	A	A	*A	*A	A
χ_7	3	3	3	3	-1	-1	-1	-1	-1	-1	*A	*A	A	A	*A
χ_8	4	4	4	4	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_9	4	4	4	4	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_{10}	5	5	5	5	-1	-1	-1	-1	-1	-1	1	1	1	1	1
χ_{11}	5	5	5	5	-1	-1	-1	-1	-1	-1	1	1	1	1	1
χ_{12}	6	-3	-3	0	0	-2	0	0	B	0	-A	*B	0	0	-*A
χ_{13}	6	-3	-3	0	0	-2	0	0	1	*B	0	-*A	B	0	-A
χ_{14}	8	-4	-2	0	-1	0	0	0	-2	0	0	1	-2	1	0
χ_{15}	10	-5	-2	0	-1	2	0	0	-1	0	0	0	0	0	0

$$A = \frac{1-\sqrt{5}}{2}, \qquad {}^*A = \frac{1+\sqrt{5}}{2}, \qquad B = 2A, \qquad {}^*B = 2{}^*A$$

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Thank you

Grazie!