On Infinitely Supported Groups

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Overview

Dedicated to the memory of

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Figure: F. De Giovanni

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What is an Infinitely Supported Group?

Definition

A group G with a generating set X is **infinitely supported** (**IS**-group) if every infinite subset of X also generates G.

- **IS**-groups are countable.
- Closely connected to **countably dominated** (**CD**) groups.
- Arise naturally in classifying infinite groups.

Our Motivation

Definition

A group G is called *barely transitive* group, if G has a subgroup H such that $|G:H|$ is infinite, $\mathrm{Core}_G(H) = g^{-1}Hg = 1$ and $|K : K \cap H|$ is finite for every proper subgroup of H.

- G is an **IS**-group
- So this property deserves to investigate in general.

Definition

Now let $\mathcal{S}$ and $\mathcal{T}$ be non-empty sets of proper subgroups of a group G. Then $\mathcal{T}$ is said to *dominate* $\mathcal{S}$ if each member of $\mathcal{S}$ is contained in a member of $\mathcal{T}$. If there is a countable set of proper subgroups of G that dominates $\mathcal{S}$, then $\mathcal{S}$ is said to be *countably dominated* or **CD** in G. Also if the set of all proper subgroups of G be dominated by a countable subset $\mathcal{S}$, then G is called *countably dominated* or a **CD**-group for short. **CD**-group are mainly considered in [A. Arikan, G. Cutolo, D.J.S. Robinson, On groups that are dominated by countably many proper subgroups. *J. Algebra* 509 (2018), 445–466]. **IS**-groups are closely related to **CD**-groups.

Characterizations and Key Lemmas

- G is an **IS**-group $\iff$ $G = \langle X \rangle$ and $H \cap X$ is finite for every proper subgroup H of G.
- If $G = \bigcup_{i=1}^{n} H_i$ for proper subgroupsH_i $(1 \leq i \leq n) \Rightarrow G$ is not an IS-group.

Theorem 3.4

1. If G has a finite non-cyclic image, then G is not an **IS**-group.
2. If G is **CD** and has no such image, then G is **IS**.

Examples of IS-Groups

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ $\mathbb{Z}_{p_1^\infty} \oplus \cdots \oplus$
- $\mathbb{Z}_{p_n^\infty} \oplus E$where E is a finite cyclic group
- Direct sums of distinct $\mathbb{Z}_{p^\infty}$
- Ol'shanskii and Obraztsov simple groups
- Heineken-Mohamed type groups
- Special Černikov groups
- Čarin's groups with *min*-n

Non-Examples of IS-Groups

- Free groups (residually finite)
- $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$
- Groups with non-cyclic finite quotients
- Braid groups B_n for $n \geq 5$
- First Grigorchuk group

Abelian IS-Groups

Lemma.

If G is an abelian **IS**-group, then it has finite (Prüfer) rank.

Theorem.

If G is an abelian **IS**-group by $X = \{x_i : \ i \in \omega\}$, then X has a finite subset F such that $\overline{G} = G/\langle F\rangle$ is periodic and every infinite p-component $\overline{G}_p$ is a direct sum of a Prüfer group and a cyclic group.

Complete Characeriztion of Periodic Abelian **IS**-Groups

Theorem.

Let G be a periodic abelian **IS**-group. Then G is in the following form:

$$G \cong (\oplus_{p\in\pi_1}\mathbb{Z}_{p^\infty}) \oplus (\oplus_{\substack{q\in\pi_2\\ n\in\mathbb{Z}^+}}\mathbb{Z}/q^n\mathbb{Z})$$

where $\pi_1,\ \pi_2$, are sets of primes at least one is non-empty.

Soluble and Special IS-Groups

There is little prospect of classifying soluble **IS**-groups, as there are too many different types. We mostly consider some celebrated constructions having **IS** property.

- Heineken-Mohamed type groups (nilpotent-by-cyclic, locally finite)
- Non-perfect minimal non-FC-groups
- Locally dihedral 2-group

These groups are **not abelian**, yet still satisfy IS conditions.

Conclusion and Open Question

- **IS**-groups offer a rich framework in group theory.
- Many celebrated constructions qualify.
- Abelian **IS**-groups $\Rightarrow$ **CD**, but what about the converse?

Open Question

Is every **IS**-group also a **CD**-group?

References

- R.S. Altınkaya, A. Arıkan, A. Arıkan. *On infinitely supported groups*, Ricerche di Matematica, 2025.

Thanks

THANK YOU

TEŞEKKÜRLER

Figure: